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A cactus group action on a shifted tableau crystal and a shifted Berenstein-Kirillov group

"Documento Definitivo"

Doutoramento em Matemática

Especialidade de Álgebra, Lógica e Fundamentos

Inês Martins Rodrigues

Tese orientada por:
Doutora Olga Maria da Silva Azenhas
Doutora Maria Manuel Correia Torres

Documento especialmente elaborado para a obtenção do grau de doutor

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Júri:

Presidente:

- Doutor Fernando Jorge Inocêncio Ferreira, Professor Catedrático e Presidente do Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa.

Vogais:

- Doutor Jake Levinson, Assistant Professor, do Department of Mathematics da Simon Fraser University, Canadá.
- Doutora Olga Maria da Silva Azenhas, Professora Auxiliar, da Faculdade de Ciências e Tecnologia da Universidade de Coimbra (orientadora).
- Doutor António José Mesquita da Cunha Machado Malheiro, Professor Associado com Agregação, da Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa.
- Doutor Carlos Alberto Martins André, Professor Associado, da Faculdade de Ciências da Universidade de Lisboa.

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Abstract

Gillespie, Levinson and Purbhoo recently introduced a crystal-like structure for shifted tableaux, called the shifted tableau crystal. This structure may be regarded as a directed acyclic weighted graph, with coloured double edges, having vertices the shifted semistandard tableaux. It decomposes into connected components, each one having unique source vertex, whose weight is a strict partition, and sink vertex, with reverse weight. The character of each connected component is the Schur Q-function indexed by the said strict partition. Following a similar approach as Halacheva, for crystals of finite-dimensional representations of the quantized universal enveloping algebra of a finite-dimensional complex reductive Lie algebra, we exhibit a natural internal action of the n-fruit cactus group on the shifted tableau crystal, realized by the restrictions of the shifted Schützenberger involution to all primed intervals of the underlying crystal alphabet. This includes the shifted crystal reflection operators, which agree with the restrictions of the shifted Schützenberger involution to single-coloured connected components, but unlike the case for type A crystals, these do not need to satisfy the braid relations of the symmetric group. In addition, we define a shifted version of the Berenstein–Kirillov group, by considering shifted Bender–Knuth involutions. Paralleling the works of Halacheva and Chmutov, Glick and Pylyavskyy for type A semistandard tableaux of straight shape, we exhibit another occurrence of the cactus group action on shifted tableau crystals of straight shape via the action of the shifted Berenstein–Kirillov group. We also conclude that the shifted Berenstein–Kirillov group is isomorphic to a quotient of the cactus group. Not all known relations that hold in the classic Berenstein–Kirillov group need to be satisfied by the shifted Bender–Knuth involutions, but the ones implying the relations of the cactus group are verified, thus we have another presentation for the cactus group in terms of shifted Bender–Knuth involutions. We also use the shifted growth diagrams due to Thomas and Yong, together with the semistandardization process of Pechenik and Yong, to provide an alternative proof concerning the mentioned cactus group action.

Keywords: Shifted tableaux, crystal graphs, Schützenberger involution, cactus group, Berenstein–Kirillov group.

RESUMO ALARGADO

As funções P e Q de Schur são funções simétricas que surgem no contexto da teoria de representação projetiva de grupos simétricos. Estas funções são indexadas por partições com partes distintas, chamadas partições estritas. Ambas são especializações de funções simétricas de Hall–Littlewood e são somas de monómios que admitem uma descrição combinatória através de certos tableaux desviados. Estes tableaux correspondem a diagramas desviados, associados a partições estritas, semelhantes a diagramas de Ferrers, nos quais cada linha é *desviada* uma unidade para a direita, relativamente à linha anterior, e são preenchidos num alfabeto marcado $[n]' := \{1' < 1 < \cdots < n' < n\}$, satisfazendo certas condições. A presente tese restringe-se aos tableaux desviados que geram as funções Q de Schur.

Gillespie, Levinson e Purbhoo (2017, 2020) introduziram recentemente uma estrutura de cristal para estes tableaux desviados. Esta estrutura, denotada $\mathsf{ShST}(\lambda/\mu, n)$, pode ser vista como um grafo dirigido acíclico, com arestas duplas coloridas. O conjunto dos seus vértices, no qual está definida uma função de peso, é formado por tableaux desviados semistandard de forma λ/μ preenchidos em $[n]'$, e as suas arestas são definidas usando os operadores de cristal marcados e não-marcados, que comutam com o *jeu de taquin*.

Ao contrário da estrutura de cristal para tableaux de Young, motivada pela teoria de representações finitas da álgebra envolvente quantizada $U_q(\mathfrak{gl}_n)$ da álgebra de Lie linear geral $\mathfrak{gl}_n$, e que constitui um modelo para os cristais de tipo A_{n-1}, a estrutura de cristal em $\mathsf{ShST}(\lambda/\mu, n)$ tem a sua origem no cálculo de Schubert em tipo B (ou tipo C). No entanto, não é conhecido se forma bases cristalinas para as representações de alguma álgebra envolvente quantizada, ao contrário de outras estruturas de cristal com tableaux desviados para a super-álgebra de Lie *queer* $\mathfrak{q}(n)$, que têm as funções P de Schur como caráteres.

A estrutura de cristal em $\mathsf{ShST}(\lambda/\mu, n)$ apresenta propriedades semelhantes às dos cristais normais de tipo A, que podem ser inteiramente descritos em termos de tableaux de Young semistandard. Com efeito, esta estrutura decompõe-se em componentes conexas, que são isomorfas por retificação a cristais de tableaux desviados de forma retificada,

$$\mathsf{ShST}(\lambda/\mu, n) \simeq \bigsqcup_{\nu} \mathsf{ShST}(\nu, n)^{f^{\lambda}_{\mu\nu}},$$

onde $f^{\lambda}_{\mu\nu}$ denota um coeficiente de Littlewood–Richardson desviado. Cada componente conexa possui um único vértice fonte, cujo peso é uma partição estrita, e um único vértice sumidouro cujo peso é o reverso dessa partição. O caráter de cada componente conexa é a função Q de Schur indexada pela partição estrita do vértice fonte. Assim, esta decomposição permite recuperar a regra de Littlewood–Richardson para funções Q de Schur enviesadas,

$$Q_{\lambda/\mu}(x) = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_{\nu}(x).$$

A involução de Schützenberger–Lusztig define-se nesta estrutura de cristal de modo semelhante ao dos cristais de tableaux de Young. Trata-se da única involução que reflete o grafo do cristal através de um eixo horizontal, revertendo o sentido das arestas, as suas cores, e o peso de cada vértice. Esta involução coincide com a operação *reversal* em tableaux desviados, e no caso particular de formas retificadas, com a evacuação. A operação *reversal* define-se num tableau através da interseção de certas classes de equivalência dual e de Knuth, e admite uma descrição explícita baseada no *jeu de taquin*.

Halacheva (2016, 2020) mostrou que existe uma ação natural interna do grupo *cactus* $J_{\mathfrak{g}}$ em cristais das representações finitas de $U_q(\mathfrak{g})$, onde $\mathfrak{g}$ é uma álgebra de Lie complexa redutiva de dimensão finita, através de restrições da involução de Schützenberger aos subconjuntos conexos não-vazios formados por nós do diagrama de Dynkin de $\mathfrak{g}$. Em particular, para os cristais de tipo A_{n-1}, onde $J_{\mathfrak{gl}_n} = J_n$, estas restrições correspondem aos subintervalos conexos de $[n-1]$. Seguindo uma abordagem semelhante, exibimos uma ação natural interna do grupo *cactus* J_n na estrutura de cristal em $\mathsf{ShST}(\lambda/\mu, n)$, que é realizada pelas restrições da involução de Schützenberger a todos os subintervalos *marcados* de $[n-1]$. Isto inclui, em particular, os operadores de reflexão do cristal σ_i, correspondendo às restrições da involução de Schützenberger às componentes conexas de uma só cor. Ao contrário do caso para os cristais de tipo A_{n-1}, este operadores não satisfazem necessariamente as relações $(\sigma_i\sigma_{i+1})^3 = 1$, para $i \in [n-2]$, do grupo simétrico $\mathfrak{S}_n$ (o grupo de Weyl de $\mathfrak{gl}_n$), pelo que a ação não se fatoriza através das relações correspondentes. É uma questão em aberto, tanto no tipo A como no caso desviado, saber se os operadores de reflexão do cristal satisfazem outras relações. Importa notar que existe também uma ação externa do grupo *cactus* no produto tensorial de cristais normais, no entanto, não é conhecido um produto tensorial para a estrutura de cristal de tableaux desviados.

Adicionalmente, definimos uma versão desviada das involuções de Bender–Knuth $\mathtt{t}_i$, utilizando o algoritmo de tableau *switching* para tableaux desviados, introduzido por Choi, Nam e Oh (2017), ou, equivalentemente, a infusão de tipo C de Thomas e Yong (2009) em tableaux desviados standard, juntamente com o processo de semistandardização de Pechenik e Yong (2017). Utilizando os operadores $\mathtt{t}_i$, introduzimos uma versão em termos de tableaux desviados do grupo de Berenstein–Kirillov $\mathcal{BK}$, que denotamos por $\mathcal{SBK}$. O grupo $\mathcal{BK}$ foi introduzido por Berenstein e Kirillov (1995), como o grupo livre gerado pelas involuções de Bender–Knuth, sujeito às relações que estas satisfazem em tableaux de Young semistandard. O grupo $\mathcal{BK}_n$ é o subgrupo de $\mathcal{BK}$ gerado pelas involuções de Bender–Knuth $\mathtt{t}_1, \dots, \mathtt{t}_{n-1}$. Este grupo atua naturalmente nos cristais de $\mathfrak{gl}_n$ de tableaux de Young semistandard de forma retificada, e esta ação coincide com a já referida ação de J_n. Como consequência, o grupo $\mathcal{BK}_n$ é isomorfo a um quociente de J_n.

Os grupos de Berenstein–Kirillov desviados $\mathcal{SBK}$ e $\mathcal{SBK}_n$ são definidos de forma análoga, através das involuções de Bender–Knuth desviadas. Em paralelo com os trabalhos de Halacheva (2016, 2020) e Chmutov, Glick e Pylyavskyy (2016, 2020) para cristais do tipo A de tableaux de Young semistandard de forma retificada, provamos que o grupo $\mathcal{SBK}_n$ atua também de forma natural na estrutura de cristal em $\mathsf{ShST}(\nu, n)$, e que esta ação coincide com a do grupo *cactus*. Como consequência, $\mathcal{SBK}_n$ é também isomorfo a um quociente do grupo *cactus* J_n. No tipo A, este quociente não é trivial, uma vez que existe pelo menos uma relação válida em $\mathcal{BK}$ que não é equivalente a nenhuma relação do grupo *cactus*. Com efeito, a relação $(t_1 t_2)^6 = 1$ é válida em $\mathcal{BK}$, sendo equivalente à relação $(\varsigma_i \varsigma_{i+1})^3 = 1$, para qualquer $i \in [n-2]$, em que ς_i denota o operador de reflexão de um cristal de tipo A. Esta equivalência é também válida para os operadores desviados, contudo, as relações de trança não são necessariamente satisfeitas. Não obstante, os operadores desviados $\mathtt{t}_i$ satisfazem todas as relações que são equivalentes a relações do grupo *cactus*, pelo que temos uma apresentação alternativa para o grupo *cactus*, através das involuções de Bender–Knuth desviadas. É uma questão em aberto saber se existem outras relações, tanto em $\mathcal{BK}$ como em $\mathcal{SBK}$, que não sejam equivalentes às do grupo *cactus*.

A prova de que o grupo *cactus* atua no cristal de tableaux desviados através de restrições da involução de Schützenberger utiliza a sua formulação enquanto única involução satisfazendo certas condições em termos de operadores dos cristal $\mathsf{ShST}(\lambda/\mu, n)$. Esta involução e as suas restrições coincidem com as involuções *reversal*, pelo que podem ser descritas enquanto operadores explícitos em tableaux desviados. Assim, é possível utilizar diagramas de crescimento

para tableaux desviados standard, introduzidos por Thomas e Yong (2016), que generalizam os diagramas de crescimento de Fomin para tableaux de Young, juntamente com o processo de semistandardização, de Pechenik e Yong (2017), para obter uma prova alternativa de que o grupo *cactus* J_n atua em $\mathsf{ShST}(\lambda/\mu, n)$ através de restrições da involução *reversal*.

Palavras-chave: Tableaux desviados, grafos de cristais, involução de Schützenberger, grupo *cactus*, grupo de Berenstein–Kirillov.

Symbols and notations

$\mathbb{N}$	set of natural numbers $\{1, 2, \ldots\}$	
$\mathbb{Z}$	set of integer numbers	
$\mathbb{Z}_{\geq 0}$	set of non-negative integers $\{0, 1, 2, \ldots\}$	
$\mathbb{Q}$	set of rational numbers	
$[n]$	alphabet $\{1 < \cdots < n\}$	2
$[i, j]$	alphabet $\{i < \cdots < j\}$, for $i < j$	6
$[n]'$	alphabet $\{1' < 1 < \cdots < n' < n\}$	15
$[i, j]'$	alphabet $\{i' < i < \cdots < j' < j\}$, for $i < j$	55
α_i	usually denotes the simple roots for the A_{n-1} root system	39
$\mathcal{BK}$	Berenstein–Kirillov group	87
$\mathcal{BK}_n$	subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$	87
$\mathsf{c}_n(T)$	complement of $T \in \mathsf{ShST}(\lambda/\mu, n)$	25
D_w	diagonally-shaped shifted tableau with word w	17
η	shifted Schützenberger–Lusztig involution	52
$\eta_{i,j}$	partial shifted Schützenberger–Lusztig involution	56
E'_i, F'_i	primed shifted tableau crystal raising operators	39
E_i, F_i	unprimed shifted crystal lowering operators	42
ε_i, φ_i	shifted tableau crystal total length functions	44
$\varepsilon'_i, \varphi'_i$	shifted tableau crystal primed length functions	44
$\hat{\varepsilon}_i, \hat{\varphi}_i$	shifted tableau crystal unprimed length functions	44
evac	shifted evacuation	26
evac_i	restriction of the shifted evacuation	36
$f^{\lambda}_{\mu\nu}$	shifted Littlewood–Richardson coefficient	8
GL_n	general linear group of degree n over $\mathbb{C}$	3

$\mathfrak{gl}_n$	general linear Lie algebra of GL_n	5
I	usually denotes $[n-1]$	39
$\mathsf{infusion}$	shifted infusion	34
$\equiv_k$	Knuth equivalence	23
λ	usually denotes a strict partition	15
$\lvert\lambda\rvert$	sum of the parts of λ	15
$\ell(\lambda)$	length of λ	15
P_λ	Schur P-function	7
$\mathsf{p_i}$	shifted promotion operator	83
Q_λ	Schur Q-function	7
q_i	$t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$	88
$q_{i,j}$	$q_{j-1}q_{j-i}q_{j-1}$	88
$\mathsf{q_i}$	$\mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)$	85
$\mathsf{q}_{i,j}$	$\mathsf{q}_{j-1}\mathsf{q}_{j-i}\mathsf{q}_{j-1}$	92
$\mathsf{rect}(T)$	rectification of T	19
$\mathfrak{S}_n$	symmetric group of $[n]$	17
s_λ	Schur function	
ς_i	type A crystal reflection operator	12
σ_i	shifted crystal reflection operator	58
$\mathcal{SBK}$	shifted Berenstein–Kirillov group	90
$\mathcal{SBK}_n$	subgroup of $\mathcal{SBK}$ generated by $\mathsf{t}_1, \ldots, \mathsf{t}_{n-1}$	90
$\mathsf{SP}(A, B)$	shifted tableau switching of a perforated pair	29
$\mathsf{SP}_{i,j}(T)$	shifted tableau switching of T^i and T^j	80
$\mathsf{SW}(S, T)$	shifted tableau switching of a pair of shifted tableaux	31
$\mathsf{SW}_{i\|i+1,\ldots,j}$	shifted tableau switching of T^i and $T^{i+i,j}$	80
$\mathsf{ShST}(\lambda/\mu, n)$	set of shifted semistandard tableaux of shape λ/μ in alphabet $[n]'$	17
$\mathsf{SSYT}(\lambda/\mu, n)$	set of semistandard Young tableaux of shape λ/μ in alphabet $[n]$	2
$\mathsf{std}(T)$	standardization of T	19
$\mathsf{sstd}_\nu(T)$	semistandardization of T with respect to ν	20
t_i	Bender–Knuth involution for Young tableaux	2
t_i	shifted Bender–Knuth involution	81
T^e	shifted reversal of T	26

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CHAPTER 1

INTRODUCTION

Schur Q- and P-functions were firstly introduced by Schur [62], to study projective representations of symmetric groups. They are symmetric functions, indexed by strict partitions and have a combinatorial description in terms of certain shifted tableaux [68]. These combinatorial objects carry many interesting parallels with the classical Young tableaux [58, 72]. There are different definitions of shifted tableaux, resulting from considering different rules to fill shifted diagrams (see, for instance, [11, 23, 66, 68]). The present thesis focus on the ones generating the Schur Q-functions. These tableaux may be organized into a crystal-like structure [23] called a *shifted tableau crystal*, that in many aspects resembles the one for normal Kashiwara crystals of type A. This structure differs from other crystals on shifted decomposition tableaux [25, 26] and on shifted tableaux for Schur P-functions [1, 24], which are crystals for the quantum queer superalgebra $U_q(\mathfrak{q}(n))$. It is not known whether the crystal-like structure due to Gillespie–Levinson–Purbhoo forms crystal bases for the representations of some quantized universal enveloping algebra. Unlike the case for crystals for $\mathfrak{gl}_n$, we do not have a natural action of the symmetric group $\mathfrak{S}_n$ on these shifted tableau crystals. However, analogous to crystals for $\mathfrak{gl}_n$, we will show that there exists a natural internal action of the cactus group, which is realized by the partial shifted Schützenberger involutions. This action has another occurrence via generators of the shifted Berenstein–Kirillov group that we introduce.

Young tableaux and Schur functions

Schur functions are a well-known family of symmetric functions, appearing in many areas of mathematics. More precisely, let $\Lambda = \Lambda_{\mathbb{Z}}$ denote the *algebra of symmetric functions* over $\mathbb{Z}$. As a set, it consists of the bounded-degree formal power series with coefficients in $\mathbb{Z}$, in countably many infinite variables $x = \{x_1, x_2, \dots\}$, that are invariant under any permutation of the variables. This algebra has a natural grading $\Lambda = \bigoplus_{n\in\mathbb{N}} \Lambda^n$, where Λ^n denotes the $\mathbb{Z}$-

module of symmetric functions homogeneous of degree n. We also consider $\Lambda_{\mathbb{Q}}$ and $\Lambda_{\mathbb{Q}}^n$ to be the corresponding $\mathbb{Q}$-algebra and $\mathbb{Q}$-vector space.

A *partition* of a positive integer m is a sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ displayed in weakly decreasing order and such that $\lambda_1 + \cdots + \lambda_k = m$. A partition λ of m is associated with a *Young diagram* (or *Ferrers diagram*), consisting of m boxes disposed in k left-justified rows, such that i-th row has λ_i boxes (following the English or matrix notation). Given $\mu \subseteq \lambda$, the skew diagram λ/μ is defined as the set of boxes of λ that are not boxes in μ. A *semistandard Young tableau* is a filling of a Young diagram with a totally ordered alphabet such that rows are weakly increasing and columns are strictly increasing. We denote by $\mathsf{SSYT}(\lambda/\mu, n)$ the set of semistandard Young tableaux of shape λ/μ filled in $[n] := \{1, \ldots, n\}$, and by $\mathsf{SSYT}(\lambda/\mu)$ the (infinite) set of the ones filled in $\mathbb{N}$. Given a set of countably many infinite variables $x = \{x_1, x_2, \ldots\}$ and a vector $\alpha = (\alpha_1, \alpha_2, \ldots)$, such that $\alpha_k = 0$ for all $k > N$, for some $N \in \mathbb{N}$, let $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. In particular, we associate to a semistandard Young tableau T the monomial $x^{wt(T)} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $wt(T) = (\alpha_1, \ldots, \alpha_m)$ is the *weight* of T, i.e., the vector such that α_i is equal to the number of i's in T.

Given $n \in \mathbb{N}$, there are many well-known linear bases for $\Lambda_{\mathbb{Z}}^n$ and $\Lambda_{\mathbb{Q}}^n$, indexed by partitions of n. For the purpose of this thesis, we only need to recall the power sum symmetric functions and Schur functions. For a more detailed introduction, we refer to [49, Chapter I], [60, Chapter 4] and [67, Chapter 7]. The *k-th power sum symmetric function* p_k, for $k \geq 1$, is defined as

$$p_k(x) = x_1^k + x_2^k + \ldots \tag{1.1}$$

and, given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k}$. The *Schur function* [45] corresponding to a partition λ is given by

$$s_\lambda(x) = \sum_{T \in \mathsf{SSYT}(\lambda)} x^{wt(T)}. \tag{1.2}$$

Unlike the power sum symmetric functions, it is not entirely obvious from the definition that the Schur functions are symmetric. One way to prove that they are symmetric functions is using the Bender–Knuth involutions t_i [4], which act on a semistandard Young tableau by swapping the multiplicities of i and $i+1$.

The sets $\{p_\lambda\}$ and $\{s_\lambda\}$, indexed by partitions λ of n, are linear bases for $\Lambda_{\mathbb{Q}}^n$, and the latter is a linear basis for $\Lambda_{\mathbb{Z}}^n$. The Littlewood–Richardson coefficients $c_{\mu\nu}^\lambda$ [46] are the structure constants that appear in the linear expansions of the product of Schur functions $s_\mu s_\nu$ and of the skew

Schur function $s_{\lambda/\mu}$ in the $\mathbb{Z}$-basis of Λ of Schur functions. These coefficients are non-negative integers and have a nice combinatorial description in terms of semistandard Young tableaux with a certain rectification (for instance, see [60, Theorem 4.9.4] or [67, Theorem A1.3.3]). The Schur functions play an important role in group representation theory of $\mathfrak{S}_n$. Namely, they are the image under the Frobenius characteristic map of the irreducible representations of $\mathfrak{S}_n$ (see, for instance, [60, Theorem 4.6.4]).

We now consider Λ_n to be the ring of symmetric *polynomials* in n variables, with coefficients in $\mathbb{Z}$. We may consider a specialization $\Lambda \longrightarrow \Lambda_n$ taking a symmetric function $f(x_1, x_2, \dots)$ in Λ to a symmetric polynomial $f(x_1, \dots, x_n) := f(x_1, \dots, x_n, 0, 0, \dots)$ in Λ_n (see [67, Section 7.8]). In what follows, we consider the *Schur polynomial* $s_\lambda(x_1, \dots, x_n)$ to be a specialization in Λ_n of the Schur function s_λ in Λ_n, for λ a partition with at most n parts. The irreducible polynomial representations ϕ^λ of $GL_n(\mathbb{C})$ are indexed by partitions with at most n parts, and their characters are given by $char(\phi^\lambda)(X) = s_\lambda(x_1, \dots, x_n)$, where $\{x_1, \dots, x_n\}$ is the set of eigenvalues of $X \in GL_n(\mathbb{C})$ [61].

The Schur polynomials are also present in the context of type A Schubert calculus. The Grassmannian $Gr(k, \mathbb{C}^n)$ is the set of all k-dimensional subspaces of $\mathbb{C}^n$, and may be regarded as a projective algebraic variety via an embedding of $Gr(k, \mathbb{C}^n)$ into the projective space $\mathbb{P}(\Lambda^k\mathbb{C}^n)$, where $\Lambda^k\mathbb{C}^n$ denotes the k-th exterior power of $\mathbb{C}^n$, called the Plücker embedding (for details see [18, 19]). The Grassmannian has a cellular decomposition into *Schubert cells* Ω_λ, consisting of the subspaces in $Gr(k, \mathbb{C}^n)$ whose associated row echelon form corresponds to a partition λ, fitting in an ambient rectangle $k \times (n-k)$. The *Schubert varieties* X_λ are obtained by taking the closure, with respect to the topology inherited from $\mathbb{P}(\Lambda^k\mathbb{C}^n)$, and the *Schubert classes* σ_λ are the fundamental classes of Schubert varieties in the cohomology ring $H^*(Gr(k, \mathbb{C}^n))$. Schubert classes form a $\mathbb{Z}$-basis for this cohomology ring. Moreover, $H^*(Gr(k, \mathbb{C}^n))$ is isomorphic, as a ring, to a quotient of Λ_k by the ideal generated by the Schur polynomials indexed by partitions that do not fit the ambient rectangle $k \times (n-k)$. Thus, the Schur polynomial s_λ is the representative of the Schubert class σ_λ, for $\lambda \subseteq k \times (n-k)$ [18, Section 9.4]. The Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ also appear as structure constants of the Schubert class σ_λ in the cup product $\sigma_\mu\sigma_\nu$ in $H^*(Gr(k, \mathbb{C}^n))$ (see, for instance, [18, 19]).

Kashiwara crystals

Kashiwara [37, 38] and Lusztig [47] independently introduced *crystal bases* (or *canonical bases*) to study representations of quantized universal enveloping algebras $U_q(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Informally speaking, crystal bases provide bases for $U_q(\mathfrak{g})$-modules at $q = 0$. We may associate to a crystal basis a unique directed, weighted, edge-coloured graph, called the *crystal graph*, and its character coincides with the character of the representation. Moreover, crystal graphs provide formulations for tensor product decomposition and branching rules.

Let E be an Euclidean space, with inner product $\langle , \rangle$. Given a root system Φ in V with index set I, let $\{\alpha_i, i \in I\}$ be the set of simple roots and $\{\alpha_i^\vee, i \in I\}$ the set of simple coroots, and let Λ be the weight lattice and Λ_+ the set of dominant weights. A *Kashiwara crystal* of type Φ [9, Definition 2.13] is a non-empty set $\mathcal{B}$ together with maps $e_i, f_i : \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{\varnothing\}$, where $\varnothing \notin \mathcal{B}$, $\varepsilon_i, \varphi_i : \mathcal{B} \longrightarrow \mathbb{Z}$, for $i \in I$, and $wt : \mathcal{B} \longrightarrow \Lambda$, satisfying the following:

1. For any $b, c \in \mathcal{B}$ and any $i \in I$, $e_i(b) = c$ if and only if $f_i(c) = b$, and in such case, $wt(c) = wt(b) + \alpha_i$, $\varepsilon_i(c) = \varepsilon_i(b) - 1$, and $\varphi_i(c) = \varphi_i(b) + 1$.

2. For any $b \in \mathcal{B}$ and any $i \in I$, $\varphi_i(b) - \varepsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$.

The maps e_i, f_i are called the *Kashiwara or crystal operators*, with e_i being a *raising operator* and f_i a *lowering operator*, the maps ε_i, φ_i are called the *length maps*, and the map wt is called the *weight map*. A Kashiwara crystal is said *seminormal* if, for each $i \in I$ and $b \in \mathcal{B}$,

$$\varepsilon_i(b) = \max\{k : e_i^k(b) \neq \varnothing\}, \quad \varphi_i(b) = \max\{k : f_i^k(b) \neq \varnothing\}.$$

To a Kashiwara crystal we associate a directed, acyclic weighted graph, called the *crystal graph*, with vertices in $\mathcal{B}$ and edges labelled in I, in which there is a i-coloured directed edge $b \xrightarrow{i} c$ if and only if $f_i(b) = c$, or equivalently, $e_i(c) = b$, for $i \in I$. The i-coloured connected components of a crystal graphs are called the *i-strings*. In particular, a crystal is seminormal if the length maps $\varphi_i(b)$ and $\varepsilon_i(b)$ measure the distance of b to the ends of its i-string, for any $i \in I$.

A crystal $\mathcal{B}$ is called a *highest weight crystal*, with highest weight $\lambda \in \Lambda_+$, if there exists an element $b_\lambda \in \mathcal{B}$ such that $wt(b_\lambda) = \lambda$, $e_i(b_\lambda) = \varnothing$, for all $i \in I$, and $\mathcal{B}$ is generated by the maps f_i acting on b_λ. Given a finite-dimensional complex reductive Lie algebra $\mathfrak{g}$ with root system Φ, each representation $V = \bigoplus_{\lambda \in \Lambda_+} V^\lambda$ of $\mathfrak{g}$ may be associated with a seminormal crystal of type Φ, with I being the set of nodes of its Dynkin diagram. For a dominant weight $\lambda \in \Lambda_+$, let

$\mathcal{B}_\lambda$ denote the connected crystal of the irreducible representation V^λ of $\mathfrak{g}$ of highest weight λ, which is a highest weight crystal. The character of $\mathcal{B}_\lambda$ coincides with the character of V^λ. A crystal is said to be *normal* if it is isomorphic to a disjoint union of $\mathcal{B}_\lambda$, for $\lambda \in \Lambda_+$. Normal crystals have nice properties, namely, the subcrystals $\mathcal{B}_J$ of a normal crystal $\mathcal{B}$, obtained from the crystal graph of $\mathcal{B}$ considering the edges labelled in a connected subset of nodes $J \subseteq I$, are also normal.

In what follows, we will consider normal crystals of type A_{n-1}, that is $\mathfrak{g} = \mathfrak{gl}_n$, the general linear Lie algebra. In this case we have $\Lambda = \mathbb{Z}^n$ and $I = [n-1]$, and for each $i \in I$, $\alpha_i = \alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, and the set of dominant weights Λ_+ is the set of partitions having at most n parts. The set $\mathsf{SSYT}(\lambda, n)$, where λ has at most n parts, has the structure of normal type A_{n-1} crystal [41], and indeed, type A normal crystals may be described entirely in terms of semistandard Young tableaux, where the crystal operators e_i, f_i are coplactic, that is, they commute with the *jeu de taquin* [9, 43].

The crystal reflection operator ς_i, originally defined by Lascoux and Schützenberger [44] for type A crystals, acts on a i-string by reflecting it through its middle axis. These operators define an action of the symmetric group $\mathfrak{S}_n$ (which is the Weyl group of $\mathfrak{gl}_n$) on those crystals, as they satisfy the braid relations. For any normal crystal, Kashiwara defined the action of the corresponding Weyl group [39, Theorem 7.2.2] [40, Theorem 11.1].

The Schützenberger involution [63] is an involution on semistandard Young tableaux, also known as the evacuation on straight shapes or the reversal on skew-shapes [5, 28]. It realizes the Lusztig involution [48] on normal crystals of type A, as a set map on $\mathcal{B}$ acting on the graph structure by "flipping" it upside down, while reverting the orientation of the arrows and its colours. More precisely, consider a normal crystal $\mathcal{B}$ of type A_{n-1} and define the map $\xi : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_\lambda$, on a connected component $\mathcal{B}_\lambda$, as the unique set of maps satisfying, for all $b \in \mathcal{B}_\lambda$,

1. $e_i\xi(b) = \xi f_{n-i}(b)$,
2. $f_i\xi(b) = \xi e_{n-i}(b)$,
3. $wt(\xi(b)) = \theta_{1,n} \cdot wt(b)$,

where $\theta_{1,n}$ denotes the longest permutation in $\mathfrak{S}_n$. In particular, the map ξ takes the highest weight b_λ element to the lowest weight element b_λ^{low}, which is the unique element in $\mathcal{B}_\lambda$ such that $f_i(b_\lambda^{\mathsf{low}}) = \varnothing$, for any $i \in I$. The *Schützenberger–Lusztig involution* is defined on $\mathcal{B}$ by applying ξ to each connected component of $\mathcal{B}$. The crystal reflection operators ς_i correspond to the restrictions of the Schützenberger–Lusztig involution to the i-strings.

The cactus group

The n-fruit cactus group J_n first appeared in the works of Devadoss [17] and Davis, Januszkiewicz and Scott [16], as the fundamental group of the quotient orbifold of $\overline{M}_0^{n+1}(\mathbb{R})$, the Deligne–Mumford moduli space of stable curves of genus 0 with $n+1$ marked points, by the action of $\mathfrak{S}_n$ that permutes the first n of those points. We recall its definition as a free group subject to certain relations, as presented by Henriques and Kamnitzer [32].

Definition 1.1 ([32, Section 3.1])**.** The *n-fruit cactus group* J_n is the free group with generators $s_{i,j}$, for $1 \leq i < j \leq n$, subject to the relations:

1. $s_{i,j}^2 = 1$.
2. $s_{i,j}s_{k,l} = s_{k,l}s_{i,j}$, for $[i,j] \cap [k,l] = \varnothing$.
3. $s_{i,j}s_{k,l} = s_{i+j-l,i+j-k}s_{i,j}$, for $[k,l] \subseteq [i,j]$.

There is an epimorphism $J_n \longrightarrow \mathfrak{S}_n$, sending $s_{i,j}$ to the longest permutation of $\mathfrak{S}_{[i,j]}$, embedded in $\mathfrak{S}_n$. The kernel of this epimorphism is known as the *pure cactus group* and denoted by PJ_n (see [32, Section 3.4]). Halacheva [29, 30] generalized the notion of cactus group by $J_\mathfrak{g}$ for any finite-dimensional complex reductive Lie algebra $\mathfrak{g}$ (see [29, Chapter 10]), where J_n corresponds to $J_{\mathfrak{gl}_n}$, and showed that there is an internal action of the cactus group $J_\mathfrak{g}$ in a normal $\mathfrak{g}$-crystal, via the partial Schützenberger involutions, which correpond to restrictions of the Schützenberger–Lusztig involution to any non-empty connected subset of nodes of the Dynkin diagram of $\mathfrak{g}$. For the type A crystal graph, these are the restrictions of the Schützenberger involution to the subgraphs corresponding to the edges coloured in connected subintervals of $[n-1]$. For the i-strings, the action of the cactus group agrees with the action of the corresponding Weyl group generators, for all $i \in I$. Indeed, the internal action of the cactus group factors through the quotient of this group by the corresponding braid relations of the Weyl group [29, 30, 31]. For type A_{n-1} crystals, these are precisely the crystal reflection operators, and thus, the action of the cactus group J_n factors through the quotient of the braid relations of $\mathfrak{S}_n$.

Shifted tableaux and Schur P- and Q-functions

Schur P- and Q-functions are symmetric functions indexed by partitions with different parts, called *strict* partitions. They are specializations of Hall–Littlewood functions [49, Chapter III], and form dual bases, with respect to a modified Hall scalar product, for the subalgebra of the symmetric functions over $\mathbb{Q}$ generated by the odd-degree power sum symmetric functions. They have a combinatorial description as a sum of monomials arising from certain shifted semistandard tableaux, which are fillings of shifted shapes (corresponding to strict partitions) in a primed alphabet $[n]' := \{1' < 1 < \cdots < n' < n\}$.

Let $\mathsf{ShST}_Q(\lambda/\mu, n)$ be the set of shifted semistandard tableaux of shape λ/μ, on the alphabet $[n]'$ that are not required to be in canonical form (for a precise definition see Section 2.1), and let $\mathsf{ShST}_P(\lambda/\mu, n)$ be the subset of $\mathsf{ShST}_Q(\lambda/\mu, n)$ of shifted tableaux without primed entries on the main diagonal. We denote by $\mathsf{ShST}_Q(\lambda/\mu)$ and $\mathsf{ShST}_P(\lambda/\mu)$ the (infinite) sets of the corresponding tableaux filled in $\{1' < 1 < \cdots\}$. As before, we associate to a shifted tableau T a monomial $x^{\mathsf{wt}(T)}$, where $\mathsf{wt}(T) = (wt_1, \ldots, wt_n)$ is the weight of T, the vector in which wt_i is equal to the total number of i and i' in T.

Definition 1.2. Let $\mu \subseteq \lambda$ be a strict partitions. The *Schur Q-function* is defined as

$$Q_{\lambda/\mu}(x) = \sum_{T \in \mathsf{ShST}_Q(\lambda/\mu)} x^{\mathsf{wt}(T)},$$

and the *Schur P-function* is defined as

$$P_{\lambda/\mu}(x) = \sum_{T \in \mathsf{ShST}_P(\lambda/\mu)} x^{\mathsf{wt}(T)}.$$

It follows from the definition that $P_{\lambda/\mu}(x) = 2^{\ell(\lambda)-\ell(\mu)} Q_{\lambda/\mu}(x)$. Both Schur Q- and P-functions are symmetric functions, as they are specializations of Hall–Littlewood functions, although there are combinatorial proofs in the same fashion as for the classical Schur functions [68, Corollary 6.2].

Let $\Omega_{\mathbb{Q}} := \langle p_1, p_3, p_5, \ldots \rangle$ be the subalgebra of $\Lambda_{\mathbb{Q}}$ generated by the odd-degree power sum symmetric functions (1.1). This algebra also has a natural grading $\Omega_{\mathbb{Q}} = \bigoplus_{n \in \mathbb{N}} \Omega_{\mathbb{Q}}^n$, where $\Omega_{\mathbb{Q}}^n$ is the set of functions of $\Omega_{\mathbb{Q}}$ that are homogeneous of degree n. Let $\Omega = \Omega_{\mathbb{Z}} := \Omega_{\mathbb{Q}} \cap \Lambda_{\mathbb{Z}}$ denote the subring of $\Omega_{\mathbb{Q}}$ with coefficients in $\mathbb{Z}$, which also has a natural grading $\Omega = \bigoplus_{n \in \mathbb{N}} \Omega^n$, with Ω^n denoting the subset of functions of Ω homogeneous of degree n.

The sets $\{Q_\lambda\}$ and $\{P_\lambda\}$, indexed by strict partitions λ of n, are linear bases of Ω^n. Moreover they are dual bases under a modified Hall scalar product [68, (5.2) and Corollary 6.2].

The *shifted Littlewood–Richardson coefficients* $f_{\mu\nu}^{\lambda}$ are the constants that appear in the linear expansion of the skew Schur Q-functions in the basis of Schur Q-functions and in the product of P-functions in the basis of Schur P-functions (a combinatorial definition is presented in Section 2.2):

$$Q_{\lambda/\mu} = \sum_{\nu} f_{\mu\nu}^{\lambda} Q_{\nu}, \qquad P_{\mu} P_{\nu} = \sum_{\lambda} f_{\mu\nu}^{\lambda} P_{\lambda}. \tag{1.3}$$

The Schur Q-functions first appeared in the context of projective representations [35, 36] of $\mathfrak{S}_n$ [62]. They provide information for the spin characters of $\mathfrak{S}_n$, which are indexed by strict partitions, on certain non-trivial conjugacy classes [50, 59, 68]. The Schur P-functions also appear in the representation theory of the queer Lie superalgebra $\mathfrak{q}(n)$ [25, 26, 65], which is a superalgebra generalization of $\mathfrak{gl}_n$.

Schur P-functions also appear in type B (or C) Schubert calculus [33, 53], as representatives for the cohomology classes of Schubert cycles in the odd orthogonal Grassmannian $OG(n, \mathbb{C}^{2n+1})$, which is the set of n-dimensional subspaces V of $\mathbb{C}^{2n+1}$ such that for any $u, v \in V$, $\langle u, v \rangle = 0$, for a fixed non-degenerate symmetric bilinear form $\langle , \rangle$. The Schubert classes τ_λ form a basis for the cohomology ring $H^*(OG(n, \mathbb{C}^{2n+1}))$, where now λ is a strict partition that fits in an ambient $n \times n$ triangle. The shifted Littlewood–Richardson coefficients also appear as structure constants of the product of Schubert classes [19, 33, 53].

A shifted tableau crystal for Schur Q-functions

In [20], Gillespie and Levinson computed the topology of real Schubert curves in $Gr(k, \mathbb{C}^n)$, using the coplactic operators of type A crystals. Motivated by the same question in the context of $OG(n, \mathbb{C}^{2n+1})$, Gillespie, Levinson and Purbhoo [23] and Gillespie and Levinson [21] introduced coplactic operators on shifted tableaux, yielding a crystal-like structure on shifted tableaux, having Schur Q-functions as characters.

This is not the first crystal-like structure concerning shifted tableaux. Indeed, the representation theory for the queer Lie superalgebra $\mathfrak{q}(n)$ motivated crystal structures for decomposition tableaux [25, 26, 66] and shifted semistandard tableaux [1, 12, 24, 34]. These crystals form canonical bases for the representations of $U_q(\mathfrak{q}(n))$ and have the Schur P-functions as characters. The crystal-like structure in [23] has its origins in the Schubert calculus for type B [22], it is non-isomorphic to the crystal for $\mathfrak{q}(n)$, and it is not known whether it forms canonical bases for the representations of some known quantized enveloping algebra. We will henceforth refer to it as a shifted tableau crystal, denoted $\mathsf{ShST}(\lambda/\mu, n)$.

The shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, n)$ has vertices the skew shifted tableaux, for a given shape λ/μ, on the primed alphabet $[n]'$, and double edges, corresponding to the action of the primed and unprimed lowering and raising operators which commute with the shifted *jeu de taquin*. This crystal-like structure has nice properties that parallel the ones for normal Kashiwara crystals of type A_{n-1}. It decomposes into connected components (3.1), and each one has an unique highest weight element, a shifted skew tableau where each primed and unprimed raising operator E'_i and E_i is equal to $\varnothing$, for any $i \in I = [n-1]$. This highest weight elements is a Littlewood–Richardson–Stembridge (LRS) tableau of shape λ/μ [68] (see Definition 2.13). Similarly, it has a unique lowest weight element, a shifted skew tableau such that each primed and unprimed lowering operator F'_i and F_i is equal to $\varnothing$, for each $i \in I$, which is the reversal of the highest weight element. The existence and uniqueness of highest and lowest weight elements is also valid for the subcrystals obtained from considering the subgraph with edges labelled in connected subsets of I.

In addition, the primed and unprimed operators considered separately yield a type A Kashiwara crystal, considering the total length functions ε_i and φ_i [23, Section 5.1] and the usual weight function on shifted tableaux. However, these are not seminormal crystals, as the total length functions do not measure the distance to the ends of a string of either F_i or F'_i operators, but rather the total distance on the string of both.

The cactus group and shifted tableau crystals

A shifted version of the type A crystal reflection operators was introduced in [54, 55] (see Definition 3.26), in terms of the shifted tableau crystal operators. In Theorem 3.30, we show that, similarly to type A, they coincide with the partial Schützenberger involution restricted to the primed alphabet of two adjacent letters $\{i', i, (i+1)', i+1\} \subseteq [n]'$, for any $i \in I$. They act on the $\{i', i\}$-coloured components of the shifted tableau crystal by a double reflection through vertical and horizontal axes, rather than a simple reflection as in the Young tableau crystal. Unlike type A crystals, they do not define a natural action of the symmetric group $\mathfrak{S}_n$ on the shifted tableau crystal, since the braid relations do not need to hold, as shown in Example 3.31.

Following a similar approach as Halacheva [29, 30], we then show in Theorem 4.1 [54, Theorem 5.7] that the restrictions of the shifted Schützenberger involution on the primed subintervals of $[n]$ yield an internal action of the cactus group J_n (Definition 1.1) on that crystal. We note that this internal action on the shifted tableau crystal, unlike the one on type A crystals, does not factor through the braid relations of the symmetric group. When the shifted Schützenberger involution is restricted to primed subintervals of two adjacent letters, the cactus group action agrees with the action of the shifted crystal reflection operators on the shifted crystal. This means that both actions agree as permutations of the vertices within each $\{i', i\}$-coloured component of the shifted crystal.

It is expected, although we have not attempted to explore it, that this combinatorial internal action of the cactus group on the shifted crystal carries some geometrical meaning, as this crystal has its origin in the orthogonal Grassmannian [22]. Moreover, the tensor product of shifted tableau crystals is not known, and consequently, nor an external action of the cactus group.

The Berenstein–Kirillov group

The Bender–Knuth moves t_i are well known involutions on semistandard Young tableaux [4], that act on adjacent letters i and $i+1$ by interchanging their multiplicity, while leaving the other letters unchanged. The tableau switching, introduced by Benkart, Sottile and Stroomer [5], is an algorithm on pairs of semistandard Young tableaux (S, T), with T extending S, that moves one through the other, obtaining a pair that is component-wise Knuth equivalent to (T, S). The tableau switching on horizontal border strips of two adjacent letters i and $i+1$, together with a swapping of the labels i and $i+1$, is known to coincide with the classical Bender–Knuth involution t_i [5, 51]. Berenstein and Kirillov [7] studied explicit relations satisfied by the involutions t_i [7, Corollary 1.1], and introduced the Berenstein–Kirillov group $\mathcal{BK}$ (also known as Gelfand–Tsetlin group), the free group generated by the classical Bender–Knuth involutions t_i, for $i \in \mathbb{Z}_{>0}$, subject to the relations they satisfy on semistandard Young tableaux of any shape [7, 8, 10]. This group is well-defined although an explicit and comprehensive set of relations is not known. Some of the relations that are held by the t_i are listed in [7, 8, 42], and [10, Theorem 1.6], and they are recalled in Section 5.2.

Chmutov, Glick and Pylyavskyy [10] studied, using semistandard growth diagrams, the relation between the group $\mathcal{BK}_n$, the subgroup of $\mathcal{BK}$ generated by $t_1, \dots, t_{n-1}$, and the cactus group J_n (Definition 1.1), concluding that $\mathcal{BK}_n$ is isomorphic to a quotient of J_n. Halacheva has remarked [30, Remark 3.9] that this may also be concluded by noting that the action of the cactus group J_n [29, Section 10.2] agrees with the one of $\mathcal{BK}_n$ on type A_{n-1} crystals of straight-shaped Young tableaux filled in $[n]$. Considering the alternative set of generators $q_1, \ldots, q_{n-1}$ for $\mathcal{BK}_n$, where $q_i := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$, then each q_i acts on a straight-shaped Young tableau via the partial Schützenberger involution, or evacuation, restricted to the alphabet $\{1, \ldots, i+1\}$ [7, Theorem 2.1]. Chmutov, Glick and Pylyavskyy also refine their results concerning the cactus group quotient in [10, Theorem 1.8] by showing precise implications between the cactus-type relations, satisfied by generators of $\mathcal{BK}_n$, and a subset of known relations (5.5) and (5.7) in the $\mathcal{BK}_n$, thereby yielding a presentation of the cactus group in terms of the Bender–Knuth generators.

A shifted Berenstein–Kirillov group

Motivated by the tableau switching characterization of the Bender–Knuth moves on semistandard Young tableaux [5], we introduced in [56, 57] a shifted version of the Bender–Knuth involutions, here denoted t_i, for shifted semistandard tableaux in the shifted tableau crystal due to Gillespie, Levinson and Purbhoo [23], using the shifted tableau switching introduced by Choi, Nam and Oh [15]. Alternatively, we may use the type C infusion on shifted standard tableaux due to Thomas and Yong [70] together with the semistandardization of Pechenik and Yong [52]. We observe that genomic Bender–Knuth involutions have also been defined in a similar way on genomic tableaux, by Pechenik and Yong [52]. The shifted Bender–Knuth involutions we present differ from the operators introduced by Stembridge [69, Section 6], which are not compatible with the canonical form requirement for the shifted tableau crystal considered (see Remark 5.18). Using the shifted Bender–Knuth involutions t_i as generators, we define a shifted analogue of the Berenstein–Kirillov group, denoted $\mathcal{SBK}$, with $\mathcal{SBK}_n$ being defined analogously.

Following [7], the elements $\mathsf{q}_i := \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)$, for $1 \leq i \leq n-1$, also constitute an alternative set of generators for $\mathcal{SBK}_n$. Similarly to the $\mathcal{BK}_n$ group, each generator q_i acts on a straight-shaped shifted semistandard tableau, via the shifted Schützenberger involution restricted to the primed alphabet $\{1, \ldots, i+1\}'$. Thereby, as in the classical case [10, 29, 30], the actions of the cactus group J_n (see Theorem 4.3) and of $\mathcal{SBK}_n$ agree on a straight-shaped shifted tableau crystal [23]. Thus, the shifted Berenstein–Kirillov group is isomorphic to a quotient of the cactus group (Theorem 5.25).

The shifted Bender–Knuth operators $\mathtt{t}_i$ also satisfy the $\mathcal{BK}$-type relations (5.5) and (5.7). Those are the relations satisfied by the generators t_i in $\mathcal{BK}$ which are equivalent to the ones of the cactus group, as shown in [10, Theorem 1.8] (here Theorem 5.20). Thus, we also have, similarly to the classical case [10], another presentation of the cactus group via the shifted Bender–Knuth moves.

Not all known relations that hold in $\mathcal{BK}$ need to be satisfied by the shifted Bender-Knuth involutions, namely the relation $(\mathtt{t}_1\mathtt{t}_2)^6 = 1$ (5.6) does not need to hold in $\mathcal{SBK}$ (see Example 5.23). As observed in [10, Remark 9], the relation $(t_1t_2)^6 = 1$ (5.6) in $\mathcal{BK}$ does not follow from any cactus group relation. In fact, it is equivalent to the braid relations of the symmetric group $\mathfrak{S}_n$, satisfied by the type A crystal reflection operators ς_i , due to Lascoux and Schützenberger [44], and rediscovered by Kashiwara [39, Theorem 7.2.2]. These operators are elements of $\mathcal{BK}$ [7, Proposition 1.4], and ς_i acts on a type A_{n-1} crystal as a middle reflection of each i-string, which agrees with the partial Schützenberger involution restricted to the alphabet $\{i, i+1\}$, for $i \in [n-1]$.

The shifted crystal reflection operators σ_i, for $1 \leq i \leq n-1$ [54, Definition 4.3] (here Definition 3.26) are also elements of $\mathcal{SBK}_n$, and σ_i acts on a shifted tableau crystal as a double reflection of each $\{i, i'\}$-coloured string, which agrees with the shifted Schützenberger involution restricted to the primed alphabet $\{i, i+1\}'$. A relation of the type $(\mathtt{t}_1\mathtt{t}_2)^{2m} = 1$ holds in $\mathcal{SBK}_n$ if and only if the relation $(\sigma_i\sigma_{i+1})^m = 1$ does, where m is a positive integer (see Proposition 5.22). However, unlike type A crystals, the shifted crystal reflection operators do not define an action of the symmetric group, thus none of the aforesaid relations holds for $m = 3$. It is not known whether some $m > 3$ exists (see Appendix A). It is an open question to find explicit relations in $\mathcal{SBK}$, beyond those listed in Proposition 5.26, that do not follow from the cactus group relations. Further relations for $\mathcal{SBK}$ seem to be intimately related with further relations satisfied by the shifted crystal reflection operators.

Shifted growth diagrams

The proof of Theorem 4.1 in Section 4.1 concerning a cactus group action on a shifted tableau crystal relies on the formulation of Schützenberger involution as the unique set involution on a shifted tableau crystal satisfying certain conditions in terms of the shifted crystal operators (Proposition 3.20). Thus, the partial Schützenberger involutions, corresponding to the restrictions of the Schützenberger involutions to all primed subintervals of $[n]$, are also described in a similar way (Lemma 3.23) to what is done in [31, Definition 5.17]. These unique maps coincide with the reversal map and its restrictions, and thus they are explicit involutions on shifted tableaux. Sticking to this algorithmic formulation, we may use type C growth diagrams, introduced by Thomas and Yong [71], together with the semistandardization process due to Pechenik and Yong [52], to obtain an alternative proof that the cactus group acts on a shifted tableau crystal via the restrictions of the reversal involution.

The type C growth diagrams, for shifted standard tableaux, were introduced by Thomas and Yong [71], together with generalizations for other cominuscule posets, and they generalize the classical growth diagrams for standard Young tableaux due to Fomin [67]. These diagrams consist of saturated chains of shifted shapes encoding the shifted *jeu de taquin* for shifted standard tableaux. Thus, they define type C infusion, as well as the shifted promotion, evacuation and reversal, and the adequate restrictions. Like the classical growth diagrams [67, Proposition A1.2.7], the shifted ones may be computed via local growth rules [71, Theorem 2.1]. The symmetry of those rules shows that the type C infusion, evacuation and reversal are involutions.

Unlike the case for type A, shifted semistandard tableaux, being filled in a primed alphabet, are not encoded by a sequence of strict shapes and thus we do not have a semistandard-like growth diagrams as in [10]. However, the shifted semistandardization due to Pechenik and Yong [52] allows us to extend these notions for semistandard shifted tableaux. Thus, we are able to obtain an alternative proof, in Chapter 6, for the cactus group action on a shifted tableau crystal (Theorem 4.1), relying on the combinatorial description of the shifted reversal.

Structure of the thesis

This thesis is organized as follows:

- Chapter 2 provides the background notions on shifted tableaux, as well as operations and algorithms among them. In particular, we recall the shifted *jeu de taquin*, the Worley–Sagan insertion algorithm, and state their relation with shifted Knuth and dual equivalences. We then present the notions of evacuation and reversal. In Section 2.5 we recall the shifted tableau switching algorithm due to Choi, Nam and Oh [15], which produces the same result as the type C infusion of Thomas and Yong [70] together with the semistandardization due to Pechenik and Yong [52].

- In Chapter 3 we present the basic definitions and main results concerning the shifted tableau crystal of Gillespie, Levinson and Purbhoo [23], highlighting its i-string decomposition. We then recall the definition of the Schützenberger involution and its restrictions. In Section 3.3 we introduce the notion of shifted crystal reflection operators using the shifted crystal operators. We then prove, in Theorem 3.30, that these operators coincide with the restrictions of the Schützenberger involution to the primed alphabet of two adjacent letters.

- Chapter 4 is intended to prove, in Theorem 4.1 ([54, Theorem 5.7]), that the cactus group J_n acts on the shifted tableau crystal via the partial Schützenberger involutions.

- In Chapter 5 we introduce a shifted version of the Berenstein–Kirillov group. We begin by defining shifted Bender–Knuth involutions using the shifted tableau switching algorithm. Then, as in the classical case, we use those shifted Bender–Knuth moves to define a shifted Berenstein–Kirillov group. Proposition 5.26 shows that the known relations (5.5) and (5.7) satisfied by the classical Bender–Knuth involutions also hold among the shifted counterparts, with the exception of the relation $(\mathtt{t}_1\mathtt{t}_2)^6 = 1$. We then prove, in Theorem 5.25, that the shifted Berenstein–Kirillov group is isomorphic to a quotient of the cactus group ([56, Theorem 4.25]) and exhibit in (5.14) an alternative presentation for the cactus group in terms of the shifted Bender–Knuth moves.

- In Chapter 6 we recall the notion of growth diagrams for shifted standard tableaux, as well as the local growth rules. Using the semistandardization, we recover the shifted *jeu de taquin*, type C infusion, evacuation and reversal (as well as its restrictions) to semistandard shifted tableaux. We then provide, in Section 6.3.1, an alternative proof for the cactus group action on the shifted tableau crystal.

CHAPTER 2

SHIFTED TABLEAUX AND THEIR OPERATIONS

In this chapter we recall the basic notions on shifted tableaux and related combinatorics. We follow the notation in [23].

2.1 Words and shifted tableaux

A *strict partition* is a sequence $\lambda = (\lambda_1 > \cdots > \lambda_k)$ of distinct positive integers displayed in strictly decreasing order. The entries λ_i are called the *parts* of λ and the *length* of λ, denoted $\ell(\lambda)$, is the number of parts of λ. We denote by $|\lambda| := \lambda_1 + \cdots + \lambda_k$ the *sum* of the parts of λ. A strict partition λ is identified with its *shifted shape* $S(\lambda)$ which consists of $|\lambda|$ boxes placed in $\ell(\lambda)$ rows, with the i-th row having λ_i boxes and being shifted $i-1$ units to the right. We use the English (or matrix) notation. The boxes in $\{(1,j),(2,j+1),(3,j+2),\ldots\}$ form a *diagonal*, for $j \geq 1$. If $j = 1$ it is called the *main diagonal*.

Given strict partitions λ and μ such that $S(\mu) \subseteq S(\lambda)$, we write $\mu \subseteq \lambda$ and define the *skew shifted shape* of λ/μ as $S(\lambda/\mu) = S(\lambda) \setminus S(\mu)$ (see Figure 2.1). Shapes of the form $\lambda/\varnothing$ are called *straight* (or *normal*). Any shifted shape λ lies naturally in the ambient triangle of the *shifted staircase shape* $\delta = (\lambda_1, \lambda_1 - 1, \ldots, 1)$. The *complement* of λ is the strict partition $\lambda^\vee$ whose set of parts is the complement in $\{\lambda_1, \lambda_1 - 1, \ldots, 1\}$ of the set of parts of λ. Pictorially, this is the partition corresponding to the empty spaces in the staircase shape, after flipping across the anti-diagonal (see Figure 2.1). In particular, $\varnothing^\vee = \delta$.

We consider the alphabet $[n] := \{1 < \cdots < n\}$ and define the *primed* alphabet to be

$$[n]' = \{1' < 1 < \cdots < n' < n\}.$$

Following the notation in [15], we will write $\mathbf{i}$ when referring to the letters i or i' without specifying whether they are primed. Given a string $w = w_1 \cdots w_m$ in the alphabet $[n]'$, the *canonical form* [23, Definition 2.1] of w is the string obtained from w by replacing the leftmost

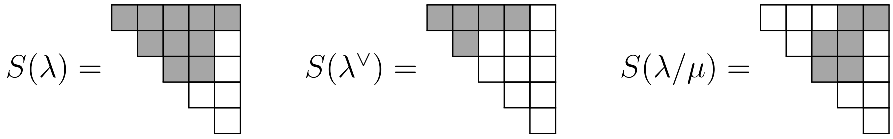


Figure 2.1: The shapes of λ, $\lambda^\vee$ and λ/μ are shaded in gray, for $\lambda = (5,3,2)$ and $\mu = (3,1)$. They are represented within the ambient triangle $\delta = (5,4,3,2,1)$.

i, if it exists, with i, for all $1 \leq i \leq n$. Two strings w and v are said to be *equivalent* if they have the same canonical form (this is indeed an equivalence relation).

A *word* $\hat{w}$ is an equivalence class of the strings equivalent to w. If w is in canonical form, then it is said to be the *canonical representative* of $\hat{w}$, while the other strings are called the *representatives* of $\hat{w}$ [23, Definition 2.2]. The *weight* of a word $\hat{w}$ is $\mathsf{wt}(w) = (wt_1, \ldots, wt_n)$, where wt_i is equal to the total number of i and i' in w, any representative of $\hat{w}$. We often refer to $\hat{w}$ by its canonical representative w. We remark that the weight of a word does not depend on the choice of representative, as the number of i and i' is the same for all representatives, for $i \in [n]$.

Example 2.1. The string $w = 122'132'$ is equivalent to $12'2'13'2'$, the former being in canonical form. The equivalence class of w is given by

$$\hat{w} = \{122'132', 1'22'132', 12'2'132', 122'13'2', 1'2'2'132', 1'22'13'2', 12'2'13'2', 1'2'2'13'2'\},$$

and we refer to it by its canonical representative w. The weight of w is $\mathsf{wt}(w) = (2,3,1)$.

Definition 2.2. Given strict partitions λ and μ such that $\mu \subseteq \lambda$, a *shifted semistandard tableau* T of shape λ/μ is a filling of $S(\lambda/\mu)$ with letters in $[n]'$ such that:

1. The entries are weakly increasing in each row and in each column.

2. There is at most one i per column, for each $i \in [n]$.

3. There is at most one i' per row, for each $i \in [n]$.

The *(row) reading word* $w(T)$ of a shifted tableau is obtained by concatenating its rows, going from bottom to top. The *weight* of T is defined as $\mathsf{wt}(T) := \mathsf{wt}(w(T))$. A word, or a shifted tableau, is said *standard* if its weight is $(1,\ldots,1)$.

Example 2.3. The following is a shifted semistandard tableau, with its reading word and weight:

$$T = \begin{array}{|c|c|c|c|c|} \hline & 1 & 1 & 2' & 2 \\ \hline \end{array} \atop \begin{array}{c|c|c|} & 2 & 3' \\ \hline & & 3 \end{array} \qquad w(T) = 323'112'2 \qquad \mathsf{wt}(T) = (2,3,2).$$

We say that a tableau T is in *canonical form* if its reading word is in canonical form and, in that case, it is identified with its set of *representatives*, that are obtained by possibly priming the entry corresponding to the leftmost i in $w(T)$, for all i [23, Definition 2.6]. The set of shifted semistandard tableaux of shape λ/μ, on the alphabet $[n]'$, in canonical form, is denoted by $\mathsf{ShST}(\lambda/\mu, n)$.

Example 2.4. The tableau of the previous example is in canonical form, as the first occurrences of each letter is unprimed. Some of its representatives are listed below. Their reading words are representatives of the class of $w(T)$.

$$\begin{array}{ccccc} \; & 1 & 1 & 2' & 2 \\ & 2 & 3' & & \\ & & 3 & & \end{array} \qquad \begin{array}{ccccc} \; & 1' & 1 & 2' & 2 \\ & 2 & 3' & & \\ & & 3 & & \end{array} \qquad \begin{array}{ccccc} \; & 1 & 1 & 2' & 2 \\ & 2' & 3' & & \\ & & 3 & & \end{array} \qquad \begin{array}{ccccc} \; & 1 & 1 & 2' & 2 \\ & 2' & 3' & & \\ & & 3' & & \end{array} \qquad \cdots$$

A *diagonally-shaped tableau* is a shifted tableau of shape $(2n-1, 2n-3, \ldots, 1)/(2n-2, 2n-4, \ldots, 2)$. Every word $w = w_1 \ldots w_n$ may be regarded as a shifted tableau D_w having this shape and word w.

Example 2.5. The word $w = 2311'$ is the reading word of

$$D_w = \begin{array}{ccccccc} \; & \; & \; & \; & \; & \; & 1' \\ & \; & \; & \; & \; & 1 & \\ & & \; & \; & 3 & & \\ & & & 2 & & & \end{array} .$$

We consider the *symmetric group* $\mathfrak{S}_n$ to be the Coxeter group generated by $\theta_1, \ldots, \theta_{n-1}$, subject to the relations

$$\theta_i^2 = 1, \qquad \theta_i\theta_j = \theta_j\theta_i, \text{ for } |i-j| > 1, \qquad (\theta_i\theta_{i+1})^3 = 1, \text{ for } 1 \leq i \leq n-2. \tag{2.1}$$

The elements of $\mathfrak{S}_n$ are explicitly described by the permutations of $[n]$, where its generators θ_i are the simple transpositions $(i, i+1)$, for $1 \leq i \leq n-1$, using cycle notations. A permutation $\tau \in \mathfrak{S}_n$ acts naturally on a vector of $\mathbb{Z}^n$ as $\tau(v_1, \ldots, v_n) := (v_{\tau^{-1}(1)}, \ldots, v_{\tau^{-1}(n)})$. This action is extended to letters of the primed alphabet $\mathbf{x} \in [n]'$ as

$$\tau(\mathbf{x}) := \begin{cases} \tau(x) & \text{if } \mathbf{x} = x \\ \tau(x)' & \text{if } \mathbf{x} = x' \end{cases} . \tag{2.2}$$

According to this action, given $\tau \in \mathfrak{S}_n$ and a word $w = w_1 \cdots w_k$ in the alphabet $[n]'$, we define $\tau(w_1 \cdots w_k)$ as the word $\tau(w_1) \cdots \tau(w_k)$, after canonicalizing it, for $w_i \in [n]'$. Similarly, the action of τ is extended to fillings T in $[n]'$ of a shifted shape (in particular, this includes shifted semistandard tableaux), defining $\tau(T)$ by the action of τ on the word of T. Given $1 \leq i < j \leq j$, we denote by $\theta_{i,j}$ the *longest permutation* in $\mathfrak{S}_{\{i,\ldots,j\}}$ embedded in $\mathfrak{S}_n$, i.e, $\theta_{i,j} = \theta_i(\theta_{i+1}\theta_i) \cdots (\theta_{j-1} \cdots \theta_i)$. In particular, $\theta_{1,n}$ is the longest permutation in $\mathfrak{S}_n$, also known as the order reversing permutation.

2.2 Shifted *jeu de taquin* and Worley–Sagan insertion

The shifted *jeu de taquin* [58, 72] is defined similarly to the one for ordinary Young tableaux. A skew shape $S(\lambda/\mu)$ is said to be a *border strip* if it contains no subset of the form

$$\{(i,j),(i+1,j+1)\}.$$

Definition 2.6. Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $i \in [n]$. The tableau obtained from T considering only the letters i and i' is called the *i-border strip* of T, and is denoted by T^i.

Given strict partitions $\nu \subseteq \mu \subseteq \lambda$, we say that λ/μ *extends* μ/ν, and, in this case, we define

$$(\mu/\nu) \sqcup (\lambda/\mu) := \lambda/\nu.$$

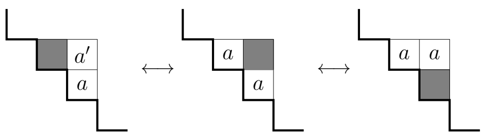
Given S and T shifted semistandard tableaux, we say that T *extends* S if the shape of T extends the shape of S. In this case, we denote by $S \sqcup T$ the *(disjoint) union* of S and T, obtained by overlapping the two tableaux, which is not necessarily a valid semistandard tableau. A shifted semistandard tableau T filled in $[n]'$ is clearly the union of its i-border strips, for $i \in [n]$.

Example 2.7. Considering $T = \begin{array}{ccccc} \square & 1 & 1 & 2' & 2 \\ & 2 & 3' & & \\ & & 3 & & \end{array}$, we have

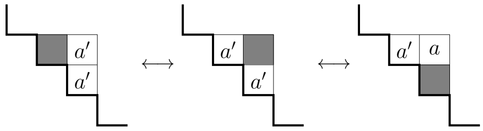
$$T = \begin{array}{ccc} \square & 1 & 1 \end{array} \sqcup \begin{array}{ccccc} \square & \square & & 2' & 2 \\ & 2 & & & \end{array} \sqcup \begin{array}{ccccc} \square & \square & \square & \square & \square \\ & \square & 3' & & \\ & & 3 & & \end{array} = T^1 \sqcup T^2 \sqcup T^3.$$

A single box b is said to be an *inner corner* of a shape λ/μ if λ/μ extends b, and an *outer corner* if b extends λ/μ.

Definition 2.8 ([72, Section 6.4])**.** Let $T \in \mathsf{ShST}(\lambda/\mu, n)$. An *inner jeu de taquin slide* is the process in which an empty inner corner of the skew shape of T is chosen and then either the entry to its right or the one below it is chosen to slide into the empty square, maintaining semistandardness. The process is then repeated on the obtained new empty square until it is an outer corner. An *outer jeu de taquin slide* is the reverse process, starting with an outer corner. This process has an exception to the sliding rules when the empty box of an inner or outer slide enters in the diagonal. If an inner slide moves a box with a' to the left into the diagonal and then moves a box with a up from the diagonal, to the right of it, the former becomes unprimed (and vice versa for the corresponding outer slide), as illustrated by the following slide:



If T is not in the canonical form, there is another exception to consider illustrated below (observe that result is in the same canonical class of the former case):



The *rectification* $\mathsf{rect}(T)$ of T is the tableau obtained by applying any sequence of inner slides until a straight shape is obtained (it is known that any chosen sequence of slides produces the same straight-shaped tableau [58, Theorem 11.1]). The *rectification* of a word w is the word of the rectification of any tableau with reading word w. Two tableaux are said to be *shifted jeu de taquin equivalent* if they have the same rectification. An operator on shifted tableaux that commutes with the shifted *jeu de taquin* is called *coplactic*.

The *standardization* of a word w, denoted $\mathsf{std}(w)$, is obtained by replacing the letters of any representative of w with $1, \dots, \ell(w)$, where $\ell(w)$ denotes the lenght of w, from least to greatest, reading right to left for primed entries, and left to right for unprimed entries [23, Definition 2.8]. This process does not depend on the choice of the representative. The standardization of a shifted tableau T, denoted $\mathsf{std}(T)$, is defined as the tableau with the same shape as T with reading word $\mathsf{std}(w(T))$.

Example 2.9. Let $T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$, with reading word $w = 322112'2$ and $\ell(w) = 7$. Then,

$$\mathsf{std}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

Lemma 2.10 ([23, Lemma 3.5])**.** *If s is a standard word in $[m]$, with $m = a_1 + \cdots + a_k$, then there is at most one word w of weight $(a_1, \dots, a_k)$ with standardization $\mathsf{std}(w) = s$.*

As a consequence, we have that any shifted semistandard tableau is completely determined (up to canonical form) by its shape, weight and standardization. Thus, given a standard tableau T of shape λ/μ and a composition ν (i.e., a vector of positive integers) such that $|\nu| = |\lambda| - |\mu|$, there exists at most one semistandard tableau with the same shape of T and weight ν. The process to obtain it, if it exists, is known as *shifted semistandardization* and was introduced by Pechenik and Yong [52, Section 9.1]. Let ν be a composition and define, for $k = 1, \dots, \ell(\nu)$,

$$\mathcal{P}_k(\nu) := \left\{ 1 + \sum_{i<k} \nu_i, 2 + \sum_{i<k} \nu_i, \dots, \sum_{i \leq k} \nu_i \right\}. \tag{2.3}$$

That is, $\mathcal{P}_1 = \{1, \dots, \nu_1\}$, $\mathcal{P}_2 = \{\nu_1 + 1, \dots, \nu_1 + \nu_2\}$, etc. By construction, each $\mathcal{P}_k(\nu)$ has cardinality ν_k.

Definition 2.11 ([52, Section 9.1]). Given a shifted standard tableau T, its *semistandardization (with respect to ν)*, denoted $\mathsf{sstd}_\nu(T)$, is given by the following process:

1. Replace each letter i with k_i, for the unique k such that $i \in \mathcal{P}_k(\nu)$.
2. Then, replace each k_i with k', if there exists a k_j south-west of k_i with $i < j$, or with k, otherwise.
3. If the obtained filling is a semistandard tableau, then ν is said to be *admissible* for T and $\mathsf{sstd}_\nu(T)$ is set to be that tableau. Otherwise, $\mathsf{sstd}_\nu(T)$ is said to be undefined.

Note that, if ν is admissible for T, then $\mathsf{wt}(\mathsf{sstd}_\nu(T)) = \nu$. Moreover, if $T \in \mathsf{ShST}(\lambda/\mu, n)$ has weight ν, then ν is admissible for $\mathsf{std}(T)$ and $\mathsf{sstd}_\nu(\mathsf{std}(T)) = T$ [52, Lemma 9.5]. A shifted tableau in these conditions is said to be *ν-Pieri filled*. As a consequence, we have that std defines a bijection between the set of shifted semistandard tableaux of shape λ/μ and weight ν and the set of ν-Pieri filled shifted semistandard tableaux of the same shape, whose inverse is given by sstd_ν [52, Theorem 9.6].

Example 2.12. Let $T = \begin{ytableau} 1 & 2 & 3 & 6 \\ \none & 4 & 5 \\ \none & \none & 7 \end{ytableau}$ be a shifted standard tableau and let $\nu = (2, 4, 1)$. We have:

$$\mathcal{P}_1(\nu) = \{1, 2\} \qquad \mathcal{P}_2(\nu) = \{3, 4, 5, 6\} \qquad \mathcal{P}_3(\nu) = \{7\}.$$

Then, the semistandardization of T with respect to ν is obtained as follows:

$$\begin{ytableau} 1 & 2 & 3 & 6 \\ \none & 4 & 5 \\ \none & \none & 7 \end{ytableau} \longrightarrow \begin{ytableau} 1_1 & 1_2 & 2_3 & 2_6 \\ \none & 2_4 & 2_5 \\ \none & \none & 3_7 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 1 & 2' & 2 \\ \none & 2 & 2 \\ \none & \none & 3 \end{ytableau}.$$

Given ν a strict partition, there exists a unique shifted tableau in canonical form of shape and weight equal to ν. This is known as the *Yamanouchi tableau* Y_ν, and its i-th row consists only of unprimed i's. A word w on the alphabet $[n]'$ with weight ν, a strict partition, is said to be *ballot* (or *lattice*, or *Yamanouchi*) if its rectification is $w(Y_\nu)$ (for another formulation of ballot word, see [68, Section 8]).

Definition 2.13. A shifted semistandard tableau T of weight ν is said to be *Littlewood–Richardson–Stembridge* (LRS) if it is in canonical form and $\mathsf{rect}(T) = Y_\nu$, or, equivalently, if its reading word is *ballot* and has weight ν.

Recall that the *shifted Littlewood–Richardson coefficients* $f^\lambda_{\mu\nu}$ are the constants that appear in the linear expansion of the skew Schur Q-functions in the basis of Schur Q-functions and in the product of P-functions in the basis of Schur P-functions (1.3). These are non-negative integers, that are equal to zero whenever $|\lambda| \neq |\mu| + |\nu|$, and they are precisely the number of LRS tableaux of shape λ/μ and weight ν [68, Theorem 8.3].

We now recall the notion of shifted tableau insertion, introduced independently by Worley [72] and Sagan [58], providing a shifted version of the well-known Schensted insertion for Young tableaux. We remark that this algorithm was originally presented for shifted tableaux enumerated by Schur Q-functions, which are not required to be in canonical form. However, the algorithm is compatible with canonicalizing, thus we present a simpler version, as in [23, Definition 5.23]. Given letters $\mathbf{a}, \mathbf{b} \in [n]'$ we say that $\mathbf{a} \prec_{\mathsf{row}} \mathbf{b}$ if either $\mathbf{a} = a'$ and $a' \leq \mathbf{b}$, or if $\mathbf{a} = a$ and $a < \mathbf{b}$. We say that $\mathbf{a} \prec_{\mathsf{col}} \mathbf{b}$ if either $\mathbf{a} = a'$ and $a' < \mathbf{b}$, or if $\mathbf{a} = a$ and $a \leq \mathbf{b}$.

Definition 2.14. Let T be a straight-shaped shifted tableau and let $\mathbf{a} \in [n]'$. The *Worley–Sagan insertion* of $\mathbf{a}$ into T is the tableau obtained as follows:

1. If there is $\mathbf{b}$ on the first row of T such that $\mathbf{a} \prec_{\mathsf{row}} \mathbf{b}$, then place $\mathbf{a}$ at the end of that row. Otherwise, let $\mathbf{b}$ be the leftmost entry in that row such that $\mathbf{a} \prec_{\mathsf{row}} \mathbf{b}$ and replace it with $\mathbf{a}$, "bumping" $\mathbf{b}$.

2. If $\mathbf{b}$ was not in the main diagonal of T before being "bumped", repeat the first step, now inserting $\mathbf{b}$ on the next row.

3. Otherwise, insert $\mathbf{b}$ on the next column to its right, in the following way:

   (a) If there is no $\mathbf{y}$ in the said column such that $\mathbf{b} \prec_{\mathsf{col}} \mathbf{y}$, place $\mathbf{b}$ at the bottom of the column.

   (b) Otherwise, let $\mathbf{y}$ be the topmost entry such that $\mathbf{b} \prec_{\mathsf{col}} \mathbf{y}$ and replace it with $\mathbf{b}$, "bumping" $\mathbf{y}$. Repeat this step, now inserting $\mathbf{y}$ into the next column.

The insertion of $\mathbf{a}$ into T is said to be *Schensted* if no entry in the main diagonal is ever "bumped", and *non-Schensted* otherwise.

Example 2.15. Consider the following shifted tableau, of straight shape,

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

The insertion of 2 into T is computed as follows: the entry 3 in the first row is the leftmost entry such that $2 \prec_{\mathsf{row}} 3$, and then it is "bumped" and replaced by 2. Then, 3 is inserted on the next row, and since there are no entries $\mathbf{b}$ such that $3 \prec_{\mathsf{row}} \mathbf{b}$, then 3 is placed at the end of that row. Thus, the resulting tableau is given by

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

and since no entries of the main diagonal, the insertion of 2 into T is Schensted. We now compute the insertion of 1 into the obtained tableau. Now $2'$ is the leftmost entry in the first row such that $1 \prec_{\mathsf{row}} 2'$, and thus $2'$ is "bumped" and replaced by 1. Moving to the second row, we have that 2 is the leftmost entry such that $2' \prec_{\mathsf{row}} 2$, and thus 2 (which is an entry of the main diagonal) is replaced by $2'$, and the process continues now in the next column. Since we have $2 \prec_{\mathsf{col}} 3'$, then $3'$ is replaced by 2, and is then placed at the end of the next column, as there are no entries $\mathbf{y}$ there such that $3' \prec_{\mathsf{col}} \mathbf{y}$. The resulting tableau, after being canonicalized, is

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 3' & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

and since an entry of the main diagonal was moved, this insertion is non-Schensted.

Similarly to the Robinson–Schensted and RSK correspondences, there is a bijection between the set of words in $[n]'$ and $\mathsf{ShST}(\nu, n) \times \mathsf{ShST}_P(\nu, n)$, where, we recall, $\mathsf{ShST}_P(\nu, n)$ denotes the set of shifted tableaux of shape ν filled in $[n]'$, not necessarily in canonical form, that has no primed entries on the main diagonal [58, Theorem 8.1]. This bijection is known as the *shifted RSK* and it is defined as follows.

Definition 2.16. Let $w = w_1 \cdots w_k$ be a word in $[n]'$. The *shifted RSK* of w is a pair of shifted tableaux with the same straight shape (P, Q), where Q is not required to be in canonical form and has no primed entries on its main diagonal. To obtain (P, Q), we consider a sequence of pairs

$$(\varnothing, \varnothing) =: (P_0, Q_0), (P_1, Q, 1), \ldots (P_k, Q_k) = (P, Q)$$

where P_i is obtained by inserting w_i into P_{i-1} (and canonicalizing it) and Q_i is obtained by placing i at the resulting new box, if that insertion was Schensted, or i' otherwise, for $i \in [k]$. The tableau P is known as the *insertion tableau* and also denoted by $P(w)$, and Q is known as the *recording tableau* and is denoted by $Q(w)$.

Example 2.17. Let $w = 2112'$. To compute the shifted RSK of w, we have

$$\varnothing \longrightarrow \begin{array}{|c|}\hline 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} =: P(w)$$

$$\varnothing \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2' \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2' & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} =: Q(w).$$

2.3 Shifted Knuth and dual equivalences

Definition 2.18 ([58])**.** Two words w and v on an alphabet $[n]'$ are said to be *shifted Knuth equivalent*, denoted $w \equiv_k v$, if one can be obtained from the other by applying a sequence of the following Knuth moves on adjacent letters

(K1) $bac \longleftrightarrow bca$ if, under the standardization ordering, $a < b < c$.

(K2) $acb \longleftrightarrow cab$ if, under the standardization ordering, $a < b < c$.

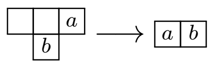
(SK1) $ab \longleftrightarrow ba$ if these are the first two letters.

(SK2) $aa \longleftrightarrow aa'$ if these are the first two letters.

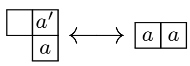
Example 2.19. Let $w = 212'21$. Then $\mathsf{std}(w) = 41352$, and since $2 < 3 < 5$, we have $w \equiv_k 212'12$.

The shifted Knuth moves may be regarded as (inner or outer) *jeu de taquin* slides. For instance, if $a < b < c$ in standardization order, then the Knuth moves (K1) and (K2) are illustrated by:

For the Knuth move (SK1), assume, without loss of generality, that $a < b$ in standardization ordering. Then,



Finally the Knuth move (SK2) is illustrated by the exception slide



If w and v are shifted Knuth equivalent words, the diagonally-shaped tableaux D_w and D_v have the same rectification. Thus D_w can be transformed into D_v via some sequence of *jeu de taquin* slides.

Theorem 2.20 ([72, Theorem 4.4.4])**.** *Two shifted semistandard tableaux are jeu de taquin equivalent if and only if their reading words are shifted Knuth equivalent.*

Theorem 2.21 ([58, Theorem 12.2])**.** *Let w and u be words in $[n]'$. Then, w and u are shifted Knuth equivalent if and only if their insertion tableaux under Worley–Sagan insertion coincide, i.e., $P(w) = P(v)$.*

Two tableaux in $\mathsf{ShST}(\lambda/\mu, n)$ are said to be shifted Knuth equivalent if so are their reading words, or equivalently, if their words have the same insertion tableaux. Shifted Knuth equivalence classes and *jeu de taquin* classes on words coincide and are in one-to-one correspondence with shifted semistandard tableaux of straight shape, via rectification or Worley–Sagan insertion. Unlike the classic Knuth relations for unprimed alphabets, the shifted Knuth equivalence is not a congruence, due to rules (SK1) and (SK2), since $w \equiv_k v$ does not necessarily imply that $tw \equiv_k tw$ for any letter $t \in [n]'$. For instance, $22'1 \equiv_k 221$ but $322'1 \not\equiv_k 3221$. However, under certain conditions we have the following results.

Lemma 2.22. *Let w and v be two words in $[n]'$ such that $w \equiv_k v$. Let $t \in [n]'$. Then,*

$$wt \equiv_k vt$$

Proof. Let D_w and D_v be diagonally-shaped shifted tableaux with words w and v, respectively. By Theorem 2.20, $\mathsf{rect}(D_w) = \mathsf{rect}(D_v)$. Let T be this straight-shaped tableau, of shape λ. Hence, we may consider the tableau T^0 of shape $(\lambda_1 + 2, \lambda_1, \ldots, \lambda_k)/(\lambda_1 + 1)$ consisting of t on the entry $(1, \lambda_1 + 2)$ and T on the remaining part. Clearly, $\mathsf{rect}(T^0) = \mathsf{rect}(D^0_w) = \mathsf{rect}(D^0_v)$ where D^0_w and D^0_v are the diagonally-shaped shifted tableaux with words wt and vt respectively. □

Lemma 2.23. *Let w and v be two words in $[n]'$ such that $w \equiv_k v$ and such that there exists a sequence of Knuth relations turning w into v using only $(K1)$ and $(K2)$. Let $t \in [n]'$. Then,*

$$tw \equiv_k tv$$

Proof. If the rules $(SK1)$ and $(SK2)$ are not used, then w and v are Knuth equivalent as Young tableau words, considering the standardization to avoid primed entries. □

We recall the notion of shifted dual equivalence on words and tableaux. Recall that Lemma 2.10 ensures that the shifted *jeu de taquin* commutes with standardization.

Definition 2.24 ([28])**.** Two standard shifted tableaux are *shifted dual equivalent* (or coplactic equivalent) if they have the same shape after applying any sequence (including the empty sequence) of inner or outer *jeu de taquin* slides to both. Two shifted semistandard tableaux are shifted dual equivalent if so are their standardizations.

In particular, considering the empty sequence of jeu de taquin slides, we have that shifted tableaux that are dual equivalent must have the same shape. This notion is extended to words, with two words being *shifted dual equivalent* if their corresponding diagonally-shaped tableaux are shifted dual equivalent.

The following characterizes dual equivalence on straight-shaped shifted tableaux, in which the dual equivalence classes are determined by the (straight) shapes. Considering mixed-insertion, the recording tableau of a dual class is the recording tableau of the unique shifted Yamanouchi tableau in that class.

Proposition 2.25 ([28, Corollary 2.5])**.** *Two tableaux of the same straight shape are dual equivalent.*

Shifted dual equivalent words also have a characterization in terms of Worley–Sagan insertion.

Theorem 2.26 ([27, Theorem 2.12])**.** *Let w and u be words in $[n]'$. Then, w and u are shifted Knuth equivalent if and only if their recording tableaux under Worley–Sagan insertion coincide, i.e., $Q(w) = Q(v)$.*

2.4 Shifted evacuation and reversal

Definition 2.27. Let $T \in \mathsf{ShST}(\lambda/\mu, n)$. The *complement* of T in $[n]'$ is the tableau $\mathsf{c}_n(T)$ obtained by reflecting T along the anti-diagonal in the shifted stair shape $\delta = (\lambda_1, \lambda_1 - 1, \ldots, 1)$, i.e., sending each box in (i, j) to $(\lambda_1 - j + 1, \lambda_1 - i + 1)$, replacing each unprimed entry i with $\theta_{1,n}(i)'$ and each primed entry i' with $\theta_{1,n}(i)$, where, we recall, $\theta_{1,n}$ denotes the longest permutation in $\mathfrak{S}_n$.

Hence, if T is of shape λ/μ, then $\mathsf{c}_n(T)$ is of shape $\mu^\vee/\lambda^\vee$, and if $\mathsf{wt}(T) = (wt_1, \ldots, wt_n)$, then $\mathsf{wt}(\mathsf{c}_n(T)) = \theta_{1,n}(\mathsf{wt}(T)) = (wt_n, \ldots, wt_1)$. We have

$$w_{\mathsf{col}}(\mathsf{c}_n(T)) = \mathsf{c}_n(w(T)) \tag{2.4}$$

where $w_{\mathsf{col}}(T)$ denotes the *column reading word* of T, which is read along columns from bottom to top, going left to right, and where $\mathsf{c}_n(w)$ is set to be $\theta_{1,n}(w)$, for w a word in $[n]'$. For diagonally-shaped tableaux, it is clear that the row and column reading words coincide. More generally, $w(T) \equiv_k w_{\mathsf{col}}(T)$, for any $T \in \mathsf{ShST}(\lambda/\mu)$ [72, Lemma 6.4.12], and thus

$$w(\mathsf{c}_n(T)) \equiv_k \mathsf{c}_n(w(T)). \tag{2.5}$$

By construction, the operator c_n is coplactic. In particular, it preserves shifted Knuth and dual equivalences [72, Lemma 7.1.4], and it commutes with standardization. The following result, which is also valid for ordinary Young tableaux, is due to Haiman.

Theorem 2.28 ([28, Theorem 2.13]). *Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, there exists a unique shifted tableau T^e that is shifted Knuth equivalent to $\mathsf{c}_n(T)$ and shifted dual equivalent to T.*

This unique tableau is known as the *reversal* of T. If T is straight-shaped, then this is known as the *shifted evacuation* and denoted $\mathsf{evac}(T)$.

Proposition 2.29 ([72, Definition 7.1.5, Lemma 7.1.6]). *Given $T \in \mathsf{ShST}(\nu, n)$, its* (shifted) evacuation, *defined as* $\mathsf{evac}(T) := \mathsf{rect}(\mathsf{c}_n(T))$ *is the unique shifted tableau that is shifted Knuth equivalent to $\mathsf{c}_n(T)$ and shifted dual equivalent to T. In particular,* $\mathsf{evac}(T)$ *has the same shape as T and* $\mathsf{evac}^2(T) = T$.

As a consequence of evac being an involution, we have that $\mathsf{evac}(Y_\nu)$ is the unique shifted tableau in canonical form of shape ν and weight $\theta_{1,n}(\nu)$, where ν is a strict partition and $n = \ell(\nu)$. The following result provides a straightforward way to compute the evacuation of Y_ν.

Lemma 2.30. *Let $\nu = (\nu_1, \ldots, \nu_n)$ be a strict partition, with $n > 1$. Then,* $\mathsf{evac}(Y_\nu)$ *is the tableau of shape ν such that its n-th row is filled with n^{ν_n}, and its i-th row is filled with*

$$i^{\nu_n}(i+1)'(i+1)^{\nu_{n-1}-\nu_n-1}\ldots n' n^{\nu_i-\nu_{i+1}-1},$$

reading from left to right, for $i < n$.

Proof. This filling clearly defines a shifted semistandard tableau. Let T_0 be the tableau in those conditions. By construction, T_0 has shape ν and it is clear that its weight is given by $(\nu_n, \ldots, \nu_1) = \theta_{1,n}(\nu)$. Hence, $T_0 = \mathsf{evac}(Y_\nu)$. □

Example 2.31. Let $\mu = (4, 3, 1)$ and $n = 3$. Then,

$$Y_\mu = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & & 3 & \end{array} \longrightarrow \mathsf{evac}(Y_\mu) = \begin{array}{cccc} 1 & 2' & 2 & 3' \\ & 2 & 3' & 3 \\ & & 3 & \end{array}.$$

Since c_n preserves shifted Knuth equivalence, the reversal operator is the coplactic extension of evacuation, in the sense that, we may first rectify T, then apply the evacuation operator, and then perform outer *jeu de taquin* slides, in the reverse order defined by the previous rectification, to get a tableau T^e with the same shape of T. From [28, Corollaries 2.5, 2.8 and 2.9], this tableau T^e is shifted dual equivalent to T, besides being shifted Knuth equivalent to $\mathsf{c}_n(T)$. This process is detailed in Proposition 2.45, with the aid of shifted tableau switching. In particular, $\mathsf{evac}(T) = T^e$ for tableaux of straight shape.

Proposition 2.32. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$. Then, we have the following:*

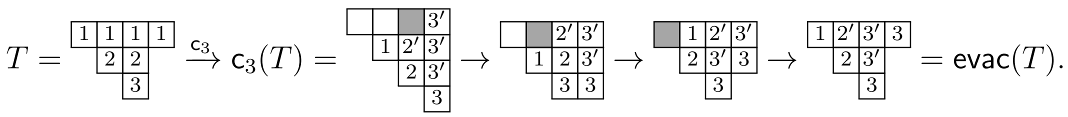
1. $(T^e)^e = T$.
2. $\mathsf{wt}(T^e) = \theta_{1,n}(\mathsf{wt}(T))$.

Proof. We have $(T^e)^e$ is shifted dual equivalent to T^e, which is shifted dual equivalent to T. The operator c_n is coplactic, and thus $(T^e)^e \equiv_k \mathsf{c}_n(T^e) \equiv_k \mathsf{c}_n^2(T) = T$. Then, by Theorem 2.28, $(T^e)^e = T$. Since shifted Knuth equivalence preserves the weight, then $\mathsf{wt}(T^e) = \mathsf{wt}(\mathsf{c}_n(T)) = \theta_{1,n}(\mathsf{wt}(T))$. $\square$

Example 2.33. Consider the following tableau in $\mathsf{ShST}(\nu, 3)$, with $\nu = (4, 2, 1)$:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\,.$$

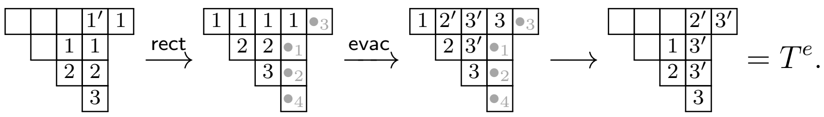
To obtain $\mathsf{evac}(T)$ we first compute $\mathsf{c}_3(T)$ and then rectify it:



Example 2.34. Consider the following tableau in $\mathsf{ShST}(\lambda/\mu, 3)$, with $\lambda = (6, 5, 3, 1)$ and $\mu = (4, 2)$:

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline \multicolumn{1}{c|}{} & & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{3-4} \multicolumn{3}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{4-4} \end{array}\,.$$

To compute the reversal T^e, we first rectify T, recording in reverse order the outer corners resulting of the sequence of inner *jeu de taquin* slides. Then, we compute the evacuation of the obtained straight-shaped tableau and perform outer *jeu de taquin* slides defined by the outer corners of the previous sequence, from the smallest to the largest. This process can be re-written with the aid of the shifted tableau switching to be introduced in the next section.



If T is a LRS tableau of shape λ/μ and weight ν, then $\mathsf{c}_n(T^e)$ is a LRS tableau of shape $\mu^\vee/\lambda^\vee$ and weight ν. Indeed, as c_n is coplactic and $T^e \equiv_k \mathsf{c}_n(T)$, then $\mathsf{c}_n(T^e) \equiv_k T \equiv_k Y_\nu$. Thus, we have the symmetry $f^{\lambda}_{\mu\nu} = f^{\mu^\vee}_{\lambda^\vee\nu}$.

2.5 Shifted tableau switching

The tableau switching algorithm for type A is an involution that, given a pair of tableaux (S, T), with S extending T, moves one through another, using switches similar to the *jeu de taquin* slides, regarding the boxes in S as inner corners, and keeping semistandardness, within each of the alphabets, in the intermediate steps [5]. Any chosen sequence of those switches produces the same final result [5, Theorem 2.2]. This is not the case for the *shifted tableau switching*, which must be performed following a determined sequence of switches, similarly to the type A *infusion map* [70, 71]. As observed in [15, Remark 8.1], the resulting pair obtained by the shifted tableau switching can be recovered alternatively, using the type C infusion map of Thomas and Yong [70] on a pair of standardized tableaux, followed by the *semistandardization* of Pechenik and Yong [52]. The infusion map on type A standard tableaux is a special case of the tableau switching process [5], in which the order to perform the switches is determined by the entries of the standardization of the inner-most tableau. Unlike the case for ordinary Young tableaux, the shifted tableau switching process comprehends a determined sequence of switches to be performed, which agrees with the one prescribed by the type C infusion map on shifted standard tableaux (Proposition 2.50). Furthermore, it is compatible with standardization [15, Remark 3.8]. This will be illustrated in Example 2.51.

We recall the definitions of the shifted tableau switching for pairs (A, B) of border strip shifted tableaux, with B extending A, and for pairs of shifted semistandard tableaux (S, T), with T extending S. We omit most of the details and proofs, and refer to [15]. Recall that $\mathbf{i}$ denotes either the letters i or $i' \in [n]'$. A skew shape $S(\lambda/\mu)$ is said to be a *double border strip* if it contains no subset of the form $\{(i,j),(i+1,j+1),(i+2,j+2)\}$.

Definition 2.35 ([15, Definition 3.1])**.** Let $S(\lambda/\mu)$ be a double border strip. A *shifted perforated* $\mathbf{a}$-*tableau* in λ/μ is a filling of some of the boxes of $S(\lambda/\mu)$ with letters a, $a' \in [n]'$ such that no a'-boxes are south-east to any a-boxes, there is at most one a per column and one a' per row, and the main diagonal has at most one $\mathbf{a}$.

The *shape* of a perforated $\mathbf{a}$-tableau A in a double border strip $S(\lambda/\mu)$ consists of the $\mathbf{a}$-filled boxes of $S(\lambda/\mu)$, and is denoted by $sh(A)$. Given a perforated $\mathbf{a}$-tableau A and a perforated $\mathbf{b}$-tableau B, the pair (A, B) is said to be a *shifted perforated* $(\mathbf{a}, \mathbf{b})$-*pair* of shape λ/μ if $S(\lambda/\mu)$ is the disjoint union of $sh(A)$ and $sh(B)$. In this case, we denote by $A \sqcup B$ the filling obtained by overlapping A and B.

Example 2.36. The following are shifted perforated $\mathbf{1}$- and $\mathbf{2}$-tableaux, that form a shifted perforated $(\mathbf{1}, \mathbf{2})$-pair of shape $(6, 4, 3)/(3, 1)$:

$$A = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 1' & 1 & \ \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|} \cline{2-5} & \ & 1' & \ & \ \\ \cline{2-5} \end{array} \quad \begin{array}{cc|c|c|c|} \cline{3-5} & & 1 & \ & 1 \\ \cline{3-5} \end{array}$$

$$B = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & 2' \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|} \cline{2-5} & \ & \ & 2' & 2 \\ \cline{2-5} \end{array} \quad \begin{array}{cc|c|c|c|} \cline{3-5} & & 2 & \ & \ \\ \cline{3-5} \end{array}$$

$$A \sqcup B = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 1' & 1 & 2' \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|} \cline{2-5} & \ & 1' & 2' & 2 \\ \cline{2-5} \end{array} \quad \begin{array}{cc|c|c|c|} \cline{3-5} & & 1 & 2 & 1 \\ \cline{3-5} \end{array}.$$

If (A, B) is a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair, one can interchange an $\mathbf{a}$-box with a $\mathbf{b}$-box in $A \sqcup B$ subject to the following moves, called *(shifted) switches* , illustrated in Figure 2.2.

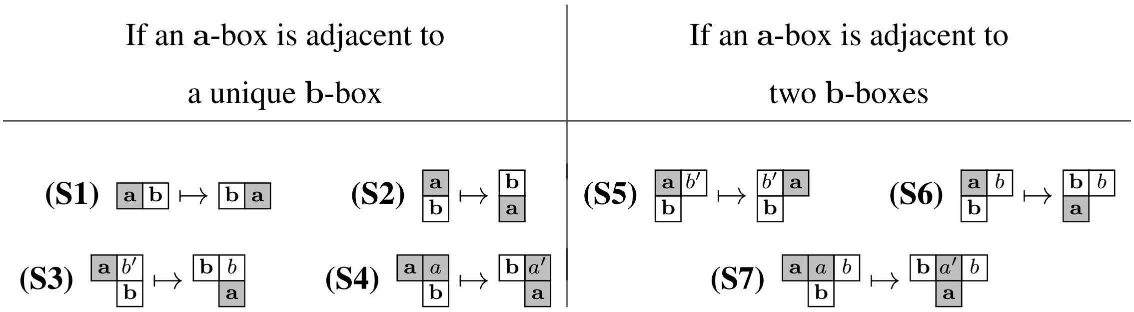


Figure 2.2: The shifted switches [15, Section 3].

The switches **(S3)**, **(S4)**, **(S7)** are called the *diagonal switches* and can only be performed when $\mathbf{a}$ and $\mathbf{b}$ are in the main diagonal. An $\mathbf{a}$-box is said to be *fully switched* if it cannot be switched with any $\mathbf{b}$-boxes, and $A \sqcup B$ is said to be *fully switched* if every $\mathbf{a}$-box is fully switched.

Remark 2.37. With the exception of **(S4)** and **(S7)**, the shifted switches in Figure 2.2 correspond to shifted *jeu de taquin* moves, regarding the $\mathbf{a}$-boxes as empty corners.

Definition 2.38 (Shifted switching process [15])**.** Let $T := A \sqcup B$ be a perforated $(\mathbf{a}, \mathbf{b})$-pair that is not fully switched. The *shifted switching process* from T to $\varsigma^m(T)$, with m the least integer such that $\varsigma^m(T)$ is fully switched, is obtained as follows: choose the rightmost a-box in A that is a neighbour to the north or west of a $\mathbf{b}$-box, if it exists, otherwise, choose the bottommost a'-box in the same conditions, and then apply the adequate switch among **(S1)-(S7)**, obtaining $\varsigma(T)$. The process is repeated until $\varsigma^m(T)$ is fully switched, where $\varsigma^i(T) := \varsigma(\varsigma^{i-1}(T))$, for $i \geq 2$. We then set $\mathsf{SP}_1(A, B) := \varsigma^m(T)^b$ and $\mathsf{SP}_2(A, B) := \varsigma^m(T)^a$, the tableaux obtained from $\varsigma^m(T)$ considering only the letters $\{b', b\}$ and $\{a', b\}$ respectively, and define

$$\mathsf{SP}(A, B) := (\mathsf{SP}_1(A, B), \mathsf{SP}_2(A, B)).$$

This process is depicted by the algorithm in Figure 2.3.

define $\mathsf{SP}(A, B)$
input (A, B) a perforated $(\mathbf{a}, \mathbf{b})$-pair, with $a, b \in [n]$, such that B extends A.
  set $F := A \sqcup B$
  while F is not fully switched, **do**
    $F := \varsigma(F)$
  set $C := F^a$ and $D := F^b$
  return (D, C)

Figure 2.3: Shifted tableau switching for shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs [15, Algorithm 1].

Example 2.39. Consider the shifted perforated $(\mathbf{1}, \mathbf{2})$-pair of the previous example, which is not fully switched.

$$A \sqcup B = \begin{ytableau} \ & \ & \ & 1' & 1 & 2' \\ \none & \ & 1' & 2' & 2 \\ \none & \none & 1 & 2 & 1 \end{ytableau}.$$

The leftmost box filled with 1 (unprimed) is in position $(1, 5)$, and it is adjacent to two $\mathbf{2}$-boxes. Hence, we apply the $(\mathbf{S5})$ switch, obtaining:

$$A \sqcup B = \begin{ytableau} \ & \ & \ & 1' & 1 & 2' \\ \none & \ & 1' & 2' & 2 \\ \none & \none & 1 & 2 & 1 \end{ytableau} \xrightarrow{(\mathbf{S5})} \begin{ytableau} \ & \ & \ & 1' & 2' & 1 \\ \none & \ & 1' & 2' & 2 \\ \none & \none & 1 & 2 & 1 \end{ytableau} = \varsigma(A \sqcup B).$$

This 1-box is now fully switched. Continuing the shifted switching process, until all $\mathbf{1}$-boxes are fully switched, we obtain:

$$\begin{aligned} \varsigma(A \sqcup B) = \begin{ytableau} \ & \ & \ & 1' & 2' & 1 \\ \none & \ & 1' & 2' & 2 \\ \none & \none & 1 & 2 & 1 \end{ytableau} &\xrightarrow{(\mathbf{S1})} \begin{ytableau} \ & \ & \ & 1' & 2' & 1 \\ \none & \ & 1' & 2' & 2 \\ \none & \none & 2 & 1 & 1 \end{ytableau} \xrightarrow{(\mathbf{S5})} \begin{ytableau} \ & \ & \ & 1' & 2' & 1 \\ \none & \ & 2' & 1' & 2 \\ \none & \none & 2 & 1 & 1 \end{ytableau} \\ &\xrightarrow{(\mathbf{S1})} \begin{ytableau} \ & \ & \ & 1' & 2' & 1 \\ \none & \ & 2' & 2 & 1' \\ \none & \none & 2 & 1 & 1 \end{ytableau} \xrightarrow{(\mathbf{S5})} \begin{ytableau} \ & \ & \ & 2' & 1' & 1 \\ \none & \ & 2' & 2 & 1' \\ \none & \none & 2 & 1 & 1 \end{ytableau} = \varsigma^5(A \sqcup B). \end{aligned}$$

Remark 2.40. Unlike the tableau switching for Young tableaux [5], the shifted version depends on the order in which the $\mathbf{a}$-boxes are chosen [14, Remark 3.7 (i)]. For instance, if one applies **(S6)** (corresponding to choose the box with $2'$) instead of **(S1)** (corresponding to the box with 1, i.e., the rightmost 1-box), the obtained filling is not a valid $(\mathbf{1}, \mathbf{2})$-pair, as the second row is not weakly increasing:

$$\begin{ytableau} \ & \ & 1' & 2 \\ \none & 1 & 2 \end{ytableau} \xrightarrow{(\mathbf{S6})} \begin{ytableau} \ & \ & 2 & 2 \\ \none & 1 & 1' \end{ytableau}.$$

This process is well defined and it is an involution [15, Theorem 3.5]. It may be extended to pairs of shifted semistandard tableaux (S, T), with T extending S. The result is denoted by

$$\mathsf{SW}(S, T) := (\mathsf{SW}_1(S, T), \mathsf{SW}_2(S, T)),$$

where $\mathsf{SW}_1(S,T) = T'$ and $\mathsf{SW}_2(S,T) = S'$ as depicted in Figure 2.4. The shifted tableau switching SW for pairs of shifted semistandard tableaux is also well defined [15, Theorem 3.6] and it is an involution [15, Theorem 4.3]. If S is straight-shaped, then $\mathsf{SW}_1(S,T) = \mathsf{rect}(T)$. Similar to the type A case [2, 5], if T is a LRS tableau, then so it is $\mathsf{SW}_2(S,T)$, for any straight-shaped shifted S extended by T [15, Theorem 4.3]. Thus, considering $S := Y_\mu$, we have a bijection that sends T, a LRS tableau of shape λ/μ and weight ν, to $\mathsf{SW}_2(Y_\mu,T)$, a LRS tableau of shape λ/ν and weight ν, giving the symmetry $f^\lambda_{\mu\nu} = f^\lambda_{\nu\mu}$.

define $\mathsf{SW}(S,T)$
input (S,T) pair of shifted tableaux, with T extending S, which are decomposed into $T = T^1 \sqcup \cdots \sqcup T^n$ and $S = S^1 \sqcup \cdots \sqcup S^m$.
  for i from m to 1, **do**
    for j from 1 to n, **do**
      $\mathsf{SP}(S^i, T^j)$
  set $T' := T^1 \sqcup \cdots \sqcup T^n$ and $S' := S^1 \sqcup \cdots \sqcup S^m$
  return (T', S')

Figure 2.4: Shifted tableau switching for pairs of shifted tableaux [15, Algorithm 2].

This shifted tableau switching is compatible with standardization [15, Remark 3.8], i.e.,

$$\begin{aligned} \mathsf{SW}(\mathsf{std}(S),T) &= (id \times \mathsf{std}) \circ \mathsf{SW}(S,T) \\ \mathsf{SW}(S,\mathsf{std}(T)) &= (\mathsf{std} \times id) \circ \mathsf{SW}(S,T) \end{aligned} \tag{2.6}$$

where $id \times \mathsf{std}$ denotes the usual Cartesian product of maps, i.e., $(id \times \mathsf{std})(S,T) = (S, \mathsf{std}(T))$. Moreover, since the switches may be regarded as *jeu de taquin* slides, the pair $\mathsf{SW}(S,T)$ is component-wise shifted Knuth equivalent to (T,S), for any pair of shifted semistandard tableaux (S,T), with T extending S. Moreover, rewriting [28, Corollaries 2.8 and 2.9] in terms of the shifted tableau switching yields the following.

Proposition 2.41 ([13, Proposition 3.2])**.** *Let S and T be shifted semistandard tableaux in the same dual equivalence class (in particular, S and T have the same shape). Let W be a semistandard shifted tableau. Then,*

1. *If S and T extend W, then $\mathsf{SW}_2(W,S) = \mathsf{SW}_2(W,T)$, and $\mathsf{SW}_1(W,S)$ is shifted dual equivalent to $\mathsf{SW}_1(W,T)$.*

2. *If W extends S and T, then $\mathsf{SW}_1(S,W) = \mathsf{SW}_1(T,W)$, and $\mathsf{SW}_2(S,W)$ is shifted dual equivalent to $\mathsf{SW}_2(T,W)$.*

Corollary 2.42. *Let S and T be shifted semistandard tableaux such that T extends S. Then,*

1. $(\mathsf{SW}_1(S,T))^e = \mathsf{SW}_1(S,T^e)$.

2. $(\mathsf{SW}_2(S,T))^e = \mathsf{SW}_2(S^e,T)$.

Proof. By definition, T is shifted dual equivalent to T^e, hence, by Proposition 2.41, we have that $\mathsf{SW}_1(S,T)$ is shifted dual equivalent to $\mathsf{SW}_1(S,T^e)$. The shifted tableau switching algorithm ensures that $\mathsf{SW}_1(S,T^e)$ is shifted Knuth equivalent to T^e, and since the operator c_n is coplactic, we have

$$\mathsf{SW}_1(S,T^e) \equiv_k T^e \equiv_k \mathsf{c}_n(T) \equiv_k \mathsf{c}_n(\mathsf{SW}_1(S,T)).$$

Since $\mathsf{SW}_1(S,T^e)$ is shifted dual equivalent to $\mathsf{SW}_1(S,T)$ and shifted dual equivalent to $\mathsf{c}_n(\mathsf{SW}_1(S,T))$, we have that $(\mathsf{SW}_1(S,T))^e = \mathsf{SW}_2(S,T^e)$. The proof for the second statement is similar. □

The following result ensures that the shifted tableau switching is also compatible with canonical form.

Proposition 2.43. *Let S, S', T and T' be shifted semistandard tableaux filled in $[n]'$, not necessarily in canonical form, such that T extends S and T' extends S'. Suppose that S and S' have the same canonical form, and so do T and T'. Then,*

1. $\mathsf{SW}_1(S,T)$, $\mathsf{SW}_1(S',T)$, $\mathsf{SW}_1(S,T')$ *and* $\mathsf{SW}_1(S',T')$ *have the same canonical form.*

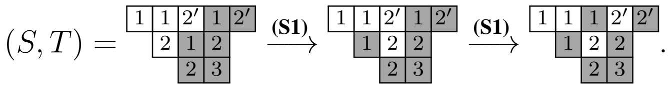
2. $\mathsf{SW}_2(S,T)$, $\mathsf{SW}_2(S',T)$, $\mathsf{SW}_2(S,T')$ *and* $\mathsf{SW}_2(S',T')$ *have the same canonical form.*

Proof. It suffices to show that for each $i \in [n]$, the southwesternmost occurrence of **i** maintains its relative position. This is verified for each switch **(S1)-(S7)**. Moreover, the switching algorithm states that one must start with the rightmost unprimed i that has neighbours to its south or east, and then proceeding to the lowest primed i'. Hence, the switching path is going from right to left, and then from bottom to top. Therefore, the lowest and leftmost **i** is either the last unprimed i or the first primed i', leaving the switching order unchanged. □

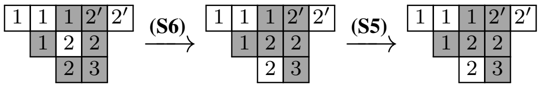
Example 2.44. Consider the following pair of shifted semistandard tableau (S,T), with T (in gray background) extending S (in white background):

$$(S,T) = \begin{array}{ccccc} 1 & 1 & 2' & 1 & 2' \\ & 2 & 1 & 2 & \\ & & 2 & 3 & \end{array}$$

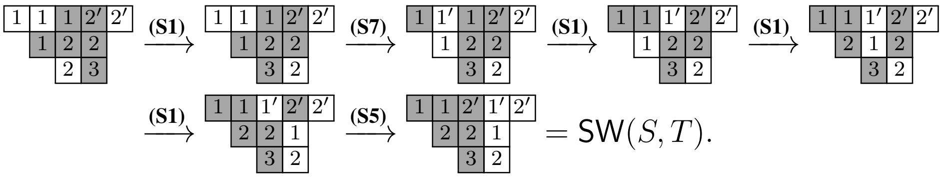
To apply the shifted tableau switching SW to (S,T), we first compute $\mathsf{SP}(S^2,T^1)$:



Then, we compute $\mathsf{SP}(S^2, T^2)$:



Continuing the process, we have:



The algorithm to compute the reversal of a shifted tableau $T \in \mathsf{ShST}(\lambda/\mu, n)$ may be described using the shifted tableau switching [13].

Proposition 2.45 ([13, Definition 4.5]). *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let U and W be shifted standard tableaux of shape μ. Let $W' := \mathsf{SW}_2(W, T)$ and $U' := \mathsf{SW}_2(U, T)$. Then,*

$$\mathsf{SW}(\mathsf{evac}(\mathsf{rect}(T)), W') = \mathsf{SW}(\mathsf{evac}(\mathsf{rect}(T)), U'),$$

and we have

$$\mathsf{SW}_2(\mathsf{evac}(\mathsf{rect}(T)), W') = T^e.$$

Proof. Since they have the same straight shape, $\mathsf{rect}(T)$ is shifted dual equivalent to $\mathsf{evac}(\mathsf{rect}(T))$. Thus, by Proposition 2.41, $\mathsf{SW}_2(\mathsf{evac}(\mathsf{rect}(T)), U')$ is dual equivalent to

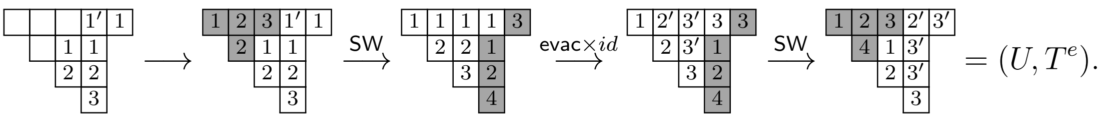
$$\mathsf{SW}_2(\mathsf{rect}(T), U') = \mathsf{SW}_2\big(\mathsf{SW}_1(U, T), \mathsf{SW}_2(U, T)\big) = \mathsf{SW}_2(\mathsf{SW}(U, T)) = T.$$

Furthermore, since $\mathsf{SW}_2(\mathsf{evac}(\mathsf{rect}(T)), U')$ is shifted Knuth equivalent to $\mathsf{evac}(\mathsf{rect}(T))$, we have

$$\mathsf{SW}_2(\mathsf{evac}(\mathsf{rect}(T)), U') \equiv_k \mathsf{evac}(\mathsf{rect}(T)) \equiv_k \mathsf{c}_n(\mathsf{rect}(T)) \equiv_k \mathsf{c}_n(T).$$

The result then follows from the uniqueness of Theorem 2.28. □

Example 2.46. To illustrate this procedure, we use the same tableau in Example 2.34, filling the inner shape μ with a standard tableau U. We note that, since $U = U^1 \sqcup \cdots \sqcup U^{|\mu|}$ is standard, then each U^i consists of a single box filled with (unprimed) i. Thus, the switches **(S4)** and **(S7)** will not be used during the shifted tableau switching process.



2.5.1 Type C infusion

As remarked before, the shifted tableau switching process could be obtained by first standardizing the involved tableaux, apply the *type C infusion involution* [70], and then the *shifted semistandardization* process [52]. This is due to the shifted tableau switching being compatible with standardization and the fact that, on shifted standard tableaux, the order in which the shifted are performed (see Figure 2.4) agrees with the one determined by the type C infusion map (see Lemma 2.48 below).

Definition 2.47 ([70])**.** Let (S, T) be a pair of shifted standard tableaux, of shapes μ/ν and λ/μ (thus, T extends S), respectively. The *type C infusion* of the pair (S, T), denoted by $\mathsf{infusion}(S,T) := (\mathsf{infusion}_1(S,T), \mathsf{infusion}_2(S,T))$, is the pair (X, Y) of standard tableaux of shapes γ/ν and λ/γ, for some strict partition γ with $|\gamma| = |\lambda| - |\mu|$, obtained in the following way:

1. Let m be the largest entry of S. Then, its box is a inner corner for λ/μ, and we perform *jeu de taquin* on T starting with that inner corner, until an outer corner is obtained. Place m on that outer corner and never move it again for the duration of the process.

2. Repeat the last step for the remaining entries of S, going from the largest to the smallest.

3. Then, X is tableau obtained after performing all the shifted *jeu de taquin* slides on T determined by the entries of S, and Y is the tableau obtained by placing the entries of S on the resulting outer corners.

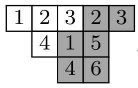
The shifted tableau $\mathsf{infusion}_1(S,T)$ is then the result of applying shifted *jeu de taquin* inner slides to T (determined by S), and $\mathsf{infusion}_2(S,T)$ encodes the order in which those slides were performed. In particular, if S has straight-shape, then $\mathsf{infusion}_1(S,T) = \mathsf{rect}(T)$.

If (S,T) is a pair of shifted standard tableaux, then there are no repeated entries, nor primed ones, thus the algorithm in Figure 2.4 to compute $\mathsf{SW}(S,T)$ requires only the switches $(\mathbf{S1})$ and $(\mathbf{S2})$ of Figure 2.2. These switches correspond to shifted *jeu de taquin* slides in a shifted standard tableau, as the exceptional slide (see Definition 2.8) cannot occur. Moreover, the algorithm for the shifted tableau switching in Figure 2.4 states that the shifted switches must be performed from the largest entry of S to the smallest, which agrees with the order defined by the type C infusion (Definition 2.47). Thus, we have the following.

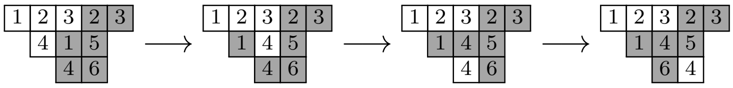
Lemma 2.48. *Let (S,T) be a pair of shifted standard tableaux, with T extending S. Then,* $\mathsf{SW}(S,T) = \mathsf{infusion}(S,T)$.

Example 2.49. Consider the following pair of shifted standard tableaux

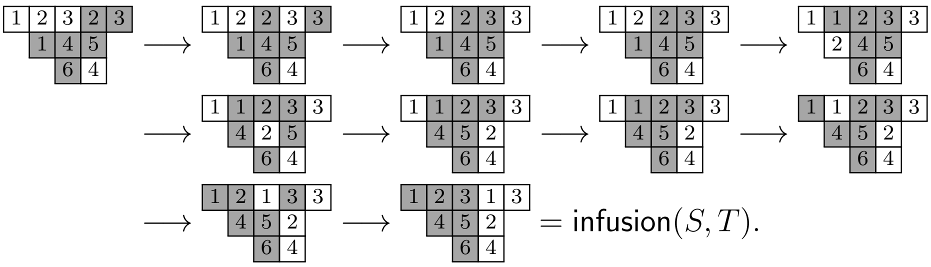
$$(S,T) =$$



To compute $\mathsf{infusion}(S,T)$ we start with the largest entry of S, and regarding its box as inner corner, perform *jeu de taquin* slides:



Continuing with the next largest entries of S, we obtain:



$= \mathsf{infusion}(S,T)$.

Proposition 2.50. *Let (S,T) be a pair of shifted semistandard tableaux, with T extending S, and such that* $\mathsf{wt}(T) = \nu_T$ *and* $\mathsf{wt}(S) = \nu_S$. *Then,*

$$\mathsf{SW}(S,T) = (\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ \mathsf{infusion}(\mathsf{std}(S), \mathsf{std}(T)).$$

Proof. Since $\mathsf{SW}_1(S,T)$ and $\mathsf{SW}_2(S,T)$ have weights ν_T and ν_S, respectively, then by [52, Lemma 9.5] we have

$$(\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ (\mathsf{std} \times \mathsf{std})(\mathsf{SW}(S,T)) = \mathsf{SW}(S,T). \tag{2.7}$$

By Lemma 2.48, (2.6) and (2.7), we have

$$\begin{aligned}
(\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ \mathsf{infusion}(\mathsf{std}(S), \mathsf{std}(T)) = \\
&= (\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ \mathsf{SW}(\mathsf{std}(S), \mathsf{std}(T)) \\
&= (\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ (id \times \mathsf{std}) \circ \mathsf{SW}(S, \mathsf{std}(T)) \\
&= (\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ (id \times \mathsf{std}) \circ (\mathsf{std} \times id) \circ \mathsf{SW}(S,T) \\
&= (\mathsf{sstd}_{\nu_T} \times \mathsf{sstd}_{\nu_S}) \circ (\mathsf{std} \times \mathsf{std}) \circ \mathsf{SW}(S,T) \\
&= \mathsf{SW}(S,T).
\end{aligned}$$

$\square$

Example 2.51. We illustrate the process with the shifted tableau pair (S,T) from a previous example:

$$(S,T) = \begin{array}{|c|c|c|c|c|} 1 & 1 & 2' & 1 & 2' \\ & 2 & 1 & 2 & \\ & & 2 & 3 & \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 2 & 3 \\ & 4 & 1 & 5 & \\ & & 4 & 6 & \end{array} = (\mathsf{std}(S), \mathsf{std}(T)).$$

From Example 2.49, we have

$$\begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 2 & 3 \\ & 4 & 1 & 5 & \\ & & 4 & 6 & \end{array} \xrightarrow{\text{infusion}} \begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 1 & 3 \\ & 4 & 5 & 2 & \\ & & 6 & 4 & \end{array}.$$

Since we have $\mathsf{wt}(T) = (2,3,1)$ and $\mathsf{wt}(S) = (2,2)$, we now apply the semistandardization process with respect to these compositions, respectively:

$$\begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 1 & 3 \\ & 4 & 5 & 2 & \\ & & 6 & 4 & \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|} 1_1 & 1_2 & 2_3 & 1_1 & 2_3 \\ & 2_4 & 2_5 & 1_2 & \\ & & 3_6 & 2_4 & \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|} 1 & 1 & 2' & 1' & 2' \\ & 2 & 2 & 1 & \\ & & 3 & 2 & \end{array} = \mathsf{SW}(S,T).$$

2.5.2 Shifted evacuation via tableau switching

The authors in [15] present another algorithm for tableaux of straight shape, that coincides with the shifted evacuation from Section 2.4, using the shifted tableau switching[1]. We consider the auxiliary alphabet $-[n]' := \{-n' < -n < \cdots < -1' < -1\}$ and $-[n]' \sqcup [n]' := \{-n' < -n < \cdots < -1' < -1 < 1' < 1 < \cdots < n' < n\}$. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $k \in [n]$, we define $\mathsf{neg}_k(T)$ to be the filling, in $-[n]' \sqcup [n]'$, obtained from T by replacing each k with $-k$ and each k' with $-k'$, leaving the remaining letters unchanged. If T is a filling of a shifted shape in $-[n]'$, we define $\mathsf{d}_n(T)$ to be the filling in $[n]'$ obtained from T by replacing each $-i$ with $\theta_{1,n}(i)$ and each $-i'$ with $\theta_{1,n}(i')$, that is,

$$\mathsf{d}_n(T) = \theta_{1,n} \mathsf{neg}_1^{-1} \cdots \mathsf{neg}_n^{-1}(T). \tag{2.8}$$

Consider the algorithm presented in Figure 2.5, defined on the alphabet $-[n]' \sqcup [n]'$ (we note that the use of negative entries ensures that those will not move again after being fully switched). This algorithm coincides with the shifted evacuation for straight-shaped tableaux [15, Theorem 5.6].

Given $T \in \mathsf{ShST}(\nu, n)$, the algorithm in Figure 2.5 may be easily modified to obtain a restriction evac_k to the alphabet $\{1, \ldots, k\}'$, for $k \leq n$, by applying evac to $T^1 \sqcup \cdots \sqcup T^k$ and maintaining $T^{k+1} \sqcup \cdots \sqcup T^n$ unchanged. This is depicted in Figure 2.6. It is clear that $\mathsf{evac}_n = \mathsf{evac}$.

[1] The authors use the terminology *shifted generalized evacuation* for this algorithm.

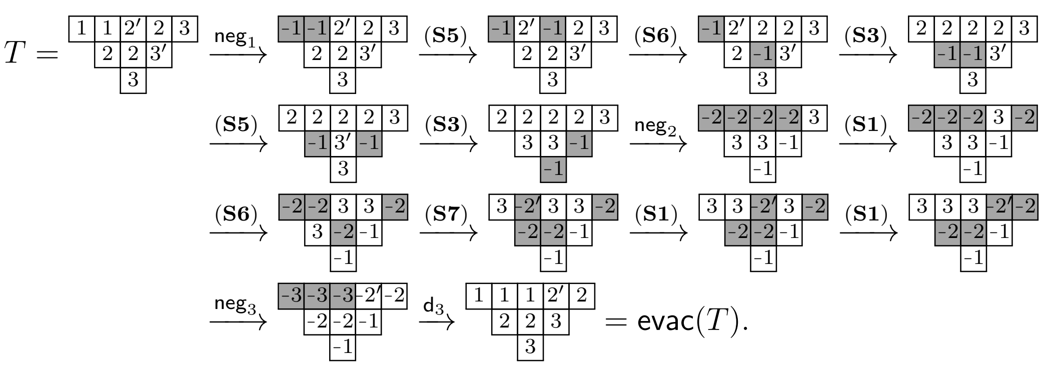
define $\mathsf{evac}(T)$
input T a shifted tableau of straight shape decomposed into $T^1 \sqcup \cdots \sqcup T^m$.
  set $T^E := \emptyset$
  for a from 1 to m, **do**
    set $T^a := \mathsf{neg}_a(T^a)$
    if $a = m$
      set $T^E := T^a \sqcup T^E$
    else
      set $(A, B) := \mathsf{SW}(T^a, T^{a+1} \sqcup \cdots \sqcup T^m)$
      set $T := A$ and $T^E := B \sqcup T^E$
  return $\mathsf{d}_m(T^E)$

Figure 2.5: The shifted evacuation algorithm [15, Algorithm 4].

define $\mathsf{evac_k}(T)$
input T a shifted tableau of straight shape and $k \leq n$.
  set $T^E := \emptyset$
  for a from 1 to k, **do**
    set $T^a := \mathsf{neg}_a(T^a)$
    if $a = k$
      set $T^E := T^a \sqcup T^E$
    else
      set $(A, B) := \mathsf{SW}(T^a, T^{a+1} \sqcup \cdots \sqcup T^k)$
      set $T := A$ and $T^E := B \sqcup T^E$
  return $\mathsf{d}_k(T^E) \sqcup T^{k+1} \sqcup \cdots \sqcup T^n$

Figure 2.6: The shifted evacuation algorithm, restricted to the letters $[1, k]'$.

Example 2.52. Let $T = \begin{array}{ccccc} 1 & 1 & 2' & 2 & 3 \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array}$. Computing $\mathsf{evac}(T)$ with the Algorithm in Figure 2.5, we have



Similarly to the ordinary Young tableaux case [51, Section 2.2, (5)] [5, Section 5], the shifted evacuation algorithms, in Figures 2.5 and 2.6, may be easily extended to skew shapes, by removing in both algorithms the requirement for the input to have a straight shape. We denote these operators by $\widetilde{\mathsf{evac}}$ and $\widetilde{\mathsf{evac}}_k$. However, we note that, similarly to the ordinary

Young tableaux case [5, Section 5], the involution $\widetilde{\mathsf{evac}}$ is different from the reversal, as in general, given $T \in \mathsf{ShST}(\lambda/\mu, n)$, we have that $\widetilde{\mathsf{evac}}(T) \neq T^e$, since $\widetilde{\mathsf{evac}}(T)$ does not need to be shifted Knuth equivalent to $\mathsf{c}_n(T)$ or to $\mathsf{evac}(\mathsf{rect}(T))$.

Example 2.53. Let $T = \begin{ytableau} \; & \; & \; & 1' & 1 \\ \none & \; & 1 & 1 \\ \none & \none & 2 & 2 \\ \none & \none & \none & 3 \end{ytableau}$. Then, $\widetilde{\mathsf{evac}}(T) = \begin{ytableau} \; & \; & \; & 2 & 3 \\ \none & \; & 1 & 3' \\ \none & \none & 2 & 3' \\ \none & \none & \none & 3 \end{ytableau} \neq T^e$ (see Example 2.34).

CHAPTER 3

A CRYSTAL-LIKE STRUCTURE ON $\mathsf{ShST}(\lambda/\mu)$

In this chapter, we recall the definition of the shifted tableau crystal introduced by Gillespie, Levinson and Purbhoo [23]. We then introduce the shifted crystal reflection operators.

3.1 Shifted tableau crystals

We recall the main results on the shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, n)$, the crystal-like structure on $\mathsf{ShST}(\lambda/\mu, n)$ introduced in [21, 23]. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ and let $\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}$ be the simple roots for the type A_{n-1} root system, for $i \in I := [n-1]$.

Definition 3.1 ([23, Definition 3.3])**.** Given a word w on $[n]'$ and $i \in I$, the *primed raising operator* $E'_i(w)$ is defined as the unique word such that

1. $\mathsf{std}(E'_i(w)) = \mathsf{std}(w)$,
2. $\mathsf{wt}(E'_i(w)) = \mathsf{wt}(w) + \alpha_i$,

if such word exists. Otherwise, $E'_i(w) = \varnothing$, and we say that E'_i is undefined on w. The *primed lowering operator* $F'_i(w)$ is defined in analogous way using $-\alpha_i$.

This notion is well defined due to Lemma 2.10, and as a direct consequence we have that $E'_i(w) = v$ if and only if $w = F'_i(v)$, for any words w and v [23, Proposition 3.4]. This definition is extended to a shifted semistandard tableau T, putting $E'_i(T)$ as the shifted semistandard tableau with the same shape as T and with (row) reading word $E'_i(w(T))$. The primed operators preserve semistandardness [23, Proposition 3.6] and they are coplactic [23, Proposition 3.7], i.e., they commute with the shifted *jeu de taquin*. Moreover, the tableaux T, $E'_i(T)$ and $F'_i(T)$ are shifted dual equivalent, since their standardization is unchanged (Definition 2.24), whenever E'_i and E_i are defined on T.

In order to simplify the notation, we will henceforth consider the alphabet $\{1,2\}'$, but the results hold for any primed alphabet $\{i, i+1\}'$ of two adjacent letters. The following propositions provide a simple way to compute the primed operators both on words and on straight-shaped shifted tableaux.

Proposition 3.2 ([23, Proposition 3.9])**.** *To compute $F'_1(w)$ consider all representatives of w. If all representatives have the property that the last 1 is left of the last $2'$ then $F'_1(w) = \varnothing$. If there exists a representative such that the last 1 is right of the last $2'$ then $F'_1(w)$ is obtained by changing the last 1 to $2'$ in that representative. The word $E'_1(w)$ is defined similarly reverting the roles of 1 and $2'$.*

Proposition 3.3 ([23, Proposition 3.11])**.** *Let $T \in \mathsf{ShST}(\lambda, n)$ a shifted semistandard tableau of straight shape. If T has one row, then $E'_1(T)$ (respectively $F'_1(T)$) is obtained by changing the leftmost 2 to 1 (respectively, 1 to 2), if possible, and it is $\varnothing$ otherwise. If T has two rows and the first row contains a $2'$, then $E'_1(T)$ is obtained by changing that $2'$ to 1 and $F'_1(T) = \varnothing$. If the first row does not contain a $2'$, then $E'_1(T) = \varnothing$ and $F'_1(T)$ is obtained by changing the rightmost 1 to $2'$.*

Example 3.4. Let $T = \begin{array}{ccccc} 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & 2 & \\ & & 3 & & \end{array}$, with $\mathsf{wt}(T) = (3,5,1)$. Then,

$$F'_2(T) = \begin{array}{ccccc} 1 & 1 & 1 & 2' & 3' \\ & 2 & 2 & 2 & \\ & & 3 & & \end{array}.$$

Observe that $\mathsf{wt}(F'_2(T)) = (3,4,2) = (3,5,1) - (0,1,-1)$ and that

$$\mathsf{std}(T) = \mathsf{std}(F'_2(T)) = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 8 \\ & 5 & 6 & 7 & \\ & & 9 & & \end{array}.$$

Given a word w on the alphabet $[n]'$ and $i \in I$, the i-th *lattice walk* of w is obtained by considering the subword w^i, consisting of the letters $\{i, i+1\}'$, and replacing each letter according to the following table, starting at the origin $(0,0)$.

$x_k y_k = 0$	$\xrightarrow{1'}$	$\xrightarrow{1}$	$\uparrow 2'$	$\uparrow 2$
$x_k y_k \neq 0$	$\xrightarrow{1'}$	$\downarrow 1$	$\xleftarrow{2'}$	$\uparrow 2$

The lattice walks of a word may be used to provide another criterion for ballotness.

Proposition 3.5 ([23, Corollary 4.5])**.** *A word w is ballot if and only if the i-th lattice walk of the subword w^i ends on the x axis, for all $i \in I$.*

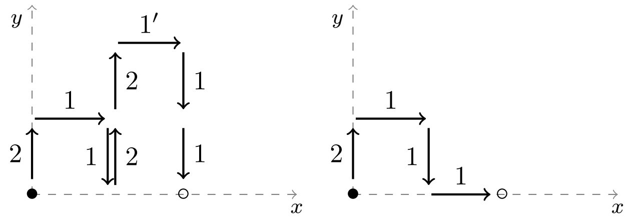


Figure 3.1: The lattice walks for w^1 and w^2, for $w = 3211221'11$.

Example 3.6. Let $w = 3211221'11$. To obtain the 1st and 2nd lattice walks of w, consider the subwords $w^1 = 211221'11$ and $w^2 = 3222$ (which corresponds to 2111, using the alphabet $\{1, 2\}'$). Replacing each letter accordingly, we obtain the lattice walks in Figure 3.1. Since both lattice walks end on the x axis, the word w is ballot.

If w is a word on the alphabet $\{1, 2\}'$ and $u = w_k w_{k+1} \cdots w_l$ is a substring of some representative of w, then u is called a *substring* of w. The coordinates (x, y) of the point of the 1-lattice walk before the start of u is called the *location* of u.

Definition 3.7 ([23, Definition 5.3])**.** A substring u is said to be a F_1-*critical substring* if any of these conditions on u and its location are satisfied (as well as the adequate transformations to apply), where ab^*c means any string of the form $ab \cdots bc$, including ac:

Type	Substring	Condition steps	Location	Transformation
1F	$u = 1(1')^*2'$	$\xrightarrow{1} \xrightarrow{1'} \uparrow_2$ $\downarrow_1 \xrightarrow{1'} \uparrow_2$	$y = 0$ $y = 1, x \geq 1$	$u \to 2'(1')^*2$
2F	$u = 1(2)^*1'$	$\xrightarrow{1} \uparrow_2 \xrightarrow{1'}$ $\downarrow_1 \uparrow_2 \xrightarrow{1'}$	$x = 0$ $x = 1, y \geq 1$	$u \to 2'(1')^*1$
3F	$u = 1$	$\xrightarrow{1}$	$y = 0$	$u \to 2$
4F	$u = 1'$	$\xrightarrow{1'}$	$x = 0$	$u \to 2'$
5F	$u = 1$ $u = 2'$	$\downarrow_1$ $\xleftarrow{2'}$	$x = 1, y \geq 1$	undefined

The *final* F_1-critical substring u of the word w is the F_1-critical substring u with the highest starting index, taking the longest in the case of a tie. If there is still a tie (due to different representatives), take any such u. Using this, we may now recall the definitions for the unprimed raising and lowering operators.

Definition 3.8 ([23, Definition 5.4]). Let w be a word. The word $F_1(w)$ is obtained by taking a representative of w containing a final F_1-critical substring and transforming it according to the previous table. If there is no F_1-critical substring or it is of type 5F, then put $F_1(w) = \varnothing$, and in this case, F_1 is said to be undefined on w.

Lemma 3.9 ([23, Proposition 5.14 (i)]). *Let w be a word on the alphabet $\{i, i+1\}'$ and let (x, y) be the endpoint of the i-th lattice walk of w. If $x = 0$, then $F_i(w)$ is undefined.*

The operators F_i are called the *unprimed raising operators*. The *unprimed lowering operators* E_i are defined on words by $E_i(w) := \mathsf{c}_n F_{n-i} \mathsf{c}_n(w)$, for $i \in I$. In particular, for $n = 2$, we have $E_1(w) = \mathsf{c}_2 F_1(\mathsf{c}_2(w))$, thus E_1 may be obtained in similar way using the following table of E_1-critical substrings:

Type	Substring	Condition steps	Location	Transformation
1E	$u = 2'(2)^*1$	$\uparrow_{2'}\ \uparrow_{2}\ \xrightarrow{1}$ $\xleftarrow{2'}\ \uparrow_{2}\ \xrightarrow{1}$	$x = 0$ $x = 1, y \geq 1$	$u \to 1(2)^*1'$
2E	$u = 2'(1')^*2$	$\uparrow_{2'}\ \xrightarrow{1'}\ \uparrow_{2}$ $\xleftarrow{2'}\ \xrightarrow{1'}\ \uparrow_{2}$	$y = 0$ $y = 1, x \geq 1$	$u \to 1(2)^*2'$
3E	$u = 2'$	$\uparrow_{2'}$	$x = 0$	$u \to 1'$
4E	$u = 2$	$\uparrow_{2}$	$y = 0$	$u \to 1$
5E	$u = 1$ $u = 2'$	$\downarrow_{1}$ $\xleftarrow{2'}$	$y = 1, x \geq 1$	undefined

These definitions can be extended to shifted tableaux. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, $F_i(T)$ is defined to be the tableau with the same shape as T, with (row) reading word $F_i(w(T))$, for $i \in I$. The definition of $E_i(T)$ is analogous. In both definitions, the row reading word of T may be replaced by the column reading word [23, Proposition 5.21]. These notions are well defined, since $E_i(T)$ and $F_i(T)$ are shifted semistandard tableaux, for all $i \in I$, whenever these operators are defined on T [23, Theorem 5.18]. Moreover, the primed and uprimed operators E'_i, E_i, F'_i and F_i commute whenever the compositions among them are defined, for each $i \in I$ [23, Proposition 5.36].

Example 3.10. Let $T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$. To compute E_2 and F_2, we consider the subword $w^2 = 323'32'$ of $w(T)$ consisting of the letters $\{2, 3\}'$, having the lattice walk on the right side of

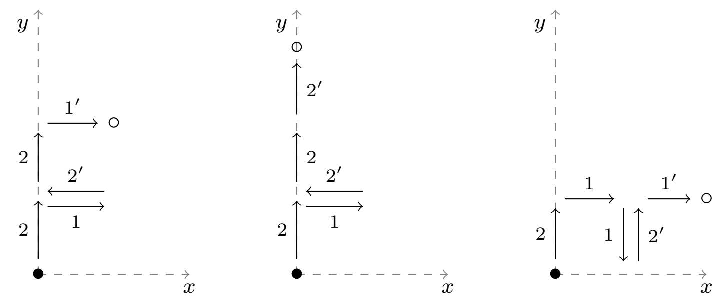


Figure 3.2: On the right, the lattice walk for $w = 212'21'$, ending at $(1, 2)$. Besides it there is the lattice walkes for $F_2(w)$, ending at $(0, 3)$, and $E_2(w)$, ending at $(2, 1)$.

Figure 3.2. Then, T has a final F_2-critical substring of type 4F, and a final E_2-critical substring of type 2E. Applying the corresponding substitutions, we obtain

$$F_2(T) = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 3' \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}, \qquad E_2(T) = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2' \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array}.$$

The primed and unprimed operators may be used to give an alternative formulation for ballot (and anti-ballot) words. Indeed, a word w on the alphabet $\{i, i+1\}'$ is ballot (respectively anti-ballot) if and only if $E_i(w) = E'_i(w) = \varnothing$ (respectively $F_i(w) = F'_i(w) = \varnothing$) [23, Proposition 5.17]. Hence, a word on the alphabet $[n]'$ is ballot (respectively anti-ballot) if and only if $E_i(w) = E'_i(w) = \varnothing$ (respectively $E_i(w) = E'_i(w) = \varnothing$) for all $i \in I$.

Whenever they are defined, $E_i(T)$ and $F_i(T)$ are shifted dual equivalent to T [23, Corollary 5.3]. Since this is also true for the primed operators, as the standardization is unchanged, then any two tableaux that differ by a sequence of any lowering or raising operators are shifted dual equivalent. Moreover, the unprimed operators are coplactic, whenever defined [23, Theorem 5.35].

A *highest weight element* (respectively *lowest weight element*) of $\mathsf{ShST}(\lambda/\mu, n)$ is a tableau T such that $E_i(T) = E'_i(T) = \varnothing$ (respectively $F_i(T) = F'_i(T) = \varnothing$), for any $i \in I$. Equivalently, T is a highest weight element (respectively lowest weight element) if and only if its reading word is ballot (respectively anti-ballot). For the next result, recall that Y_ν is the unique shifted tableau in canonical form of shape and weight ν.

Proposition 3.11 ([23, Proposition 6.4]). *The shifted tableau crystal* $\mathsf{ShST}(\nu, n)$ *has a unique highest weight element, which coincides with* Y_ν. *Then, every tableau* $\mathsf{ShST}(\nu, n)$ *may be obtained from* Y_ν *by a sequence of primed and unprimed lowering and raising operators.*

The set $\mathsf{ShST}(\lambda/\mu, n)$ is closed under the operators E_i, E'_i, F_i, F'_i, for $i \in I$. Moreover, we also have the *partial length functions* [21] given by

$$\varepsilon'_i(T) := \max\{k : E'^k_i(T) \neq \varnothing\} \qquad \widehat{\varepsilon}_i(T) := \max\{k : E^k_i(T) \neq \varnothing\}$$
$$\varphi'_i(T) := \max\{k : F'^k_i(T) \neq \varnothing\} \qquad \widehat{\varphi}_i(T) := \max\{k : F^k_i(T) \neq \varnothing\},$$

and *total length functions* $\varepsilon_i(T)$ and $\varphi_i(T)$, which are defined as the y-coordinate and x-coordinate, respectively, of the endpoints of the i-th lattice walk of T, for $i \in I$ [23, Section 5.1].

Example 3.12. Considering T as in the previous example, we have $F_2(T) \neq \varnothing$ and $E_2(T) \neq \varnothing$. The lattice walks for $F_2(T)$ and $E_2(T)$ (see Figure 3.2) have a final F_2-critical substring of type 5F and a final E_2-critical substring of type 5E, respectively. Then, $F_2^2(T) = E_2^2(T) = \varnothing$, and thus $\hat{\varphi}_2(T) = \hat{\varepsilon}_2(T) = 1$. Moreover, we have $F'_2(T) = \varnothing$, and

$$E_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array},$$

and so $\varphi'_2(T) = 0$ and $\varepsilon'_2(T) = 1$. Finally, the $2nd$ lattice walk of T ends at $(1, 2)$, and thus we have $\varphi_2(T) = 1$ and $\varepsilon_2(T) = 2$.

The set $\mathsf{ShST}(\lambda/\mu, n)$, together with primed and unprimed operators, partial and total length functions, and weight function, is called a *shifted tableau crystal*. We use the notation $\mathsf{ShST}(\lambda/\mu, n)$ for both the set and its structure of shifted tableau crystal. It may be regarded as a directed acyclic graph with weighted vertices, and i-coloured labelled double edges, solid ones being labelled with i ($x \xrightarrow{i} y$ if $F_i(x) = y$), and dashed ones with i' ($x \overset{i'}{\dashrightarrow} y$ if $F'_i(x) = y$), for $i \in I$ (see Figure 3.3). The connected components of $\mathsf{ShST}(\lambda/\mu, n)$ are the connected components of the underlying undirected and non-labelled graph. We also remark that the set $\mathsf{ShST}(\lambda/\mu, n)$ together with only the primed (or the unprimed) operators and with the same weight functions wt and total length functions φ_i, ε_i, satisfies the axioms of a type A Kashiwara crystal [23, Proposition 6.9]. However it is not a seminormal crystal, as the total length functions ε_i and φ_i measure the total distance of T to the ends of a string consisting of both F_i and F'_i operators, but not necessarily the distance to ends of a string obtained by only one of F_i or F'_i [23, Remark 1.3].

Example 3.13. For instance, considering T as in the previous examples and the string in Figure 3.4 consisting of both solid and dashed edges, then $\varepsilon_2(T) = 2$ (see Example 3.12) is the distance to the left end of that string. But considering the string of solid edges containing T, then the distance to the left end is 1.

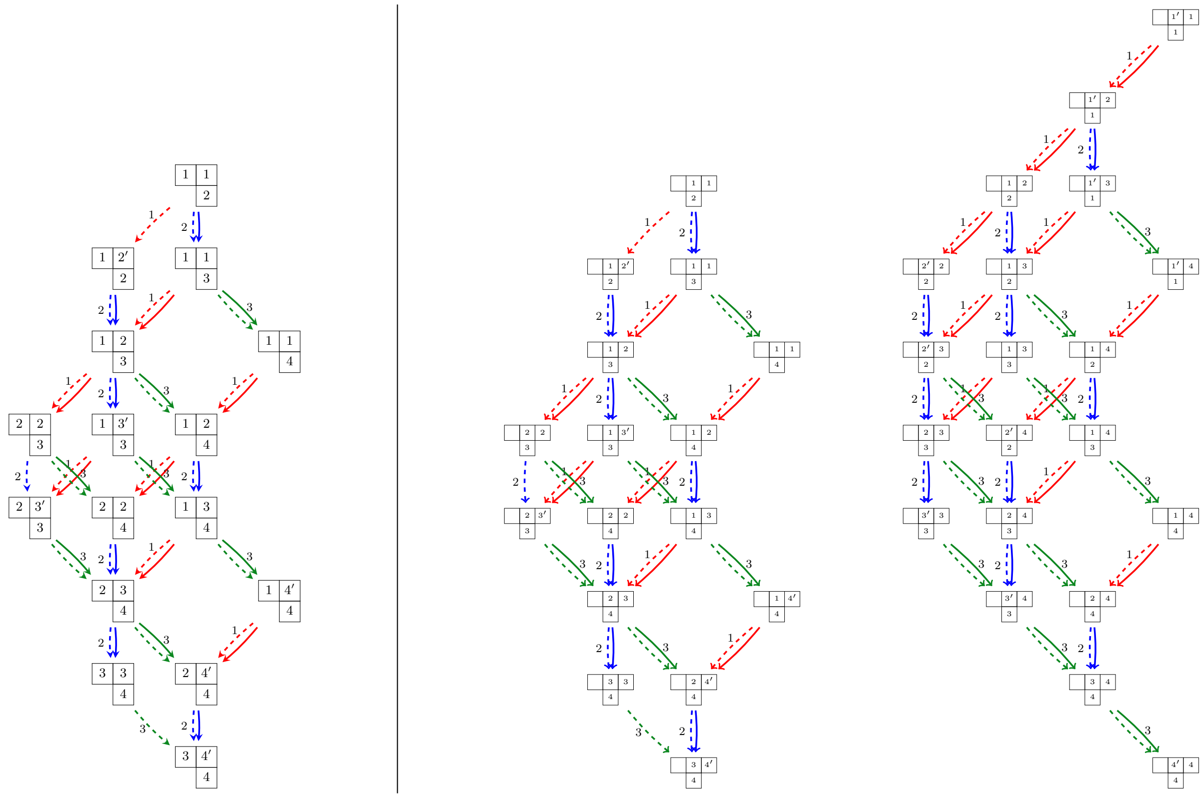


Figure 3.3: On the left, the shifted tableau crystal graph $\mathsf{ShST}(\nu, 4)$, for $\nu = (2, 1)$. On the right, the shifted tableau crystal graph $\mathsf{ShST}(\lambda/\mu, 4)$, for $\lambda = (3, 1)$ and $\mu = (1)$. The operators F_1, F_1' are in red, the F_2, F_2' in blue, and the F_3, F_3' in green. Note that $\mathsf{ShST}(\lambda/\mu, 4)$ has two connected components, one of them being isomorphic to $\mathsf{ShST}(\nu, 4)$.

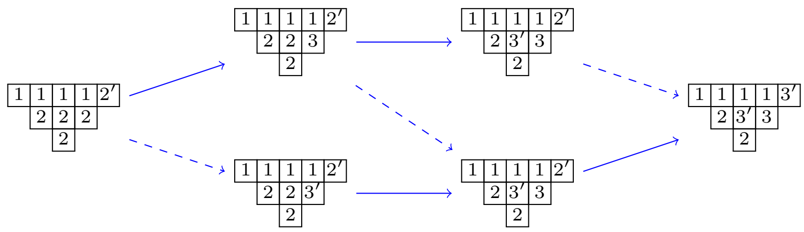


Figure 3.4: A separated 2-string

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Proposition 3.14 ([23, Corollary 6.5])**.** *Each connected component of* $\mathsf{ShST}(\lambda/\mu, n)$ *has a unique highest weight element* T^{high}*, which is a LRS tableau, and is isomorphic, as a weighted edge-labelled graph, to the shifted tableau crystal* $\mathsf{ShST}(\nu, n)$*, where* $\nu = \mathsf{wt}(T^{\mathsf{high}})$.

Proposition 3.15 ([23, Corollary 6.6])**.** *Each connected component of* $\mathsf{ShST}(\lambda/\mu, n)$ *forms a shifted dual equivalence class.*

Therefore, by decomposing $\mathsf{ShST}(\lambda/\mu, n)$ into connected components, we have the crystal graph isomorphism

$$\mathsf{ShST}(\lambda/\mu, n) \simeq \bigsqcup_{\nu} \mathsf{ShST}(\nu, n)^{f^{\lambda}_{\mu\nu}}, \tag{3.1}$$

where $f^{\lambda}_{\mu\nu}$ is the shifted Littlewood–Richardson coefficient.

Recall that given a infinite set of variables $x = \{x_1, x_2, \ldots\}$, we denote $x^{\alpha} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, for any vector $\alpha = (\alpha_1, \alpha_2, \ldots)$, such that $\alpha_k = 0$ for all $k > N$, for some $N \in \mathbb{N}$. In particular, we associate to a string w in $[n]'$, a monomial $x^{\mathsf{wt}(w)} = x_1^{wt_1} \cdots x_n^{wt_n}$, where $\mathsf{wt}(w) = (wt_1, \ldots, wt_n)$. The *character* of a shifted tableau T with word $\hat{w}$ is given by

$$\chi(T) := \sum_{w \in \hat{w}} x^{\mathsf{wt}(w)}$$

where the sum is all over the representatives of $\hat{w}$. The character of a collection of tableaux is defined as the sum of the tableaux [23, Definition 7.1]. The following result is a direct consequence of the definition of character, the definition of Schur Q-functions (Definition 1.2) and Proposition 3.11.

Proposition 3.16 ([23, Proposition 7.4]). *The character of* $\mathsf{ShST}(\lambda/\mu)$ *is the Schur* Q*-function* $Q_{\lambda/\mu}(x)$.

Consequently, taking the character in (3.1) yields the well known decomposition (1.3) of skew Schur Q-function $Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_{\nu}$ (for details, see [23, Section 7]).

3.1.1 Decomposition into i-strings

Given $i \in I$, we may consider an equivalence relation on $\mathsf{ShST}(\lambda/\mu, n)$, as a set, in which $X \sim_i Y$ if X and Y are related by any sequence (including the empty sequence) of i-labelled crystal operators F_i, F'_i, E_i or E'_i. The equivalence classes are called the *i-strings*. These are the underlying subsets of the $\{i', i\}$-connected components of the crystal graph $\mathsf{ShST}(\lambda/\mu, n)$, which are obtained by removing the edges not coloured in $\{i', i\}$.

Hence, $\mathsf{ShST}(\lambda/\mu, n)$ may be partitioned, as a set, into *i-strings* (see Figure 3.6). The i-strings have two possible arrangements [21, Section 3.1] [23, Section 8], as shown in Figure 3.5. A string consisting of two i-labelled chains of equal lenght, connected by i'-labelled edges is called a *separated i-string*. The smallest separated string is formed by two vertices connected by a i'-labelled edge. A string formed by a double chain of both i- and i'-labelled strings

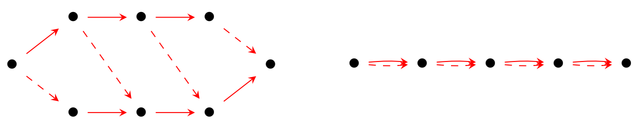


Figure 3.5: A separated i-string (left) and a collapsed i-string (right).

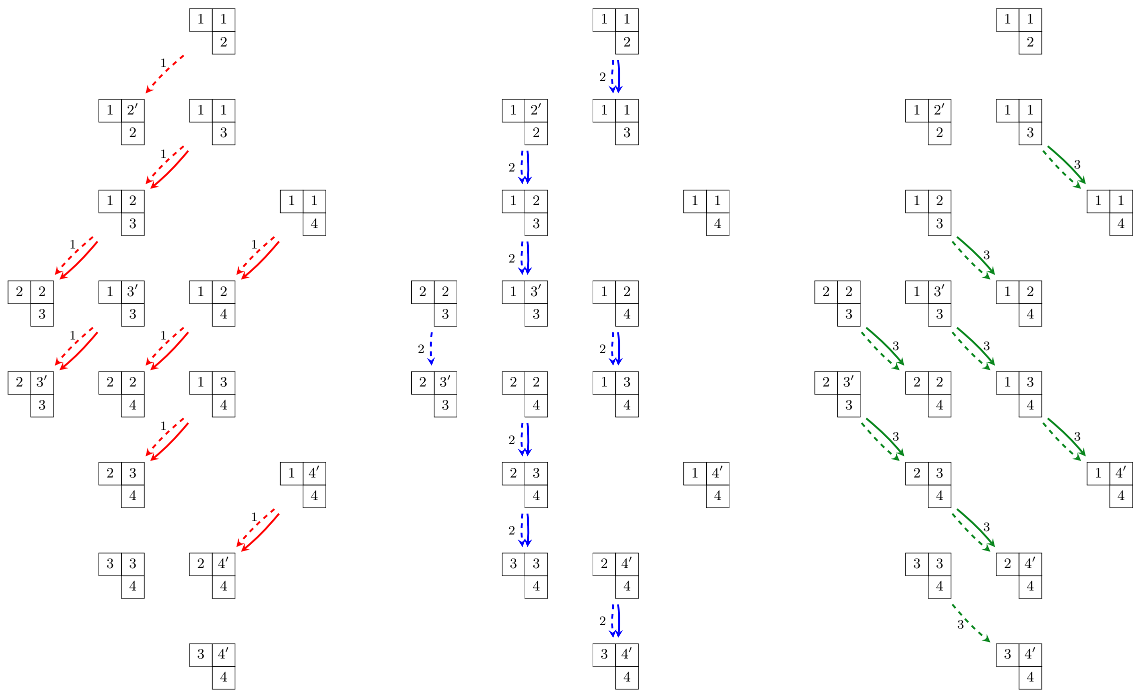


Figure 3.6: The crystal graph $\mathsf{ShST}(\nu, 4)$, for $\nu = (2, 1)$, partitioned into 1-strings (left), 2-strings (middle) and 3-strings (right).

is called a *collapsed i-string*. A single vertex (without edges) is considered as the smallest collapsed string.

The following propositions are intended to detail the possible arrangements for an i-string. This corresponds to the details of the axiom (B1) in [21]. Each i-string has a unique highest weight element, which is a shifted tableau T in that i-string such that $E_i(T) = E'_i(T) = \varnothing$, and a unique lowest weight element, which is defined similarly. In particular, the next result provides a condition for an i-string to be separated or collapsed in terms of its highest weight element.

Proposition 3.17. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $i \in I$. Suppose that T is the highest weight element of its i-string. Then, the i-string is collapsed if and only if* $\mathsf{wt}(T)_{i+1} = 0$.

Proof. To simplify the notation, we consider the alphabet $\{1,2\}'$. Suppose $\mathsf{wt}(T)_2 = 0$. If $\mathsf{wt}(T)_1 = 0$, then $F_1(T) = F_1'(T) = \varnothing$ and the 1-string is a trivial collapsed string. Thus, we may assume, without loss of generality, that $\mathsf{wt}(T)_1 > 0$. Since $\mathsf{wt}(T)_2 = 0$, there are no occurrences of $2'$ to the right of the last 1 in the word of T. Therefore, $F_1'(T)$ is defined and obtained by changing the last 1 into $2'$. Moreover, since $\mathsf{wt}(F_1'(T))_2 = 1$, this $2'$ is identified with 2 in the canonical form. We claim that $F_1(T)$ is defined, i.e., there exists a final F_1-critical substring that is not of type 5F. Since $\mathsf{wt}(T)_2 = 0$, the location of a possible substring is $y = 0$, excluding the type 5F, and since $\mathsf{wt}(T)_1 > 1$, there is necessarily a substring of type 2F (with $x = 0$) or 3F (4F would be the case where either $w(T) = 1'$, which is equal to 1 in canonical form, or $w(T) = 1'(\mathbf{1})^*$, and again the first $1'$ would be identified with 1, and the substring would be of type 2F and 3F). If $w(T)$ has a final F_1-critical substring of type 2F, then F_1 changes the substring $11'$ into $2'1$, which is identified with 21 since this is the only occurrence of 2. If it is of type 3F, then F_1 changes the substring 1 into 2. In both cases, F_1 changes the last 1 into 2, coinciding with F_1'. Since the operators commute [23, Proposition 5.36], then $F_1(T) = F_1'(T)$ implies that $F_1^k(T) = F_1'^k(T)$, for any $k \geq 1$, which means that the i-string is collapsed.

Now suppose that the i-string is collapsed. In particular, $F_1(T) = F_1'(T)$. If both F_1 and F_1' are undefined on T, then $\mathsf{wt}(T)_1 = 0$, which is a trivial case. Thus, we may assume that $F_1(T) = F_1'(T) \neq \varnothing$. Since $F_1'(T) \neq \varnothing$, we have $\mathsf{wt}(T)_1 > 0$ and there are no occurrences of $2'$ to the right of the last 1 in T. Suppose that $\mathsf{wt}(T)_2 > 0$. We have the following cases:

Case 1. We assume there are no occurrences of 2 after the mentioned 1. Since we are assuming that $\mathsf{wt}(T)_2 > 0$, the occurrences of $\mathbf{2}$ must be to the left of the last 1. Moreover, $F_1(T)$ must coincide with $F_1'(T)$, so we either have:

- F_1 changes 1 into 2, which implies that the $2'$ resulting from F_1' must be identified with 2 in canonical form. For this to happen, this 2 must be the only occurrence of $\mathbf{2}$ in $F_1(T)$. Hence, $\mathsf{wt}(F_1(T))_2 = 1$ and necessarily $\mathsf{wt}(T)_2 = 0$, contradicting the hypothesis.

- F_1 changes 1 into $2'$. For this to happen, $w(T)$ must have a final F_1-critical substring of type 1F or 2F. If it is 1F, there would be some $2'$ to the right of the last 1 and F_1' would not be defined. If it is 2F, and since there are no occurrences of $\mathbf{2}$ to the right of the last 1, by assumption, it must be the case $11' \longmapsto 2'1$. So, since $F_1(T) = F_1'(T)$, we have that the canonical form of $2'1'$ is $2'1$. For 1 and $1'$ to be identified, there must be no occurrences of

$\mathbf{1}$ to the right of the last 1. By hypothesis, $E_1(T) = E_1'(T) = \varnothing$. Clearly E_1' is undefined since the last $2'$ is not right to the last 1. Hence, it must be the case where T has no final E_1-critical substring or has some of type 5E. If it is 5E, then it is either $2'$ at $x \geq 1, y = 1$. Since $y = 1$, there must be some $\mathbf{2}$ before the $2'$ of the substring, and since $x \geq 1$, there must be some $\mathbf{1}$ after it, which contradicts the non-existence of $\mathbf{1}$ to the right of the 1 to be changed. Therefore, it must be the case where there is no final E_1-critical substring. Since we are assuming that $\mathsf{wt}(T)_2 > 0$, some $\mathbf{2}$ must appear before the last 1, yielding at least some final E_1-critical substring of type 3E or 4E, which is a contradiction.

Case 2. Assume there are some occurrences of 2 after the last 1. Then, we must have the substring 12 (the 1 appearing is the one to be changed) at either one of these locations:

- At $x \geq 0$ and $y = 0$. In this case E_1 would be defined, being type 4E.
- At $x = 0$ and $y > 0$. But then, $y > 0$ implies that there are some $\mathbf{2}$ before this string, placing it at location $y = 0$ and yielding a 3E or 4E type.
- At $y = 1$. Then, the 2 is located at $y = 0$ yielding a 4E type.
- At $y > 1$. In this case necessarily $x > 1$, otherwise this would be a final F_1-critical substring of type 5F and $F_1(T) = \varnothing$. But then, the location obtained is not a valid one for this string to be the final critical substring. Therefore, the 1 to be changed by F_1 is not the same as the one changed by F_1', which contradicts their equality.

□

The next lemmas concern the total length functions, which are the total distances from a vertex to the highest and lowest weight vertices of its i-string.

Lemma 3.18. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $i \in I$. Then,*

$$\varepsilon_i(T) = \begin{cases} \widehat{\varepsilon}_i(T) = \varepsilon_i'(T) & \text{if } T \text{ is in a collapsed } i\text{-string} \\ \widehat{\varepsilon}_i(T) + \varepsilon_i'(T) & \text{if } T \text{ is in a separated } i\text{-string.} \end{cases}$$

The result is also valid for φ_i with the adequate changes.

Proof. Suppose that T is in a collapsed i-string. Then, by Proposition 3.14, that collapsed string has a highest weight element T_0, and so $T_0 = E_i^k(T)$, for some $k \geq 0$. Since T_0 is a highest weight, then $E_i(T_0) = \varnothing$, hence $E_i^{k+1}(T) = \varnothing$. Consequently, $\widehat{\varepsilon}(T) = k$. On the other hand, T_0 is a LRS tableau (for the alphabet $\{i', i, (i+1)', i+1\}$), thus, by Proposition 3.5, the endpoint of the i-th lattice walk of its word has the y-coordinate equal to 0. The operator F_i shifts the endpoint of the i-th lattice walk by $(-1, 1)$ [23, Corollary 5.12]. Since $T_0 = E_i^k(T)$, then $T = F_i^k(T_0)$, and so the i-th lattice walk of T has the y-coordinate equal to k. Then, $\varepsilon_i(T) = k = \widehat{\varepsilon}_i(T)$.

Now suppose that T is in a separated i-string. This i-string has a highest weight element T_0, which is a LRS tableau for the mentioned alphabet. Thus, the endpoint of the i-th lattice walk of its word has the y-coordinate equal to zero. Then, we have two cases.

- Suppose that T is such that $E_i'(T) \neq \varnothing$. Let $T_1 := E_i'(T)$. By definition of E_i', T_1 is obtained from T by replacing the last $(i+1)'$ that was right to the last i with i. Hence, $E_i'(T_1) = E_i'^2(T) = \varnothing$ and so
$$\varepsilon_i'(T) = 1. \tag{3.2}$$
Since T_0 is the highest weight element, we have that $E_i^k E_i'(T) = T_0$, for some $k \geq 0$. This is equivalent to $F_i' F_i^k(T_0) = T$, and since both F_i and F_i' shift the endpoint of the i-th lattice walk by $(-1, 1)$ [23, Propostion 4.9], the y-coordinate of the i-th lattice walk of the word of T must be equal to $k+1$. Hence,
$$\varepsilon_i(T) = k + 1. \tag{3.3}$$
Since the operators E_i and E_i' commute when defined, we have $T_0 = E_i' E_i^k(T)$, and so $E_i^k(T) = F_i'(T_0) \neq \varnothing$ (recall that the shortest i-string is one with a i'-labelled edge). Thus, $E_i^{k+1}(T) = E_i F_i'(T) = \varnothing$, and we have
$$\widehat{\varepsilon}_i(T) = k. \tag{3.4}$$
By (3.2), (3.3) and (3.4), we have
$$\varepsilon_i(T) = \varepsilon_i'(T) + \widehat{\varepsilon}_i(T).$$

- Suppose now that T is such that $E_i'(T) = \varnothing$. Then, $\varepsilon_i'(T) = 0$. As in the previous case, there exists a highest weight element T_0 in this i-string, and the endpoint of the i-th lattice walk of its words has y-coordinate equal to zero. If $T = T_0$, then $\widehat{\varepsilon}_i(T) = 0$ and the proof is done. Otherwise, there exists $k > 0$ such that $T_0 = E_i^k(T)$, and so, $F_i^k(T_0) = T$.

Consequently, the endpoint of the i-th lattice walk of the word of T has its y-coordinate equal to k. Hence,

$$\varepsilon_i(T) = k. \tag{3.5}$$

Moreover, $E_i^{k+1}(T) = E_i(T_0) = \varnothing$, as T_0 is a highest weight element. So,

$$\widehat{\varepsilon}_i(T) = k. \tag{3.6}$$

Hence, by (3.5) and (3.6), and since $\varepsilon'_i(T) = 0$, we have $\varepsilon_i(T) = \varepsilon'_i(T) + \widehat{\varepsilon}_i(T)$.

$\square$

Lemma 3.19. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and suppose its i-string has highest weight element T_0^{high} and lowest weight element T_0^{low}, and that $T \neq T_0^{\text{high}}, T_0^{\text{low}}$. The following holds:*

1. *If the i-string is separated, then*

$$\begin{aligned} T_0^{\text{low}} &= F_i'^a F_i^b(T) = F_i^b F_i'^a(T) = F_i^{b-k} F_i'^a F_i^k(T), \textit{ for some } k \geq 0 \\ T_0^{\text{high}} &= E_i'^c E_i^d(T) = E_i^d E_i'^c(T) = E_i^{d-k} E_i'^c E_i^k, \textit{ for some } k \geq 0 \end{aligned}$$

   with $a = \varphi'_i(T) \in \{0,1\}$, $b = \hat{\varphi}_i(T) \geq 0$, $c = \varepsilon'_i(T) \in \{0,1\}$, and $d = \hat{\varepsilon}_i(T) \geq 0$.

2. *If the i-string is collapsed, then*

$$\begin{aligned} T_0^{\text{low}} &= F_i^a(T) = F_i'^a(T) \\ T_0^{\text{high}} &= E_i^b(T) = E_i'^b(T) \end{aligned}$$

   with $a = \varphi_i(T)$ and $b = \varepsilon_i(T)$.

Proof. We prove the case for the separated i-string and for the raising operators. For collapsed i-string, the proof is similar, noting that by Lemma 3.18 we have $\varepsilon_i(T) = \widehat{\varepsilon}_i(T) = \varepsilon'_i(T)$. Let T be in separated i-string. We have $E'_i(E_i'^c E_i^d(T)) = E_i'^{c+1} E_i^d(T) = E_i^d E_i'^{c+1}(T)$, since these operators commute. By definition of $c = \varepsilon'_i(T)$, we have $E_i'^{c+1}(T) = \varnothing$. Hence $E'_i(E_i'^c E_i^d(T)) = E_i^d(\varnothing) = \varnothing$. On the other hand, we have $E_i(E_i'^c E_i^d(T)) = E_i'^c E_i^{d+1}(T)$. By definition of $d = \widehat{\varepsilon}_i(T)$, $E_i^{d+1} = \varnothing$. Consequently, $E_i(E_i'^c E_i^d(T)) = E_i'^c(\varnothing) = \varnothing$. Thus, $E_i'^c E_i^d(T)$ must be the highest weight element of this i-string. $\square$

3.2 The Schützenberger–Lusztig involution

The Schützenberger involution, or Lusztig involution, is defined on the shifted tableau crystal [21, Section 2.3.1] in the same fashion as for type A Young tableau crystal. Similarly, it is realized by shifted evacuation, for tableaux of straight shape, and through shifted reversal otherwise. For each $i \in I = [n-1]$, we define the shifted crystal reflection operator σ_i, using the primed and unprimed crystal operators E_i, E_i', F_i and F_i'. We also show in Example 3.31 that they do not need to satisfy the braid relations and, therefore, do not yield a natural action of $\mathfrak{S}_n$ on this crystal. Throughout this section ν will denote a strict partition.

Proposition 3.20. *There exists a unique map of sets $\eta : \mathsf{ShST}(\nu, n) \longrightarrow \mathsf{ShST}(\nu, n)$ that satisfies the following, for all $T \in \mathsf{ShST}(\nu, n)$ and for all $i \in I$:*

1. $E_i'\eta(T) = \eta F'_{\theta_{1,n-1}(i)}(T)$.

2. $E_i\eta(T) = \eta F_{\theta_{1,n-1}(i)}(T)$.

3. $F_i'\eta(T) = \eta E'_{\theta_{1,n-1}(i)}(T)$.

4. $F_i\eta(T) = \eta E_{\theta_{1,n-1}(i)}(T)$.

5. $\mathsf{wt}(\eta(T)) = \theta_{1,n}(\mathsf{wt}(T))$.

This map may is also defined on $\mathsf{ShST}(\lambda/\mu, n)$, using the coplacity of the crystal operators, by applying η to its connected components. Moreover, it coincides with the evacuation evac in $\mathsf{ShST}(\nu, n)$, and with the reversal e on the connected components of $\mathsf{ShST}(\lambda/\mu, n)$.

The map η is called the *Schützenberger or Lusztig involution*, and we use the notation η for both straight-shaped and skew tableaux. The map η is indeed an involution on the set of vertices of $\mathsf{ShST}(\nu, n)$, that reverses all arrows and indices. In particular, it sends the highest weight element to the lowest, i.e. $\eta(T^{\mathsf{high}}) = T^{\mathsf{low}}$, and vice versa.

The Schützenberger involution is coplactic and a weight-reversing, shape-preserving involution. Note that the operator c_n also acts on $\mathsf{ShST}(\nu, n)$ by reversing arrows and indices, however it does not preserve the shape, although the resulting crystal $\mathsf{c}_n(\mathsf{ShST}(\nu, n))$ is in a bijective correspondence, as sets, with $\mathsf{ShST}(\nu, n)$ and with $\mathsf{evac}(\mathsf{ShST}(\nu, n))$, being isomorphic as crystal graph to the latter (see Figure 3.7).

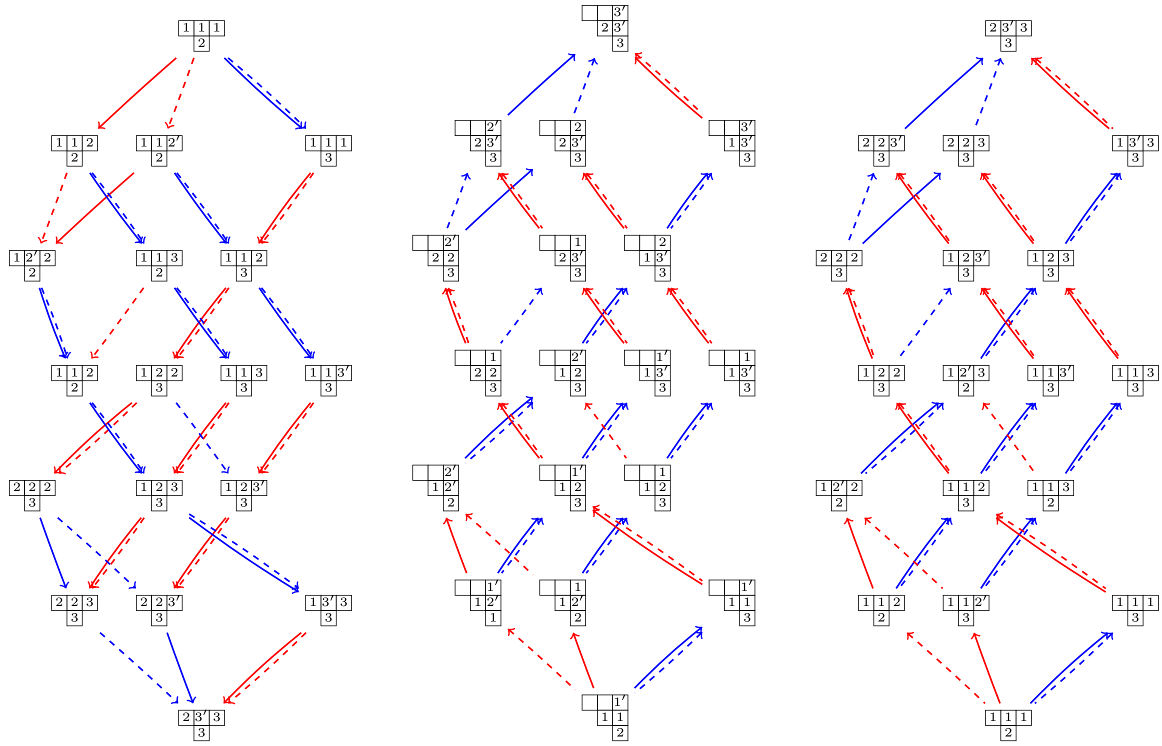


Figure 3.7: The shifted tableau crystal $\mathsf{ShST}(\nu, 3)$, for $\nu = (3, 1)$, on the left [23, Figure 6.1]. In the middle, $\mathsf{c}_3(\mathsf{ShST}(\nu, 3))$, which is a connected component of $\mathsf{ShST}(\lambda/\mu, 3)$, for $\lambda = (3, 2, 1)$ and $\mu = (2)$. On the right, $\mathsf{evac}(\mathsf{ShST}(\nu, 3)) = \mathsf{rect}(\mathsf{c}_n(\mathsf{ShST}(\nu, 3)))$.

Proof of Proposition 3.20. We prove that the evacuation evac satisfies the aforementioned conditions. Let $T \in \mathsf{ShST}(\nu, n)$ and let $i \in I$. By definition, $\mathsf{wt}(\mathsf{evac}(T)) = \theta_{1,n}(\mathsf{wt}(T))$. And since evac is an involution, it suffices to prove the first two conditions. By definition of the primed operators, $\mathsf{std}(E'_i\mathsf{evac}(T)) = \mathsf{std}(\mathsf{evac}(T))$. Therefore, since standardization commutes with evacuation, we have

$$\mathsf{std}(\mathsf{evac}E'_i\mathsf{evac}(T)) = \mathsf{std}(\mathsf{evac}^2(T)) = \mathsf{std}(T).$$

Moreover, we have $\alpha_{\theta_{1,n-1}(i)} = -\theta_{1,n}(\alpha_i)$, and thus

$$\begin{aligned}
\mathsf{wt}(\mathsf{evac}E'_i\mathsf{evac}(T)) &= \theta_{1,n}(\mathsf{wt}(E'_i\mathsf{evac}(T))) \\
&= \theta_{1,n}(\mathsf{wt}(\mathsf{evac}(T)) + \alpha_i) \\
&= \theta_{1,n}(\theta_{1,n}(\mathsf{wt}(T)) - \theta_{1,n}(\alpha_{\theta_{1,n-1}(i)})) \\
&= \theta_{1,n}^2(\mathsf{wt}(T) - \alpha_{\theta_{1,n-1}(i)}) \\
&= \mathsf{wt}(T) - \alpha_{\theta_{1,n-1}(i)}.
\end{aligned}$$

Hence, by the definition of F'_{n-i}, we have $\mathsf{evac}E'_i\mathsf{evac}(T) = F'_{n-i}(T)$ and consequently,

$$E'_i\mathsf{evac}(T) = \mathsf{evac}F'_{n-i}(T).$$

To prove that $E_i\mathsf{evac}(T) = \mathsf{evac}F_{\theta_{1,n-1}(i)}(T)$ we note that $E_i\mathsf{evac}(T)$ and $F_{\theta_{1,n-1}(i)}(T)$ are in the same connected component of $\mathsf{ShST}(\nu, n)$, hence they are shifted dual equivalent, due to Proposition 3.15. Thus, it remains to show that $E_i\mathsf{evac}(T)$ and $\mathsf{c}_n(F_{\theta_{1,n-1}(i)}(T))$ are shifted Knuth equivalent. We have that $w(\mathsf{evac}(T)) \equiv_k \mathsf{c}_n(T)$ and since E_i is coplactic, we have $E_i(w(\mathsf{evac}(T))) \equiv_k E_i(\mathsf{c}_n(T))$. Then,

$$\begin{aligned} w(E_i(\mathsf{evac}(T))) &= E_i(w(\mathsf{evac}(T))) \\ &\equiv_k E_i(w(\mathsf{c}_n(T))) \\ &= \mathsf{c}_n F_{\theta_{1,n-1}(i)}\mathsf{c}_n(w(\mathsf{c}_n(T))). \end{aligned} \tag{3.7}$$

By (2.4), we have $\mathsf{c}_n w(\mathsf{c}_n(T)) = w_{\mathsf{col}}(\mathsf{c}_n^2(T)) = w_{\mathsf{col}}(T)$. Moreover, the row and column words of a shifted semistandard tableau are shifted Knuth equivalent (see, for instance, [72, Lemma 6.4.12]). Thus, since c_n and $F_{\theta_{1,n-1}(i)}$ are coplactic,

$$\begin{aligned} \mathsf{c}_n F_{\theta_{1,n-1}(i)}\mathsf{c}_n(w(\mathsf{c}_n(T))) &= \mathsf{c}_n F_{\theta_{1,n-1}(i)}(w_{\mathsf{col}}(T)) \\ &\equiv_k \mathsf{c}_n F_{\theta_{1,n-1}(i)}(w(T)) \\ &= \mathsf{c}_n w(F_{\theta_{1,n-1}(i)}(T)). \end{aligned} \tag{3.8}$$

Finally, by (2.4), we have $\mathsf{c}_n w(F_{\theta_{1,n-1}(i)}(T)) = w_{\mathsf{col}}(\mathsf{c}_n F_{\theta_{1,n-1}(i)}(T)) \equiv_k w(\mathsf{c}_n F_{\theta_{1,n-1}(i)}(T))$. Thus, from (3.7) and (3.8) we have

$$w(E_i\mathsf{evac}(T)) \equiv_k w(\mathsf{c}_n F_{\theta_{1,n-1}(i)}(T))$$

and, consequently, $\mathsf{evac}(F_{\theta_{1,n-1}(i)}(T)) = E_i(\mathsf{evac}(T))$. For the uniqueness part, suppose that there is another involution ξ on $\mathsf{ShST}(\nu, n)$ satisfying the previous properties and let Y_ν be the highest weight element of $\mathsf{ShST}(\nu, n)$. By Proposition 3.14, we have $T = H_{i_1} \cdots H_{i_k}(Y_\nu)$, where $H_i \in \{F'_i, F_i, E'_i, E_i\}$, with $i_k \in I$. Moreover, let $\tilde{H}_i$ be E'_i (respectively, E_i, F'_i and F_i) if H_i is F'_i (respectively F_i, E'_i and E_i). Then,

$$\begin{aligned} \xi(T) &= \xi H_{i_1} \cdots H_{i_k}(Y_\nu) \\ &= \tilde{H}_{\theta_{1,n-1}(i_1)} \cdots \tilde{H}_{\theta_{1,n-1}(i_k)}\xi(Y_\nu) \\ &= \tilde{H}_{\theta_{1,n-1}(i_1)} \cdots \tilde{H}_{\theta_{1,n-1}(i_k)}\mathsf{evac}(Y_\nu) \\ &= \mathsf{evac}H_{i_1} \cdots H_{i_k}(Y_\nu) \\ &= \mathsf{evac}(T). \end{aligned}$$

□

As a direct consequence of Proposition 3.11 and Proposition 3.20, we have that the evacuation of a Yamanouchi tableau Y_ν is the lowest weight of $\mathsf{ShST}(\nu, n)$, i.e., F_i' and F_i are undefined on $\mathsf{evac}(Y_\nu)$, for all $i \leq I$.

3.2.1 The partial Schützenberger involutions

Given $1 \leq i < j \leq n$, let $[i,j] := \{i < \cdots < j\}$ and $[i,j]' := \{i' < i < \cdots < j' < j\}$. We may define an equivalence relation in $\mathsf{ShST}(\lambda/\mu, n)$, in which $X \sim_{i,j} Y$ if X and Y are related by any sequence of $[i, j-1]$-labelled crystal operators. The equivalence classes are the underlying subsets of the $[i, j-1]'$-connected components of the crystal graph $\mathsf{ShST}(\lambda/\mu, n)$, which are obtained by removing the edges not coloured in $[i, j-1]'$. We denote by $\mathcal{B}_{i,j}$ the collection of these equivalence classes. Since, in particular, $X \sim_{i,i+1} Y$ if and only if $X \sim_i Y$, $\mathcal{B}_{i,i+1}$ is the collection of all the i-strings of $\mathsf{ShST}(\lambda/\mu, n)$. Moreover, we have $\mathcal{B}_{1,n} = \mathsf{ShST}(\lambda/\mu, n)$.

A *highest weight element* of $\mathcal{B}_{i,j}$ is a shifted tableau $T \in \mathsf{ShST}(\lambda/\mu, n)$ such that E_k' and E_k are undefined on T, for all $k \in [i, j-1]$. A *lowest weight element* is defined analogously, using F_k' and F_k.

Lemma 3.21. *Let $1 \leq i < j \leq n$. Each connected component of $\mathcal{B}_{i,j}$ is isomorphic, as a weighted edge-labelled graph, to $\mathsf{ShST}(\nu, n)$, for some ν. In particular, each connected component has unique highest and lowest weight elements.*

Proof. Let $\mathcal{C}$ be a connected component of $\mathcal{B}_{i,j}$ and let $T \in \mathcal{C}$. Then, T may be regarded as a shifted tableau of skew shape λ_0/μ_0, for strict partitions such that $\mu \subseteq \mu_0 \subseteq \lambda_0 \subseteq \lambda$, where μ_0/μ corresponds to the boxes of T consisting of the letters $[1, i-1]'$ and λ/λ_0 corresponding to the boxes filled in $[j+1, n]'$. These indeed define the same shapes within each connected component, since the operators corresponding to the edges coloured in $[i, j-1]$ leave the shapes unchanged. Then, we may relabel the filling of T by replacing each letter $\mathbf{k} \in [i,j]'$ with $\rho_{i,j}(\mathbf{k})$, where $\rho_{i,j}$ is defined as

$$\begin{aligned} \rho_{i,j} \colon [i,j]' &\to [1, j-i+1]' \\ k &\mapsto k - i + 1 \\ k' &\mapsto (k-i+1)' \end{aligned} \tag{3.9}$$

obtaining a tableau $\rho_{i,j}(T)$ of shape λ_0/μ_0 in the alphabet $[1, j-i+1]'$. Thus, $\mathcal{C}$ may be identified with $\mathsf{ShST}(\lambda_0/\mu_0, j-i+1)$, which is isomorphic via rectification to $\mathsf{ShST}(\nu_0, n)$, where $\nu_0 = (\mathsf{wt}(T)_i, \dots, \mathsf{wt}(T)_j)$. Thus, by Proposition 3.14, $\mathcal{C}$ has unique highest and lowest weight elements.

□

Given $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $1 \leq i < j \leq n$, we denote $T^{i,j} := T^i \sqcup \cdots \sqcup T^j$, the tableau obtained from T considering only the entries in $[i,j]'$. Let $\mathcal{C}$ be the (unique) connected component of $\mathcal{B}_{i,j}$ containing T. By Lemma 3.21, $\mathcal{C}$ is identified with a shifted tableau crystal $\mathsf{ShST}(\lambda_0/\mu_0, j-i+1)$, for some $\mu \subseteq \mu_0 \subseteq \lambda_0 \subseteq \lambda$. Then, when we write $\eta(T^{i,j})$, this means that we first apply the Schützenberger involution η defined in in $\mathsf{ShST}(\lambda_0/\mu_0, j-i+1)$ to $\rho_{i,j}(T^{i,j})$ (as defined by (3.9)) and then apply $\rho_{i,j}^{-1}$ to the obtained tableau. Thus, we have the following definition.

Definition 3.22. Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $1 \leq i < j \leq n$. The *partial Schützenberger involution* restricted to the interval $[i,j]'$ is the map $\eta_{i,j} : \mathsf{ShST}(\lambda/\mu, n) \longrightarrow \mathsf{ShST}(\lambda/\mu, n)$ defined as

$$\eta_{i,j}(T) := T^{1,i-1} \sqcup \eta(T^{i,j}) \sqcup T^{j+1,n}.$$

In particular, we have $\eta_{1,n}(T) = \eta(T)$.

Lemma 3.23. *Let $1 \leq i < j < n$ and let $k \in [i,j]$. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, we have, whenever the operators are defined:*

1. $E'_k \eta_{i,j}(T) = \eta_{i,j} F'_{\theta_{i,j-1}(k)}(T)$.
2. $E_k \eta_{i,j}(T) = \eta_{i,j} F_{\theta_{i,j-1}(k)}(T)$.
3. $F'_k \eta_{i,j}(T) = \eta_{i,j} E'_{\theta_{i,j-1}(k)}(T)$.
4. $F_k \eta_{i,j}(T) = \eta_{i,j} E_{\theta_{i,j-1}(k)}(T)$.
5. $\mathsf{wt}(\eta_{i,j}(T)) = \theta_{i,j}(\mathsf{wt}(T))$.

Proof. If $[i,j] = [1,n]$, then this is Proposition 3.20. Otherwise, Lemma 3.21 and Definition 3.22 ensure that $\eta_{i,j}$ may be regarded as η defined in $\mathsf{ShST}(\lambda_0/\mu, j-i+1)$, and the result follows from Proposition 3.20. □

As a direct consequence of Lemma 3.23 and the fact that η is a involution, we have the next result.

Corollary 3.24. *Let $1 \leq i < j < n$ and let $k \in [i, j-1]$. Then, the operators $\eta_{i,j}$ satisfy the following:*

1. $\eta_{i,j}^2 = 1$.

2. *$\eta_{i,j}$ takes each connected component of $\mathcal{B}_{i,j}$ to itself.*

3. *$\eta_{i,j}$ takes each k-string to a $\hat{k}$-string, with $\hat{k} = \theta_{i,j-1}(k)$.*

Proposition 3.25. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $1 \leq i < j < n$. Then, putting $\hat{k} := \theta_{i,j-1}(k)$, we have, for any $k \in [i, j-1]$,*

1. $\varphi_k(T) = \varepsilon_{\hat{k}}\eta_{i,j}(T)$.

2. $\varepsilon_k(T) = \varphi_{\hat{k}}\eta_{i,j}(T)$.

In particular, $\eta_{i,j}$ interchanges the highest and lowest weight elements within each connected component of $\mathcal{B}_{i,j}$.

Proof. We prove the first assertion, the second one is analogous. Using Lemma 3.19, there are two cases:

1. Suppose that the k-component in which T lies is a collapsed k-string $\mathcal{S}_k$. Then, by Lemma 3.19, $F_k^{\varphi_k(T)}(T)$ is the lowest weight element of $\mathcal{S}_k$. We have that $\eta_{i,j}F_k^{\varphi_k(T)}(T)$ is in a $\hat{k}$-string $\mathcal{S}_{\hat{k}}^0$ (which is also collapsed) and by Lemma 3.23,

$$\eta_{i,j}F_k^{\varphi_k(T)}(T) = E_{\hat{k}}^{\varphi_k(T)}(\eta_{i,j}(T)).$$

Hence, by the definition of $\varepsilon_{\hat{k}}$, we have that

$$\varepsilon_{\hat{k}}(\eta_{i,j}(T)) \geq \varphi_k(T).$$

On the other hand, since $E_{\hat{k}}^{\varepsilon_{\theta_{i,j-1}(k)}(\eta_{i,j}(T))}(\eta_{i,j}(T))$ is in $\mathcal{S}_{\hat{k}}^0$, then $\eta_{i,j}E_{\hat{k}}^{\varepsilon_{\hat{k}}(\eta_{i,j}(T))}(\eta_{i,j}(T))$ must be in $\mathcal{S}_k$. By Lemma 3.23, and since $\eta_{i,j}$ is an involution, we have

$$\eta_{i,j}E_{\hat{k}}^{\varepsilon_{\hat{k}}(\eta_{i,j}(T))}(\eta_{i,j}(T)) = F_i^{\varepsilon_{\hat{k}}(\eta_{i,j}(T))}(T).$$

Consequently, by the definition of φ_k, we have

$$\varphi_k(T) \geq \varepsilon_{\hat{k}}(\eta_{i,j}(T)).$$

2. Now suppose that T is in $\mathcal{S}_k$, a separated k-string. Then, by Lemma 3.19 $F_k'^{\varphi_k'(T)} F_k^{\hat{\varphi}_k(T)}(T)$ is the lowest weight element of $\mathcal{S}_k$. Consequently, $\eta_{i,j} F_k'^{\varphi_k(T)} F_k^{\hat{\varphi}_k(T)}(T)$ is in a $\hat{k}$-string $\mathcal{S}_{\hat{k}}$ (which is also separated). As before, we have

$$\eta_{i,j} F_k'^{\varphi_k'(T)} F_k^{\hat{\varphi}_k(T)}(T) = E_{\hat{k}}'^{\varphi_k'(T)} E_{\hat{k}}^{\hat{\varphi}_k(T)}(\eta_{i,j}(T)),$$

and by definition of $\varepsilon_{\hat{k}}$, we have

$$\varepsilon_{\hat{k}}(\eta_{i,j}(T)) \geq \varphi_k'(T) + \hat{\varphi}_k(T) = \varphi_k(T).$$

Since $E_{\hat{k}}'^{\varepsilon_{\hat{k}}'(\eta_{i,j}(T))} E_{\hat{k}}^{\hat{\varepsilon}_{\hat{k}}(\eta_{i,j}(T))}(\eta_{i,j}(T))$ is in $\mathcal{S}_{\hat{k}}$, we have that $\eta E_{\hat{k}}'^{\varepsilon_{\hat{k}}'(\eta_{i,j}(T))} E_{\hat{k}}^{\hat{\varepsilon}_{\hat{k}}(\eta_{i,j}(T))}(\eta_{i,j}(T))$ is in $\mathcal{S}_{\hat{k}}$. By Proposition 3.20, and since $\eta_{i,j}$ is an involution, we have

$$\eta E_{\hat{k}}'^{\varepsilon_{\hat{k}}'(\eta_{i,j}(T))} E_{\hat{k}}^{\hat{\varepsilon}_{\hat{k}}(\eta_{i,j}(T))}(\eta_{i,j}(T)) = F_k'^{\varepsilon_{\hat{k}}'(\eta_{i,j}(T))} F_k^{\hat{\varepsilon}_{\hat{k}}(\eta_{i,j}(T))}(T),$$

and then,

$$\varphi_k(T) \geq \varepsilon_{\hat{k}}'(\eta_{i,j}(T)) + \hat{\varepsilon}_{\hat{k}}(\eta_{i,j}(T)) = \varepsilon_{\hat{k}}(\eta_{i,j}(T)).$$

□

3.3 The shifted reflection crystal operators

We now introduce a shifted version of the *crystal reflection operators* σ_i (see [9, Definition 2.35]) on $\mathsf{ShST}(\nu, n)$, for each $i \in I$. Crystal reflection operators were originally defined by Lascoux and Schützenberger [44] in the Young tableau crystal of type A. They are involutions, on type A crystals, so that each i-string is sent to itself by reflection over its middle axis, for all $i \in I$. It coincides with the restriction of the Schützenberger involution to the tableaux consisting of the letters $\{i, i{+}1\}$, ignoring the remaining ones. On $\mathsf{ShST}(\nu, n)$, collapsed strings are similar to the i-strings of type A crystals, hence the shifted reflection operator σ_i is expected to resemble the one for Young tableaux. However, for separated strings, a sole reflection of the i-string would not coincide with the restriction of the Schützenberger involution to $\{i, i+1\}'$, hence we have the next definition. We recall that $\alpha_i = e_i - e_{i+1}$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Definition 3.26. Let $i \in I$ and $T \in \mathsf{ShST}(\nu, n)$. Let $k = \langle \mathsf{wt}(T), \alpha_i \rangle$ (usual inner product in $\mathbb{R}^n$). The *shifted crystal reflection operator* σ_i is defined as follows

$$\sigma_i(T) = \begin{cases} F_i' F_i^{k-1}(T) & \text{if } k > 0 \text{ and } F_i'(T) \neq \varnothing \\ E_i' F_i^{k+1}(T) & \text{if } k > 0 \text{ and } F_i'(T) = \varnothing \\ E_i F_i'(T) & \text{if } k = 0 \text{ and } F_i'(T) \neq \varnothing \\ E_i' F_i(T) & \text{if } k = 0 \text{ and } F_i'(T) = \varnothing \text{ and } F_i(T) \neq \varnothing \\ T & \text{if } k = 0 \text{ and } F_i'(T) = F_i(T) = \varnothing \\ E_i^{-k+1} F_i'(T) & \text{if } k < 0 \text{ and } F_i'(T) \neq \varnothing \\ E_i^{-k-1} E_i'(T) & \text{if } k < 0 \text{ and } F_i'(T) = \varnothing \end{cases}$$

As the definition suggests, the shifted reflection operator σ_i must do a double reflection, by vertical and horizontal middle axes (see Figure 3.8). As we will see in Theorem 3.30, a simple reflection in the same fashion as the type A crystal fails to ensure the coincidence of the shifted crystal reflection operators with the adequate restriction of the Schützenberger involution, on separated strings. By coplacity, the operator σ_i is extended to $\mathsf{ShST}(\lambda/\nu, n)$, for $i \in I$.

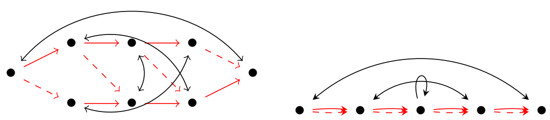


Figure 3.8: The action of a crystal reflection operator in separated and collapsed strings, which coincides with the action of the Schützenberger involution on these strings.

We remark that this definition is the same for both separated or collapsed strings. However, for the latter there is simpler formulation, as stated in the following lemma, since in this case the primed and unprimed operators coincide.

Lemma 3.27. *Let $i \in I$ and let $T \in \mathsf{ShST}(\lambda/\mu, n)$ be such that $F_i(T) = F_i'(T)$ (i.e., T is in a collapsed i-string). Let $k = \langle \mathsf{wt}(T), \alpha_i \rangle$. Then,*

$$\sigma_i(T) = \begin{cases} F_i^k(T) & \text{if } k > 0 \\ T & \text{if } k = 0 \\ E_i^{-k}(T) & \text{if } k < 0 \end{cases}$$

Example 3.28. Let $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$. To compute $\sigma_2(T)$ we have $k = \langle \mathsf{wt}(T), \alpha_2 \rangle = \langle (3,4,2), (0,1,-1) \rangle = 2 > 0$. Moreover, $F_2(T)$ and $F_2'(T)$ are both defined on T, thus, we have

$$\sigma_2(T) = F_2' F_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3' \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}\,.$$

Proposition 3.29. *For $i \in I$, the operator σ_i satisfies the following:*

1. *σ_i sends each connected component of $\mathsf{ShST}(\lambda/\mu, n)$ to itself, and each i-string to itself.*

2. *$\sigma_i^2 = 1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$.*

3. *$\mathsf{wt}(\sigma_i(T)) = \theta_i(\mathsf{wt}(T))$, where $\theta_i = (i, i+1) \in \mathfrak{S}_n$.*

Proof. The first assertion results directly from the definition of the raising and lowering operators. For the second assertion, it is clear that $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$, since each σ_k acts only on the primed subinterval of adjacent letters $\{k, k+1\}'$, leaving the remaining ones unchanged. To prove that σ_i is an involution, we must analyse various cases according to Definition 3.26. Let $T \in \mathsf{ShST}(\lambda/\mu, n)$.

Case 1. Suppose that $k > 0$ and that $F_i(T) \neq \varnothing$. Let $S = \sigma_i(T) = F_i' F_i^{k-1}(T)$. Then, $F_i'(S) = \varnothing$. By definition of σ_i, we have

$$\begin{aligned}\mathsf{wt}(S) &= \mathsf{wt}(F_i' F_i^{k-1}(T))\\ &= \mathsf{wt}(F_i^{k-1}(T)) - \alpha_i\\ &= \mathsf{wt}(T) - (k-1)\alpha_i - \alpha_i\\ &= \mathsf{wt}(T) - k\alpha_i.\end{aligned}$$

Hence, putting $\tilde{k} := \mathsf{wt}(S)_i - \mathsf{wt}(S)_{i+1}$ we have

$$\begin{aligned}\tilde{k} &= (\mathsf{wt}(T)_i - (k\alpha_i)_i) - (\mathsf{wt}(T)_{i+1} - (k\alpha_i)_{i+1})\\ &= \mathsf{wt}(T)_i - k - \mathsf{wt}(T)_{i+1} - k\\ &= -k < 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\sigma_i(S) &= E_i^{-(-k)-1} E_i'(S)\\ &= E_i^{k-1} E_i'(S)\\ &= E_i^{k-1} E_i' F_i' F_i^{k-1}(T)\\ &= E_i^{k-1} F_i^{k-1}(T) = T.\end{aligned}$$

Case 2. Now suppose T is such that $k > 0$ and $F_i'(T) = \varnothing$. Let $S = \sigma_i(T)$. We have $\mathsf{wt}(S) = \mathsf{wt}(T) + \alpha_i - (k+1)\alpha_i = \mathsf{wt}(T) - k\alpha_i$. Using the same notation as before, we have

$$\begin{aligned}\tilde{k} &= (\mathsf{wt}(T)_i - (k\alpha_i)_i) - (\mathsf{wt}(T)_{i+1} - (k\alpha_i)_{i+1}) \\ &= \mathsf{wt}(T)_i - \mathsf{wt}(T)_{i+1} - 2k = -k < 0.\end{aligned}$$

Moreover, since $E_i' F_i^{k+1}(T)$ is defined, this means that the last $(i+1)'$ was to the right of the last i in the word of $F_i^{k+1}(T)$, and was then changed to i, due to Proposition 3.2. Consequently, in S the last i is to the right of the last $(i+1)'$, which means that $F_i'(S) \neq \varnothing$. Hence,

$$\sigma_i(S) = E_i'^{-(-k)+1} F_i'(S) = E_i^{k+1} F_i' E_i' F_i^{k+1}(T) = E_i^{k+1} F_i^{k+1}(T) = T.$$

Case 3. Suppose that T is such that $k = 0$ and $F_i'(T) \neq \varnothing$. Let $S = \sigma_i(T)$. We have $\mathsf{wt}(S) = \mathsf{wt}(T)$, hence $\tilde{k} = k$. Since $F_i'(T) \neq \varnothing$, we have $F_i'(S) = \varnothing$. Then,

$$\sigma_i(S) = E_i F_i'(S) = E_i F_i' E_i' F_i(T) = E_i F_i(T) = T.$$

Case 4. Suppose that $k = 0$ and that $F_i'(T) = F_i'(T) = \varnothing$. Then, $\sigma_i^2(T) = \sigma_i(T) = T$.

The remaining cases are dual to the first three, which concludes the proof that $\sigma_i^2 = 1$. Finally, using the same notation as before, for the first case we have

$$\begin{aligned}\mathsf{wt}(\sigma_i(T)) &= \mathsf{wt}(T) - k\alpha_i \\ &= \mathsf{wt}(T) - (\mathsf{wt}(T)_i - \mathsf{wt}(T)_{i+1})\alpha_i \\ &= (\mathsf{wt}(T)_1, \dots, \mathsf{wt}(T)_i - \mathsf{wt}(T)_i + \mathsf{wt}(T)_{i+1}, \\ &\quad \mathsf{wt}(T)_{i+1} + \mathsf{wt}(T)_i - \mathsf{wt}(T)_{i+1}, \dots, \mathsf{wt}(T)_n) \\ &= (\mathsf{wt}(T)_1, \dots, \mathsf{wt}(T)_{i+1}, \mathsf{wt}(T)_i, \dots, \mathsf{wt}(T)_n) \\ &= \theta_i(\mathsf{wt}(T)).\end{aligned}$$

The remaining cases are proved analogously. □

Given $i \in I$, recall that $\mathcal{B}_{i,i+1}$ denotes the collection of the i-strings of $\mathsf{ShST}(\lambda/\mu, n)$ and that

$$\eta_{i,i+1}(T) := T^{1,i-1} \sqcup \eta(T^{i,i+1}) \sqcup T^{i+2,n}. \tag{3.10}$$

The next result, which is proved on Section 3.3.1, states that the shifted crystal reflection operator coincides with the partial Schützenberger involution restricted to $\{i, i+1\}'$.

Theorem 3.30. *Let T be a shifted semistandard tableau on the alphabet $[n]'$. Then, for any $i \in I$,*

$$\sigma_i(T) = \eta_{i,i+1}(T).$$

Unlike the type A crystals, the reflection operators σ_i, for $i \in I$, do not define an action of the symmetric group $\mathfrak{S}_n$ on $\mathsf{ShST}(\lambda/\mu, n)$. In particular, the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ do not need to hold, as shown in the next example.

Example 3.31. Let $\mathsf{ShST}(\nu, 3)$ where $\nu = (5, 3, 1)$, and consider the shifted semistandard tableau

$$T = \begin{ytableau} 1 & 1 & 1 & 1 & 3' \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau}\,.$$

The weight of T is given by $\mathsf{wt}(T) = (4, 2, 3)$. Then, since $\langle \mathsf{wt}(T), (1, -1, 0)\rangle = 4 - 2 = 2 > 0$ and $F_1'(T) \neq \varnothing$, we have

$$\sigma_1(T) = F_1'F_1(T) = \begin{ytableau} 1 & 1 & 2' & 2 & 3' \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau}\,.$$

Putting $T_1 := \sigma_1(T)$, we have that $\langle \mathsf{wt}(T_1), (0, 1, -1)\rangle = 4 - 3 = 1 > 0$. As $F_2'(T_1) = \varnothing$, we have

$$\sigma_2\sigma_1(T) = E_2'F_2^2(T_1) = E_2'\left(\begin{ytableau} 1 & 1 & 2' & 3' & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}\right) = \begin{ytableau} 1 & 1 & 2' & 2 & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}\,,$$

and putting $T_2 := \sigma_2\sigma_1(T)$, we have $\langle \mathsf{wt}(T_2), (1, -1, 0)\rangle = 2 - 3 = -1 < 0$ and $F_1'(T_2) = \varnothing$, and thus,

$$\sigma_1\sigma_2\sigma_1(T) = \sigma_1(T_2) = E_1'(T_2) = \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}\,. \tag{3.11}$$

On the other hand, we have that $\langle \mathsf{wt}(T), (0, 1, -1)\rangle = 2 - 3 = -1 < 0$ and $F_2'(T) = \varnothing$, hence

$$\sigma_2(T) = E_2'(T) = \begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau}\,.$$

We put $T_3 := \sigma_2(T)$, thus we have $\langle \mathsf{wt}(T_2), (1, -1, 0)\rangle = 4 - 3 = 1 > 0$ and $F_1'(T_3) \neq \varnothing$ and consequently

$$\sigma_1\sigma_2(T) = \sigma_1(T_3) = F_1'(T_3) = \begin{ytableau} 1 & 1 & 1 & 2' & 2 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau}\,.$$

Finally, putting $T_4 := \sigma_1\sigma_2(T)$, we have that $\langle \mathsf{wt}(T_4), (0, 1, -1)\rangle = 3 - 2 = 1 > 0$ and $F_2'(T_4) \neq \varnothing$, thus

$$\sigma_2\sigma_1\sigma_2(T) = \sigma_2(T_4) = F_2'F_2(T_4) = \begin{ytableau} 1 & 1 & 1 & 2' & 3' \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau} \tag{3.12}$$

Then, by (3.11) and (3.12), we have that $\sigma_1\sigma_2\sigma_1(T) \neq \sigma_2\sigma_1\sigma_2(T)$.

However, we have the following result, as in [3, Section 3.2] for ordinary LR tableaux, ensuring that the longest permutation of $\mathfrak{S}_n$ acts on a connected component of $\mathsf{ShST}(\lambda/\mu, n)$ by sending the highest weight element to the lowest weight element.

Theorem 3.32. *Let T^{high} be a LRS tableau in $\mathsf{ShST}(\lambda/\mu, n)$. Let $\theta_{1,n} = \theta_{i_1} \cdots \theta_{i_k}$ denote the longest permutation in $\mathfrak{S}_n$. Then, $\theta_{1,n}$ acts on a connected component of $\mathsf{ShST}(\lambda/\mu, n)$ by sending its highest weight element T^{high} to the lowest weight element T^{low}, i.e.,*

$$\theta_{1,n} \cdot T^{\mathsf{high}} = \sigma_{i_1} \cdots \sigma_{i_k}(T^{\mathsf{high}}) = \eta(T^{\mathsf{high}}) = T^{\mathsf{low}}.$$

Proof. Since the operators σ_i are coplactic, we may consider $Y_\nu = \mathsf{rect}(T)$, $\nu = \mathsf{wt}(T)$. By Proposition 3.29, σ_i permutes the entries i and $i+1$ on the weight and keeps the shape ν, and as $\theta_{1,n}$ is the longest permutation, $\sigma_{i_1} \cdots \sigma_{i_k}$ reverts the weight of T. Then, the uniqueness of $\mathsf{evac}(Y_\nu)$ implies that $\sigma_{i_1} \cdots \sigma_{i_k}(Y_\nu) = \mathsf{evac}(Y_\nu)$. □

Remark 3.33. Let $G_n := \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ be the free group generated by the shifted crystal reflection operators σ_i, for $i \in I$, modulo the relations satisfied by them on shifted semistandard tableaux. We know from Proposition 3.29 that the relations $\sigma_i^2 = 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| > 1$, hold on G_n, but not the braid relations of $\mathfrak{S}_n$, $(\sigma_i \sigma_{i+1})^3 = 1$, for $i \in [n-2]$. However, since $\mathsf{ShST}(\nu, n)$ is finite, we have that, given $T \in \mathsf{ShST}(\nu, n)$, there must exist some $m > 3$ such that $(\sigma_i \sigma_{i+1})^m(T) = T$, for $i \in [n-2]$. We computed some examples on the alphabet $\{1, 2, 3\}'$, which show that $(\sigma_1 \sigma_2)^m = 1$, for m a multiple of at least 90 [54, Appendix A], but we do not know if an upper bound valid for any shape ν exists.

3.3.1 Proof of Theorem 3.30

It suffices to prove Theorem 3.30 for tableaux on the primed alphabet of two adjacent letters and we consider it to be $\{1, 2\}'$, to simplify the notation. Moreover, the raising and lowering operators are coplactic, thus σ_1 is also coplactic. Hence, it suffices to prove the result for tableaux of straight shape. We remark that such tableaux have at most two rows. Furthermore, T and $\sigma_1(T)$ are in the same 1-string (which, in particular, is a connected component), hence by Proposition 3.15, T and $\sigma_1(T)$ are shifted dual equivalent. It remains to show that $\mathsf{c}_2(T)$ and $\sigma_1(T)$ are shifted Knuth equivalent. We remark that another proof may be done by directly verifying the conditions on Proposition 3.20. The one we present highlights some of the properties of straight-shaped tableaux with at most two rows. First, we introduce some technical results on shifted Knuth equivalence.

Lemma 3.34. *Let $a_1, \ldots, a_n, b_1, \ldots, b_m, c \in [n]'$, with $m, n \geq 1$.*

1. *If $b_m < \cdots < b_1 < c < a_1 < \cdots < a_n$ in standardization ordering, then*

$$c a_1 \cdots a_n b_1 \cdots b_m \equiv_k c b_1 \cdots b_m a_1 \cdots a_n.$$

2. *If* $a_1 < \cdots < a_n < c < b_m < \cdots < b_1$ *in standardization ordering, then*

$$a_1 \cdots a_n b_1 \cdots b_m c \equiv_k b_1 \cdots b_m a_1 \cdots a_n c.$$

Proof. We prove the first part by induction on n, the second part is proved similarly. If $n = 1$, we have

$$\begin{aligned}
\underline{ca_1}b_1b_2 \cdots b_m &\equiv_k c\underline{b_1a_1b_2} \cdots b_m && \text{(K1)}\\
&\equiv_k cb_1\underline{b_2a_1b_3} \cdots b_m && \text{(K1)}\\
&\cdots\\
&\equiv_k cb_1 \cdots \underline{b_{m-1}a_1b_m} && \text{(K1)}\\
&\equiv_k cb_1 \cdots b_{m-1}b_ma_1 && \text{(K1)}
\end{aligned}$$

Now suppose the result is true for some $n \geq 1$ and let $a_{n+1} > a_n$ in standardization ordering. Then,

$$\begin{aligned}
ca_1 \cdots \underline{a_na_{n+1}b_1}b_2 \cdots b_m &\equiv_k ca_1 \cdots a_n\underline{b_1a_{n+1}b_2}b_3 \cdots b_m && \text{(K1)}\\
&\cdots\\
&\equiv_k ca_1 \cdots a_nb_1 \cdots \underline{b_{m-1}a_{n+1}b_m} && \text{(K1)}\\
&\equiv_k ca_1 \cdots a_nb_1 \cdots b_ma_{n+1} && \text{(K1)}\\
&\equiv_k cb_1 \cdots b_ma_1 \cdots a_na_{n+1} && \text{Induction hypothesis and Lemma 2.23}
\end{aligned}$$

□

Lemma 3.35. *Let* $a \in [n]'$*. Then, for any* $m \geq 1$, $a(a')^m \equiv_k a^{m+1}$.

Proof. For $m = 1$ the result corresponds to the (SK2) relation. Suppose the result holds for some $m \geq 1$. Then,

$$\begin{aligned}
a(a')^{m+1} &= a(a')^ma'\\
&\equiv_k (a)^{m+1}a' && \text{Induction hypothesis and Lemma 2.23}\\
&= a(a)^ma'\\
&\equiv_k \underline{aa'}(a)^m && \text{Lemma 3.34}\\
&\equiv_k aa(a)^m && \text{(SK2)}\\
&= (a)^{m+2}.
\end{aligned}$$

□

In order to prove that $\mathsf{c}_2(T)$ and $\sigma_1(T)$ are shifted Knuth equivalent, we will present sequences of Knuth moves between their words. We have remarked that T has at most two rows. The case where it has one row is in the following result.

Proposition 3.36. *Let T be a straight-shaped shifted semistandard tableau with one row, filled with in the alphabet $\{1,2\}'$. Then, $\sigma_1(T) = \mathsf{evac}(T)$.*

Proof. As stated before, if suffices to show that $\sigma_1(T)$ is shifted Knuth equivalent to $\mathsf{c}_2(T)$. Suppose that $w(T) = 1^a$, for $a \geq 1$. If $a = 1$, then $w(\mathsf{c}_2(T)) = 2 = w(F_1(T)) = w(\sigma_1(T))$. If $a > 1$, then $w(\mathsf{c}_2(T)) = 2(2')^{a-1}$ and $w(\sigma_1(T)) = w(F_1^a(T)) = 2^a$. Hence, by Lemma 3.35, $w(\mathsf{c}_2(T)) \equiv_k w(\sigma_1(T))$. Now suppose that $w(T) = 1^a 2^b$, with $a, b \geq 1$. Then, $w(\sigma_1(T)) = 1^b 2^a$ and $w(\mathsf{c}_2(T)) = 2(2')^{a-1} 1 (1')^{b-1}$. There are two cases:

Case 1. If $a = 1$, we have

$$\begin{aligned} \underline{21}(1')^{b-1} &\equiv_k 12(1')^{b-1} && \text{(SK1)} \\ &\equiv_k 1(1')^{b-1}2 && \text{Lemma 2.23} \\ &\equiv_k 1^b 2. && \text{Lemmas 2.23 and 3.35} \end{aligned}$$

Case 2. If $a > 1$, then we have

$$\begin{aligned} 2(2')^{a-1}1(1')^{b-1} &\equiv_k 2^a 1 (1')^{b-1} && \text{Lemmas 2.23 and 3.35} \\ &= 22^{a-1}1(1')^{b-1} && \\ &\equiv_k \underline{21}(1')^{b-1}2^{a-1} && \text{Lemma 3.34} \\ &\equiv_k 12(1')^{b-1}2^{a-1} && \text{(SK1)} \\ &\equiv_k 1(1')^{b-1}22^{a-1} && \text{Lemmas 2.23 and 3.34} \\ &\equiv 1^b 2^a. && \text{Lemmas 2.23 and 3.35} \end{aligned}$$

□

If T has two rows, we remark that it suffices to verify the case where the second row has only one box. To make this statement rigorous, we need to introduce some notation. A shifted semistandard tableau T is called *detached* if its main diagonal has exactly one box. Then, we may define the following operator on shifted semistandard tableaux:

$$\mathsf{r}(T) = \begin{cases} T & \text{if } T \text{ is detached} \\ \widehat{T} & \text{otherwise} \end{cases}$$

where $\widehat{T}$ is obtained from T removing its main diagonal and shifting every box one unit to the left (so that its second diagonal becomes the main diagonal).

Example 3.37. If $T = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & 2 & & \end{array}$, then

$$\mathsf{r}(T) = \begin{array}{ccccc} 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & & \end{array}, \quad \mathsf{r}^2(T) = \begin{array}{cccc} 1 & 1 & 2' & 2 \\ & 2 & & \end{array}, \quad \mathsf{r}^m(T) = \begin{array}{ccc} 1 & 2 & 2 \end{array},$$

for $m \geq 3$.

The following lemma states that, if T is not detached and its $(l+1)$-th diagonal is the first with one box, then $\sigma_1(T)$ is determined by $\mathsf{r}^{l-1}(T)$, i.e., one may temporarily remove the first $l-1$ diagonals with two elements, compute σ_1 on the remaining tableau, and then place the diagonals back.

Lemma 3.38. *Let T be a shifted semistandard tableau of straight shape, with two rows, filled in the alphabet $\{1,2\}'$. Let l be such that $\{(1,l),(2,l+1)\}$ and $\{(1,l+1)\}$ are adjacent diagonals of T with two boxes and one box, respectively. Then,*

$$\mathsf{r}^{l-1}\sigma_1(T) = \sigma_1\mathsf{r}^{l-1}(T).$$

Proof. If $T = Y_\nu$, for $\nu = (\nu_1, \nu_2)$, then $\mathsf{wt}(\sigma_1(Y_\nu)) = (\nu_2, \nu_1)$. Consequently, $\sigma_1(Y_\nu) = \mathsf{evac}(Y_\nu)$. Then, $\mathsf{r}^{\nu_2-1}\mathsf{evac}(Y_\nu) = \mathsf{evac}(Y_{\nu^0})$, where $\nu^0 = (\nu_1 - \nu_2, 1)$. Similarly, $\mathsf{r}^{\nu_2-1}(Y_\nu) = Y_{\nu^0}$, and using the same argument with the weight, $\sigma_1\mathsf{r}^{\nu_2-1}(Y_\nu) = \mathsf{evac}(Y_{\nu^0})$. The proof for $\mathsf{evac}(Y_\nu)$ is similar.

Suppose now that T is neither Y_ν nor $\mathsf{evac}(Y_\nu)$. Suppose that the word of T is given by $w(T) = 2^a 1^{a+1} 1^b \mathbf{2} 2^c$, with $a \geq 1$ and $b, c \geq 0$. Then, $\mathsf{wt}(T) = (a+b+1, a+c+1)$ and considering Definition 3.26, we have $k = (a+b+1) - (a+c+1) = b - c$ (note that it does not depend on a). We show the case when $k > 0$ and $\mathbf{2} = 2$. The proof for the other cases is analogous. If $\mathbf{2} = 2$, then $F_1(T) \neq \varnothing$ and we have

$$\begin{aligned} \sigma_i(T) &= F_1' F_1^{b-c-1}(T) \\ &= F_1' F_1^{b-c-1}(2^a 1^{a+1} 1^b 2^{c+1}) \\ &= F_1' F_1^{b-c-1}(2^a 1^{a+1} 1^{b-(b-c-1)} 2^{(c+1)+(b-c-1)}) \\ &= F_1'(2^a 1^{a+1} 1^{c+1} 2^b) \\ &= 2^a 1^{a+1} 1^c 2' 2^b \end{aligned}$$

and so, $\mathsf{r}^{a-1}\sigma_1(T) = 21^21^c2'2^b$.

On the other hand, we have $\mathsf{r}^{a-1}(T) = 21^21^b22^c$ and so

$$\begin{aligned}\sigma_1\mathsf{r}^{a-1}(T) &= F_1'F_1^{b-c-1}(21^21^b2^{c+1})\\ &= F_1'(21^21^{c+1}2^b)\\ &= 21^21^c2'2^b.\end{aligned}$$

$\square$

In what follows, we consider T to be of shape $\nu = (m,1)$, i.e., such that its second row has only one box. To show that $\mathsf{c}_2(T)$ is shifted Knuth equivalent to $\sigma_1(T)$ is equivalent to show that $\mathsf{rect}(\mathsf{c}_2(T)) = \sigma_1(T)$, since T is of straight shape. Moreover, we ask for T to be neither Y_ν nor $\mathsf{evac}(Y_\nu)$, since the result for those cases is already proved. We have the following lemma, which is easy to prove.

Lemma 3.39. *Let $\nu = (m,1)$, for $m \geq 3$. Let $T \in \mathsf{ShST}(\nu,2)$ such that $T \neq Y_\nu, \mathsf{evac}(Y_\nu)$ and let $k = \varepsilon_1(T)$. Then,*

1. *If $T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&\cdots&1&2'&2&\cdots&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}$, with $\mathsf{wt}(T) = (m-k,k+1)$, then*
$$\sigma_1(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&\cdots&1&2&2&\cdots&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}$$
with $\mathsf{wt}(\sigma_1(T)) = (k+1,m-k)$.

2. *If $T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&\cdots&1&2&2&\cdots&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}$, with $\mathsf{wt}(T) = (m-k,k+1)$, then*
$$\sigma_1(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&\cdots&1&2'&2&\cdots&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}$$
with $\mathsf{wt}(\sigma_1(T)) = (k+1,m-k)$.

The rectification does not depend on the sequence of inner corners, so, for simplicity, we may fix that we always choose the rightmost inner corner in the highest-index row. Thus, we apply *jeu de taquin* slides on $\mathsf{c}_2(T)$, following this sequence, to the point where an occurrence of 2 or $2'$ on T (which correspond to $1'$ or 1 in $\mathsf{c}_2(T)$) will determine different slides on the next move. For instance, consider the following tableaux:

$$T_1 = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array} \longrightarrow \mathsf{c}_2(T_1) = \begin{array}{ccccc} \square&\square&\square&\square&1'\\ &\square&\square&\square&{\color{red}1'}\\ &&\square&\square&2'\\ &&&1&2'\\ &&&&2\end{array} \equiv_k \begin{array}{ccccc} \square&\square&\square&\square&1'\\ &\square&\blacksquare&{\color{red}1'}&2\\ &&1&2'&\\ &&&2&\end{array}$$ the box $(2,4)$ with ${\color{red}1'}$ will go **left** on the next slide.

$$T_2 = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2'&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array} \longrightarrow \mathsf{c}_2(T_2) = \begin{array}{ccccc} \square&\square&\square&\square&1'\\ &\square&\square&\square&{\color{red}1}\\ &&\square&\square&2'\\ &&&1&2'\\ &&&&2\end{array} \equiv_k \begin{array}{ccccc} \square&\square&\square&\square&1'\\ &\square&\blacksquare&{\color{red}1}&2\\ &&1&2'&\\ &&&2&\end{array}$$ the box $(3,3)$ with 1 will go **up** on the next slide.

Continuing the rectification we obtain, respectively:

$$\mathsf{rect}(\mathsf{c}_2(T_1)) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2' & 2 \\ \hline \end{array} \atop \begin{array}{c|c|} \multicolumn{1}{c}{} & 2 \\ \cline{2-2} \end{array} \qquad \mathsf{rect}(\mathsf{c}_2(T_2)) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline \end{array} \atop \begin{array}{c|c|} \multicolumn{1}{c}{} & 2 \\ \cline{2-2} \end{array}.$$

We begin by stating some auxiliary results.

Lemma 3.40. *Let $T = Y_\nu$ be the highest weight of its 1-string, with $m \geq 3$, for $\nu = (m, 1)$. Then,*

1. $w(\mathsf{c}_2 F_1(T)) = 21(2')^{m-2}1 \equiv_k 212'1(2)^{m-3}$.
2. $w(\mathsf{c}_2 F_1'(T)) = 21(2')^{m-2}1' \equiv_k 212'1'(2)^{m-3}$.

Proof. We prove a more general claim that, if $a \geq 1$, then,

$$21(2')^a 1 \equiv_k 212'1(2)^{a-1}$$
$$21(2')^a 1' \equiv_k 212'1'(2)^{a-1}.$$

Then, the result follows, observing that $m \geq 3$ ensures that $a := m - 2 \geq 1$. If $a = 1$, the claim is trivial. Suppose this is true for some $a \geq 1$. We have,

$$21(2')^{a+1}1 = 21(2')^a 2'1.$$

The word $21(2')^a 2'$ has the same standardization of $21(2')^a 1$. Therefore, by induction hypothesis $21(2')^a 2' \equiv_k 212'2'(2)^{a-1}$. Then, we have

$$\begin{aligned}
21(2')^a 2'1 &\equiv_k 2\underline{12'2'}(2)^{a-1}1 && \text{Lemma 2.22} \\
&\equiv_k \underline{22'}12'(2)^{a-1}1 && \text{(K2)} \\
&\equiv_k \underline{221}2'(2)^{a-1}1 && \text{(SK2)} \\
&\equiv_k \underline{21}22'(2)^{a-1}1 && \text{(K1)} \\
&\equiv_k 1\underline{222'}(2)^{a-1}1 && \text{(SK1)} \\
&\equiv_k \underline{122}'2(2)^{a-1}1 && \text{(K1)} \\
&\equiv_k 212'2(2)^{a-1}1 && \text{(SK1)} \\
&= 212'(2)^a 1.
\end{aligned}$$

Moreover, we have $2'(2)^a 1 \equiv_k 2'1(2)^a$, by Lemma 3.34, using only (K1) Knuth moves. Hence, by Lemma 2.23, we have

$$212'(2)^a 1 \equiv_k 212'1(2)^a.$$

Consequently, $21(2')^a1 \equiv_k 212'1(2)^{a-1}$. The proof that $21(2')^a1' \equiv_k 212'1'(2)^{a-1}$ is done similarly, since Lemma 3.34 also ensures that $2'(2)^a1' \equiv_k 2'1'(2)^a$, using only (K1) Knuth moves. ☐

Corollary 3.41. *Let T be a shifted semistandard tableaux in the 1-string of Y_ν, with $\nu = (m, 1)$ and $m \geq 3$, such that T is not Y_ν neither $\mathsf{evac}(Y_\nu)$. Let $a = \varepsilon_1(T)$.*

1. *If $T = F_1^a(Y_\nu)$, then $w(\mathsf{c}_2(T)) = 21(2')^{m-a-1}1(1')^{a-1} \equiv_k 212'1(2)^{m-a-2}(1')^{a-1}$.*

2. *If $T = F_1'F_1^{a-1}(Y_\nu)$ $w(\mathsf{c}_2(T)) = 21(2')^{m-a-1}1'(1')^{a-1} \equiv_k 212'1'(2)^{m-a-2}(1')^{a-1}$.*

Proof. Since $T \neq \mathsf{evac}(Y_\nu)$ and $\mathsf{evac}(Y_\nu)$ is a lowest weight element, then

$$a = \varepsilon_1(T) < \varepsilon_1(\mathsf{evac}(Y_\nu)) = m - 1.$$

Then, $a \leq m - 2$ and so we have that $m - a - 1 \geq 1$. Therefore, using Lemma 3.40, we have

$$21(2')^{m-a-1}1 \equiv_k 212'1(2)^{m-a-2}.$$

Consequently, by Lemma 2.22, we have $21(2')^{m-a-1}1(1')^{a-1} \equiv_k 212'1(2)^{m-a-2}(1')^{a-1}$. The proof for the second case is similar. ☐

Proposition 3.42. *Let T be a shifted semistandard tableau in the 1-string of Y_ν, with $\nu = (m, 1)$ and $m \geq 3$, such that T is neither Y_ν or $\mathsf{evac}(Y_\nu)$. Let $a = \varepsilon_1(T)$.*

1. *If $T = F_1^a(Y_\nu)$, then $212'1(2)^{m-a-2}(1')^{a-1} \equiv_k 2(1)^{a+1}2(2)^{m-a-2} = \sigma_1(T)$.*

2. *If $T = F_1'F_1^a(Y_\nu)$, then $212'1'(2)^{m-a-2}(1')^{a-1} \equiv_k 2(1)^{a+1}2'(2)^{m-a-2} = \sigma_1(T)$.*

Proof. We first prove the first assertion. We have

$$
\begin{aligned}
212'1(2)^{m-a-2}(1')^{a-1} &\equiv_k \underline{212'}1(1')^{k-1}(2)^{m-a-2} && \text{Lemmas 3.34 and 2.23}\\
&\equiv_k \underline{22'}11(1')^{a-1}(2)^{m-a-2} && \text{(K2)}\\
&\equiv_k 2\underline{211}(1')^{a-1}(2)^{m-a-2} && \text{(SK2)}\\
&\equiv_k \underline{21}21(1')^{a-1}(2)^{m-a-2} && \text{(K2)}\\
&\equiv_k 1\underline{221}(1')^{a-1}(2)^{m-a-2} && \text{(SK1)}\\
&\equiv_k 1212(1')^{a-1}(2)^{m-a-2} && \text{(K1)}\\
&\equiv_k \underline{121}(1')^{a-1}2(2)^{m-a-2} && \text{Lemma 2.23}\\
&\equiv_k 211(1')^{a-1}2(2)^{m-a-2} && \text{(SK1)}\\
&\equiv_k \underline{21}(1')^{a-1}12(2)^{m-a-2} && \text{Lemmas 2.22, 2.23 and 3.34}\\
&\equiv_k 12(1')^{a-1}12(2)^{m-a-2} && \text{(SK1)}\\
&\equiv_k 1(1')^{a-1}212(2)^{m-a-2} && \text{Lemmas 3.34 and 2.22}\\
&\equiv_k (1)^{a}212(2)^{m-a-2} && \text{Lemmas 3.35 and 2.23}\\
&\equiv_k 2(1)^{a}12(2)^{m-a-2} && \text{Lemmas 3.34 and 2.23}\\
&= 2(1)^{a+1}2(2)^{m-a-2}.
\end{aligned}
$$

For the second assertion, we have

$$
\begin{aligned}
212'1'(2)^{m-a-2}(1')^{a-1} &\equiv_k 212'1'(1')^{a-1}(2)^{m-a-2} && \text{Lemma 3.34}\\
&= 212'(1')^{a}(2)^{m-a-2}\\
&\equiv_k \underline{21}(1')^{a}2'(2)^{m-a-2} && \text{Lemma 3.34}\\
&\equiv_k 12(1')^{a}2'(2)^{m-a-2} && \text{(SK1)}\\
&\equiv_k 1(1')^{a}22'(2)^{m-a-2} && \text{Lemma 3.34}\\
&\equiv_k (1)^{a+1}22'(2)^{m-a-2} && \text{Lemma 3.35}\\
&\equiv_k 2(1)^{a+1}2'(2)^{m-a-2}. && \text{Lemma 3.34}
\end{aligned}
$$

□

We are now able to prove Theorem 3.30.

Proof of Theorem 3.30. It suffices to show the result for T of straight shape with two rows. Corollary 3.41 and Proposition 3.42 ensure that the words $w(\mathsf{c}_2(T))$ and $w(\sigma_1(T))$ are shifted

Knuth equivalent, thus $\mathsf{c}_2(T) \equiv_k \sigma_1(T)$. Since T and $\sigma_1(T)$ are dual equivalent, this concludes the proof that $\sigma_1(T) = \mathsf{evac}(T)$. $\square$

CHAPTER 4

AN ACTION OF THE CACTUS GROUP

Halacheva [29] showed that there is a natural action of the cactus group $J_{\mathfrak{g}}$ on any $\mathfrak{g}$-crystal, for $\mathfrak{g}$ a complex, reductive, finite-dimensional Lie algebra. In particular, the cactus group $J_n = J_{\mathfrak{gl}_n}$ (Definition 1.1) acts internally on the type A crystal of semistandard Young tableux $\mathsf{SSYT}(\lambda/\mu, n)$ (here considering any partitions), via the partial Schützenberger involutions, which correspond to partial evacuations on $\mathsf{SSYT}(\nu, n)$. Following a similar approach, it was shown in [54, Theorem 5.7] that there is a natural action of J_n on the shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, n)$. This action is realized by the restrictions of the Schützenberger involution to all primed intervals of $[n]'$ (thus, containing in particular the shifted crystal reflection operators). We recall the definition of the cactus group as in [32, Section 3.1].

Recall that $\theta_{i,j}$ denotes the longest permutation of $\mathfrak{S}_{[i,j]}$ embedded in $\mathfrak{S}_n$, and that n-fruit cactus group J_n (Definition 1.1) is the free group generated by $s_{i,j}$, for $1 \le i < j \le n$, subject to the relations

$$s_{i,j}^2 = 1, \quad s_{i,j}s_{k,l} = s_{k,l}s_{i,j}, \text{ for } [i,j] \cap [k,l] = \varnothing, \quad s_{i,j}s_{k,l} = s_{i+j-l,i+j-k}s_{i,j}, \text{ for } [k,l] \subseteq [i,j].$$

The first and third relations ensure that the elements of the form $s_{1,k}$ generate J_n, for $1 < k \le n$, since any $s_{i,j}$ may be written as

$$s_{i,j} = s_{1,j}s_{1,j-i+1}s_{1,j}. \tag{4.1}$$

Theorem 4.1. *There is a natural action of the n-fruit cactus group J_n on the shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, n)$ given by the group homomorphism:*

$$\begin{aligned} \phi : J_n &\longrightarrow \mathfrak{S}_{\mathsf{ShST}(\lambda/\mu,n)} \\ s_{i,j} &\longmapsto \eta_{i,j} \end{aligned}$$

for $1 \le i < j \le n$.

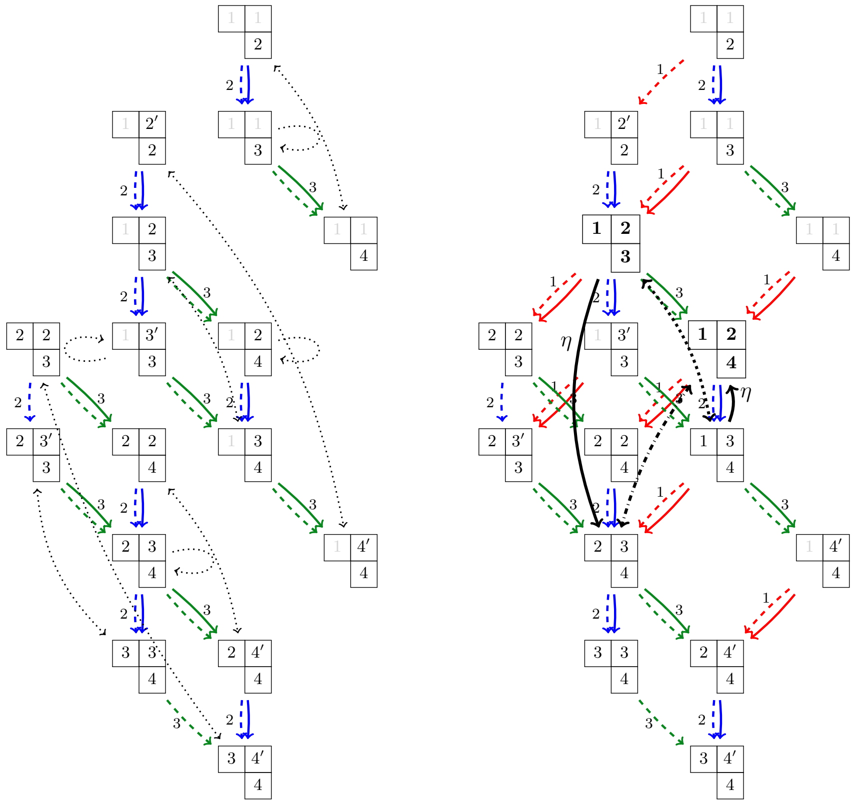


Figure 4.1: On the left, the action of $s_{2,4}$ on $\mathsf{ShST}(\nu, 4)$, with $\nu = (2,1)$. On the right, an illustration of $s_{1,3}s_{1,4} = s_{1,4}s_{2,4}$.

Recall that, given $T \in \mathsf{ShST}(\nu, n)$, $\mathsf{evac}_j(T) = \mathsf{evac}(T^{1,j}) \sqcup T^{j+1,n} = \eta_{1,j}(T)$. As a consequence, the next results follow from (4.1) and from ϕ being a homomorphism.

Corollary 4.2. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $1 \leq i < j \leq n$. Then,*

$$\eta_{i,j}(T) = \eta_{1,j}\eta_{1,j-i+1}\eta_{1,j}(T).$$

In particular, for $T \in \mathsf{ShST}(\nu, n)$, we have

$$\eta_{i,j}(T) = \mathsf{evac}_j \mathsf{evac}_{j-i+1} \mathsf{evac}_j(T).$$

Theorem 4.3. *There is a natural action of the n-fruit cactus group on the shifted tableau crystal $\mathsf{ShST}(\nu, n)$, given by the group homomorphism, for $1 < i \leq n$:*

$$\widehat{\phi} : J_n \longrightarrow \mathfrak{S}_{\mathsf{ShST}(\nu,n)}$$

$$s_{1,i} \longmapsto \mathsf{evac}_i.$$

Proof. Since ϕ is a homomorphism between J_n and $\mathfrak{S}_{\mathsf{ShST}(\lambda/\mu,n)}$, in particular it is a homomorphism between J_n and $\mathfrak{S}_{\mathsf{ShST}(\nu,n)}$. The result then follows from (4.1), as we have $\widehat{\phi}(s_{1,i}) = \mathsf{evac}_i = \eta_{1,i} = \phi(s_{1,i})$. □

4.1 Proof of Theorem 4.1

To show that ϕ in Theorem 4.1 is a group homomorphism, we show that the operators $\eta_{i,j}$ satisfy the cactus group relations of Definition 1.1, for any $1 \leq i < j \leq n$. The first two relations are trivial, and we claim that if suffices to show that these operators satisfy the third relation for any $[k,l] \subseteq [1,j]$. Indeed, we have the following.

Lemma 4.4. *Suppose that, for any* $[k,l] \subseteq [1,j]$,

$$\eta_{1,j}\eta_{k,l} = \eta_{j-l+1,j-k+1}\eta_{1,j}. \tag{4.2}$$

Then, for any $[k,l] \subseteq [i,j]$, *we have*

$$\eta_{i,j}\eta_{k,l} = \eta_{j+i-l,j+i-k}\eta_{i,j}. \tag{4.3}$$

Proof. Given $[k,l] \subseteq [i,j]$, we show that (4.2) implies the third relation (4.3) of Definition 1.1. Since, in particular, $[i,j] \subseteq [1,j]$, then (4.2) ensures that

$$\eta_{1,j}\eta_{i,j} = \eta_{1+,j-i+1}\eta_{1,j}. \tag{4.4}$$

Moreover, $[k,l] \subseteq [i,j]$ implies that $[k-i+1, l-i+1] \subseteq [1, j-i+1]$, and thus, by (4.2),

$$\eta_{1,j-i+1}\eta_{k-i+1,l-i+1} = \eta_{j-l+1,j-k+1}\eta_{1,j-i+1}. \tag{4.5}$$

Similarly, we have $[j+i-l, j+i-k] \subseteq [1,j]$, and thus, by (4.2),

$$\eta_{1,j}\eta_{j+i-l,j+i-k} = \eta_{k-i+1,l-i+1}\eta_{1,j}. \tag{4.6}$$

Then, using the fact that $\eta_{i,j}$ is an involution, we have, for any $[i,j]$,

$$\begin{aligned}
\eta_{i,j}\eta_{k,l} &= \eta_{1,j}\eta_{1,j-i+1}\eta_{1,j}\eta_{1,j}\eta_{j-l+1,j-k+1}\eta_{1,j} && \text{by (4.2) and (4.4)}\\
&= \eta_{1,j}\eta_{1,j-i+1}\eta_{j-l+1,j-k+1}\eta_{1,j} \\
&= \eta_{1,j}\eta_{1,j-i+1}\big(\eta_{1,j-i+1}\eta_{k-i+1,l-i+1}\eta_{1,j-i+1}\big)\eta_{1,j} && \text{by (4.5)}\\
&= \eta_{1,j}\eta_{k-i+1,l-i+1}\eta_{1,j-i+1}\eta_{1,j} \\
&= \eta_{1,j}\big(\eta_{1,j}\eta_{j+i-l,j+i-k}\eta_{1,j}\big)\eta_{1,j-i+1}\eta_{1,j} && \text{by (4.6)}\\
&= \eta_{j+i-l,j+i-k}\eta_{i,j} && \text{by (4.4)}
\end{aligned}$$

□

Proof of Theorem 4.1. Each operator $\eta_{i,j}$ is an involution (Corollary 3.24) and the second relation is a direct consequence of the definition, as each operator $\eta_{i,j}$ acts only on the letters $[i,j]'$, leaving the others unchanged. Thus, to conclude the proof, due to Lemma 4.4, it suffices to prove that the relation (4.2) holds for any $[k,l] \subseteq [1,j]$. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, let $\mathcal{C}_0$ be the unique connected component of $\mathcal{B}_{k,l}$ containing T. If $\mathcal{C}_0$ is an isolated vertex, then by Lemma 3.23 and Corollary 3.24 the result is trivially true. Thus, we assume that $\mathcal{C}_0$ has at least two vertices. By Lemma 3.21, $\mathcal{C}_0$ has a unique highest weight element T_0^{high} and lowest weight element $T_0^{\mathsf{low}} = \eta_{k,l}(T_0^{\mathsf{high}})$. These elements are different, as $\mathcal{C}_0$ has at least two vertices. Since $l \leq j$, $\mathcal{C}_0$ is contained in a connected component $\mathcal{C}_1$ of $\mathcal{B}_{1,j}$. By Lemma 3.21, $\mathcal{C}_1$ has unique highest weight element T_1^{high} and lowest weight element $T_1^{\mathsf{low}} = \eta_{1,j}(T_1^{\mathsf{high}}) \neq T_1^{\mathsf{high}}$. Then, we have

$$\begin{aligned} T &= F_{i_1}'^{m_1} F_{i_1}^{n_1} \cdots F_{i_r}'^{m_r} F_{i_r}^{n_r} (T_0^{\mathsf{high}}) \\ T_0^{\mathsf{low}} &= E_{j_1}'^{a_1} E_{j_1}^{b_1} \cdots E_{j_s}'^{a_s} E_{j_s}^{b_s} \eta_{1,j}(T_1^{\mathsf{high}}) \end{aligned} \tag{4.7}$$

for some $i_1, \ldots, i_r \in [k, l-1]$, $j_1, \ldots, j_s \in [1, j-1]$, with $m_1, \ldots, m_r, a_1, \ldots, a_s \in \{0,1\}$, and $n_1, \ldots, n_r, b_1, \ldots, b_s \geq 0$. Thus, we have

$$\begin{aligned}
\eta_{1,j}\eta_{k,l}(T) &= \eta_{1,j}\eta_{k,l} F_{i_1}'^{m_1} F_{i_1}^{n_1} \cdots F_{i_r}'^{m_r} F_{i_r}^{n_r} (T_0^{\mathsf{high}}) && (4.7)\\
&= \eta_{1,j} E'^{m_1}_{\theta_{k,l-1}(i_1)} E^{n_1}_{\theta_{k,l-1}(i_1)} \cdots E'^{m_r}_{\theta_{k,l-1}(i_r)} E^{n_r}_{\theta_{k,l-1}(i_r)} \eta_{k,l}(T_0^{\mathsf{high}}) && \text{Lemma 3.23}\\
&= \eta_{1,j} E'^{m_1}_{\theta_{k,l-1}(i_1)} E^{n_1}_{\theta_{k,l-1}(i_1)} \cdots E'^{m_r}_{\theta_{k,l-1}(i_r)} E^{n_r}_{\theta_{k,l-1}(i_r)} (T_0^{\mathsf{low}}) \\
&= \eta_{1,j} E'^{m_1}_{\theta_{k,l-1}(i_1)} E^{n_1}_{\theta_{k,l-1}(i_1)} \cdots E'^{m_r}_{\theta_{k,l-1}(i_r)} E^{n_r}_{\theta_{k,l-1}(i_r)} \\
&\qquad E_{j_1}'^{a_1} E_{j_1}^{b_1} \cdots E_{j_s}'^{a_s} E_{j_s}^{b_s} \eta_{1,j}(T_1^{\mathsf{high}}) && (4.7)\\
&= F'^{m_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} F^{n_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} \cdots F'^{m_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} F^{n_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} \\
&\qquad F'^{a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F'^{a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (\eta_{1,j}^2(T_1^{\mathsf{high}})) && \text{Lemma 3.23}\\
&= F'^{m_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} F^{n_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} \cdots F'^{m_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} F^{n_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} \\
&\qquad F'^{a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F'^{a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (T_1^{\mathsf{high}}),
\end{aligned}$$

that is,

$$\eta_{1,j}\eta_{k,l}(T) = F'^{a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F'^{a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (T_1^{\mathsf{high}}). \tag{4.8}$$

Since $\mathcal{C}_0$ is a connected component of $\mathcal{B}_{k,l}$, given $X, Y \in \mathcal{C}_0$, we have $X \sim_{k,l} Y$, and thus $X = H_{i_1} \cdots H_{i_p}(Y)$, where $H_i \in \{F_i', F_i, E_i', E_i\}$, for some $i_1, \ldots, i_p \in [k, l-1]$. Lemma 3.23 then ensures that

$$\eta_{1,j}(X) = \eta_{1,j} H_{i_1} \cdots H_{i_p}(Y) = \tilde{H}_{\theta_{1,j-1}(i_1)} \cdots \tilde{H}_{\theta_{1,j-1}(i_p)}(\eta_{1,j}(Y)),$$

where $\tilde{H}_i$ denotes the partial inverse of H_i. Then, $\eta_{1,j}(X)$ and $\eta_{1,j}(Y)$ are related by a sequence of crystal operators labelled in $\theta_{1,j-1}[k,l-1] = [j-l+1, j-k]$, which means that $\eta_{1,j}(X) \sim_{j-l+1,j-k+1} \eta_{1,j}(Y)$. Then, $\eta_{1,j}$ takes $\mathcal{C}_0$, a connected component of $\mathcal{B}_{k,l}$, to $\eta_{1,j}(\mathcal{C}_0)$, which is a connected component of $\mathcal{B}_{j-l+1,j-k+1}$. Then, Proposition 3.25 ensures that, for any $Q \in \mathcal{C}_0$ and any $p \in [k, l-1] \subseteq [1, j-1]$, $\varepsilon_p(T) = \varphi_{\theta_{1,j-1}(p)}(\eta_{1,j}(T))$ and $\varphi_p(T) = \varepsilon_{\theta_{1,j-1}(p)}(\eta_{1,j}(T))$. This implies that $\eta_{1,j}(T_0^{\mathsf{low}})$ and $\eta_{1,j}(T_0^{\mathsf{high}})$ are, respectively, the highest and lowest weight elements of $\eta_{1,j}(\mathcal{C}_0)$. On the other hand, since $\eta_{1,j}(\mathcal{C}_0)$ is a connected component of $\mathcal{B}_{j-l+1,j-k+1}$, then $\eta_{j-l+1,j-k+1}$ interchanges its highest and lowest weight elements, that is,

$$\eta_{j-l+1,j-k+1}\eta_{1,j}(T_0^{\mathsf{high}}) = \eta_{1,j}(T_0^{\mathsf{low}}). \tag{4.9}$$

Then, noting that $\theta_{1,j-1}\theta_{k,l-1} = \theta_{j-l+1,j-k}\theta_{1,j-1}$, we may write,

$$\begin{aligned}
\eta_{j-l+1,j-k+1}\eta_{1,j}(T) &= \eta_{j-l+1,j-k+1}\eta_{1,j} F_{i_1}^{\prime m_1} F_{i_1}^{n_1} \cdots F_{i_k}^{\prime m_k} F_{i_k}^{n_k}(T_0^{\mathsf{high}}) && (4.7)\\
&= \eta_{j-l+1,j-k+1} E^{\prime m_1}_{\theta_{1,j-1}(i_1)} E^{n_1}_{\theta_{1,j-1}(i_1)} \cdots \\
&\qquad \cdots E^{\prime m_k}_{\theta_{1,j-1}(i_k)} E^{n_k}_{\theta_{1,j-1}(i_k)} (\eta_{1,j}(T_0^{\mathsf{high}})) && \text{Lemma 3.23}\\
&= F^{\prime m_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} F^{n_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} \cdots \\
&\qquad \cdots F^{\prime m_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} F^{n_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} \\
&\qquad \eta_{j-l+1,j-k+1}\eta_{1,j}(T_0^{\mathsf{high}}) && \text{Lemma 3.23}\\
&= F^{\prime m_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} F^{n_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} \cdots \\
&\qquad \cdots F^{\prime m_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} F^{n_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} \eta_{1,j}(T_0^{\mathsf{low}}) && (4.9)\\
&= F^{\prime m_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} F^{n_1}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_1)} \cdots \\
&\qquad \cdots F^{\prime m_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} F^{n_k}_{\theta_{j-l+1,j-k}\theta_{1,j-1}(i_k)} \\
&\qquad \eta_{1,j} E^{\prime a_1}_{j_1} E^{b_1}_{j_1} \cdots E^{\prime a_s}_{j_s} E^{b_s}_{j_s} \eta_{1,j}(T_1^{\mathsf{high}}) && (4.7)\\
&= F^{\prime m_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} F^{n_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} \cdots F^{\prime m_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} F^{n_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} \\
&\qquad F^{\prime a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F^{\prime a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (\eta_{1,j}^2(T_1^{\mathsf{high}})) && \text{Lemma 3.23}\\
&= F^{\prime m_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} F^{n_1}_{\theta_{1,j-1}\theta_{k,l-1}(i_1)} \cdots F^{\prime m_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} F^{n_r}_{\theta_{1,j-1}\theta_{k,l-1}(i_r)} \\
&\qquad F^{\prime a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F^{\prime a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (T_1^{\mathsf{high}}),
\end{aligned}$$

that is

$$\eta_{j-l+1,j-k+1}\eta_{1,j}(T) = F^{\prime a_1}_{\theta_{1,j-1}(j_1)} F^{b_1}_{\theta_{1,j-1}(j_1)} \cdots F^{\prime a_s}_{\theta_{1,j-1}(j_s)} F^{b_s}_{\theta_{1,j-1}(j_s)} (T_1^{\mathsf{high}}). \tag{4.10}$$

Finally, by (4.8) and (4.10), we have

$$\eta_{1,j}\eta_{k,l}(T) = \eta_{j-l+1,j-k+1}\eta_{1,j}(T).$$

$\square$

CHAPTER 5

A SHIFTED BERENSTEIN–KIRILLOV GROUP

In this chapter we introduce a shifted version of the Bender–Knuth involutions for shifted semistandard tableaux. Stembridge has defined Bender–Knuth moves for shifted tableaux [69, Section 6], but they differ from the ones we introduce, as they do not preserve classes of canonical form (see Remark 5.18). For ordinary Young tableaux, the Bender–Knuth involutions on letters $\{i, i+1\}$ are known to coincide with the tableau switching applied to horizontal border strips filled with the same letters [5, Proposition 2.6], [51, Section 4.1], together with a swapping of the letters. Thus, it is natural to use the shifted version of that algorithm, introduced by Choi, Nam and Oh [15], to define the shifted Bender–Knuth moves, or, equivalently, the type C infusion map due to Thomas and Yong [70] on standardized tableaux, followed by the shifted semistandardization process of Pechenik and Yong [52]. As in [7], we are then able to recover the shifted evacuation, promotion, and shifted crystal reflection operators.

We then use the shifted Bender–Knuth involutions to introduce a shifted version of the Berenstein–Kirillov group. Following the works of Halacheva [29, 30] and Chmutov, Glick and Pylyavskyy [10], we show, using the action of the cactus group J_n on $\mathsf{ShST}(\nu, n)$ (Chapter 4), that the shifted Berenstein–Kirillov group is isomorphic to a quotient of the cactus group. We also give an alternative presentation for the cactus group in terms of the shifted Bender–Knuth involutions.

5.1 Shifted Bender–Knuth involutions

We now introduce the *shifted Bender–Knuth involutions* $\mathtt{t}_i$, for $i \in \mathbb{Z}_{>0}$, which will yield another presentation for the cactus group J_n. We first fix some notation. Given $i \in I = [n-1]$, recall that $\theta_i \in \mathfrak{S}_n$ denotes the simple transposition $(i, i+1)$. We write the cyclic permutation $\zeta_i = \theta_i \theta_{i-1} \cdots \theta_1$ as $\zeta_i := (1, i+1, i, \ldots, 2) \in \mathfrak{S}_n$. We recall that these permutations act on letters of the marked alphabet $[n]'$ as in (2.2).

Definition 5.1. Let $T^{i_1}, \dots, T^{i_n}$ be a sequence of i_k-border strips, with $\mathbf{i_k} \in [n]'$ and such that $\{i_1, \dots, i_n\} = [n]$. Suppose that $T^{i_{k+1}}$ extends T^{i_k}, for $1 \leq k < n$. Consider $T := T^{i_1} \sqcup \cdots \sqcup T^{i_n}$, a shifted skew shape filled in the alphabet $[n]'$ (that is not necessarily a shifted semistandard filling).

1. Let $i, j \in [n]$ be such that T^j extends T^i. We define $\mathsf{SP}_{i,j}(T)$ to be the filling of the shape of T obtained by leaving each T^k unchanged, for $k \neq i, j$, and replacing $T^i \sqcup T^j$ with $\mathsf{SP}_1(T^i, T^j) \sqcup \mathsf{SP}_2(T^i, T^j)$.

2. We also define $\mathsf{SW}_{i_k|i_{k+1},\dots,i_{k+l}}(T) := \mathsf{SP}_{i_k,i_{k+l}}\mathsf{SP}_{i_k,i_{k+l-1}} \cdots \mathsf{SP}_{i_k,i_{k+1}}(T)$.

Example 5.2. Let $T = \begin{array}{ccccc} 1 & 1 & 2' & 2 & 3 \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array}$. Then, to compute $\mathsf{SP}_{2,3}(T)$ we have:

$$\begin{array}{ccccc} 1 & 1 & 2' & 2 & 3 \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S6)}} \begin{array}{ccccc} 1 & 1 & 2' & 3' & 3 \\ & 2 & 2 & 2 & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S4)}} \begin{array}{ccccc} 1 & 1 & 2' & 3' & 3 \\ & 3 & 2' & 2 & \\ & & 2 & & \end{array} \xrightarrow{\textbf{(S1),(S1)}} \begin{array}{ccccc} 1 & 1 & 3' & 3 & 2' \\ & 3 & 2' & 2 & \\ & & 2 & & \end{array} = \mathsf{SP}_{2,3}(T).$$

To compute $\mathsf{SW}_{1|2,3}(T)$, first apply the shifted tableau switching to the pair (T^1, T^2), obtaining $(\tilde{T}^2, \tilde{T}^1)$, and then apply it again to the pair $(\tilde{T}^1, T^3)$:

$$\begin{array}{ccccc} 1 & 1 & 2' & 2 & 3 \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S5)}} \begin{array}{ccccc} 1 & 2' & 1 & 2 & 3 \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S6)}} \begin{array}{ccccc} 1 & 2' & 2 & 2 & 3 \\ & 2 & 1 & 3' & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S3)}} \begin{array}{ccccc} 2 & 2 & 2 & 2 & 3 \\ & 1 & 1 & 3' & \\ & & 3 & & \end{array} = \mathsf{SP}_{1,2}(T)$$

$$\longrightarrow \begin{array}{ccccc} 2 & 2 & 2 & 2 & 3 \\ & 1 & 1 & 3' & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S5)}} \begin{array}{ccccc} 2 & 2 & 2 & 2 & 3 \\ & 1 & 3' & 1 & \\ & & 3 & & \end{array} \xrightarrow{\textbf{(S3)}} \begin{array}{ccccc} 2 & 2 & 2 & 2 & 3 \\ & 3 & 3 & 1 & \\ & & 1 & & \end{array} = \mathsf{SP}_{1,3}\mathsf{SP}_{1,2}(T) = \mathsf{SW}_{1|2,3}(T).$$

We remark that $\mathsf{SP}_{i,j}$ and $\mathsf{SW}_{K|J}$ in general do not yield shifted semistandard tableaux, as the rows and columns may not be weakly increasing, as shown in the previous example, but they may be composed with adequate permutations of $\mathfrak{S}_n$, acting as in (2.2) on the entries in $[n]'$, ensuring that the resulting filling is a valid shifted semistandard tableau.

Lemma 5.3. *Let $1 \leq i < j \leq n$ and $T \in \mathsf{ShST}(\lambda/\mu, n)$, such that T^j extends T^i. Then,*

1. $\mathsf{wt}(\mathsf{SP}_{i,j}(T)) = \mathsf{wt}(T)$[1].

2. $\mathsf{SP}_{j,i}\mathsf{SP}_{i,j} = 1$.

3. $\tau\mathsf{SP}_{i,j}(T) = \mathsf{SP}_{\tau(i),\tau(j)}\tau(T)$, *for any permutation* $\tau \in \mathfrak{S}_n$.

[1]The weight of a filling of a shifted shape, not necessarily a valid shifted semistandard tableau, is defined as before.

Proof. To prove the first statement, we note that the shifted tableau switching solely moves boxes, not changing the total weight. For the second statement, we assume, without loss of generality, that $T = A \sqcup B$, with $A = T^i$ and $B = T^j$. Then, $\mathsf{SP}_{i,j}(A \sqcup B) = \mathsf{SP}_1(A, B) \sqcup \mathsf{SP}_2(A, B)$, where $\mathsf{SP}_1(A, B)$ is filled in $\{j', j\}$ and $\mathsf{SP}_2(A, B)$ is filled in $\{i', i\}$. Then, since the shifted tableau switching is an involution [15, Theorem 4.3], we have

$$\begin{aligned}
&\mathsf{SP}_{j,i}\big(\mathsf{SP}_1(A, B) \sqcup \mathsf{SP}_2(A, B)\big) = \\
&\qquad = \mathsf{SP}_1\big(\mathsf{SP}_1(A, B), \mathsf{SP}_2(A, B)\big) \sqcup \mathsf{SP}_2\big(\mathsf{SP}_1(A, B), \mathsf{SP}_2(A, B)\big) \\
&\qquad = \mathsf{SP}_1\big(\mathsf{SP}(A, B)\big) \sqcup \mathsf{SP}_2\big(\mathsf{SP}(A, B)\big) \\
&\qquad = A \sqcup B.
\end{aligned}$$

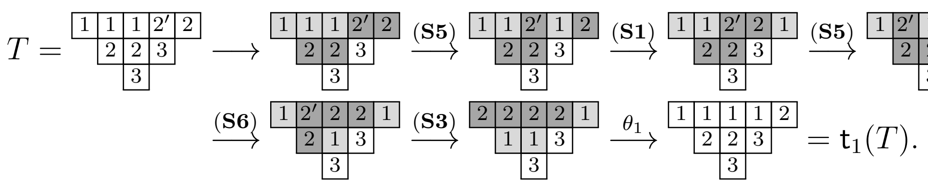
For the last assertion, we note that applying the shifted tableau switching to the pair (T^i, T^j), followed by the action of a permutation $\tau \in \mathfrak{S}_n$ is the same as first apply the permutation τ to the letters in T, and then compute the shifted tableau switching to the pair that previously corresponded to (T^i, T^j), which is now $(T^{\tau(i)}, T^{\tau(j)})$. $\square$

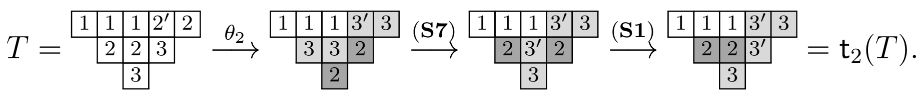
We may now define the operators $\mathtt{t}_i$, for $i \in \mathbb{Z}_{>0}$, for shifted semistandard tableaux.

Definition 5.4. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, for $n > 1$, and $i \in I$, we define the *shifted Bender–Knuth move* $\mathtt{t}_i$ as

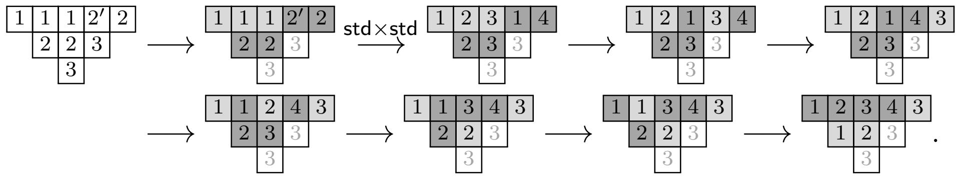
$$\mathtt{t}_i(T) := \theta_i \mathsf{SP}_{i,i+1}(T) = \mathsf{SP}_{i+1,i}\theta_i(T).$$

Example 5.5. Let $T = \begin{array}{ccccc} 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & 3 & \\ & & 3 & & \end{array}$. Then, we have

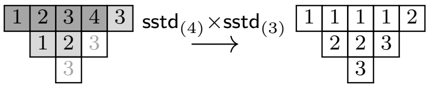




Remark 5.6. A shifted Bender–Knuth move may be formulated in terms of type C infusion and semistandardization. The tableau $\mathtt{t}_1(T)$, as in the previous example, may be computed as follows:

$T =$ 

Then, the semistandardization process with respect to $wt_2 = (4)$ and $wt_1 = (3)$ yields:

 $= \mathsf{t}_1(T).$

Proposition 5.7. *The shifted Bender–Knuth operators* t_i *satisfy the following, for any* $i \in I$*:*

1. $\mathsf{t}_i^2 = 1$.

2. $\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i$, *for* $|i-j| > 1$.

3. $\mathsf{wt}(\mathsf{t}_i(T)) = \theta_i(\mathsf{wt}(T))$, *for any* $T \in \mathsf{ShST}(\lambda/\mu, n)$.

Thus, t_i *defines a bijection between the set of shifted semistandard tableaux of shape* λ/μ *and weight* ν*, and the set of shifted semistandard tableaux of the same shape and weight* $\theta_i(\nu)$*.*

Proof. By Lemma 5.3, we have

$$\mathsf{t}_i^2 = \theta_i \mathsf{SP}_{i,i+1} \theta_i \mathsf{SP}_{i,i+1} = \mathsf{SP}_{i+1,i} \theta_i^2 \mathsf{SP}_{i,i+1} = \mathsf{SP}_{i+1,i} \mathsf{SP}_{i,i+1} = 1.$$

The second assertion results from t_i acting only on the letters $\{i, i+1\}'$, leaving the others unchanged. For the third statement, Lemma 5.3, ensures that

$$\mathsf{wt}(\mathsf{t}_i(T)) = \mathsf{wt}(\mathsf{SP}_{i+1,i}\theta_i(T)) = \mathsf{wt}(\theta_i(T)) = \theta_i(\mathsf{wt}(T)).$$

□

Remark 5.8. Since the operators t_i act on the weight of a shifted semistandard tableau T as the simple transposition θ_i, for each i, they can be used to derive a proof that the Schur Q- and P-functions are symmetric, similarly to the one for classic Schur functions using Bender–Knuth moves.

As in the ordinary case, the operators t_i do not commute with the *jeu de taquin*, as shown in Example 5.9. In general, t_i does not coincide with σ_i (although t_1 and σ_1 coincide on straight-shaped tableaux). Moreover, if T is in a i-string $\mathcal{B}_i$, it is not necessary for $\mathsf{t}_i(T)$ to be in the same i-string (see Figure 5.1).

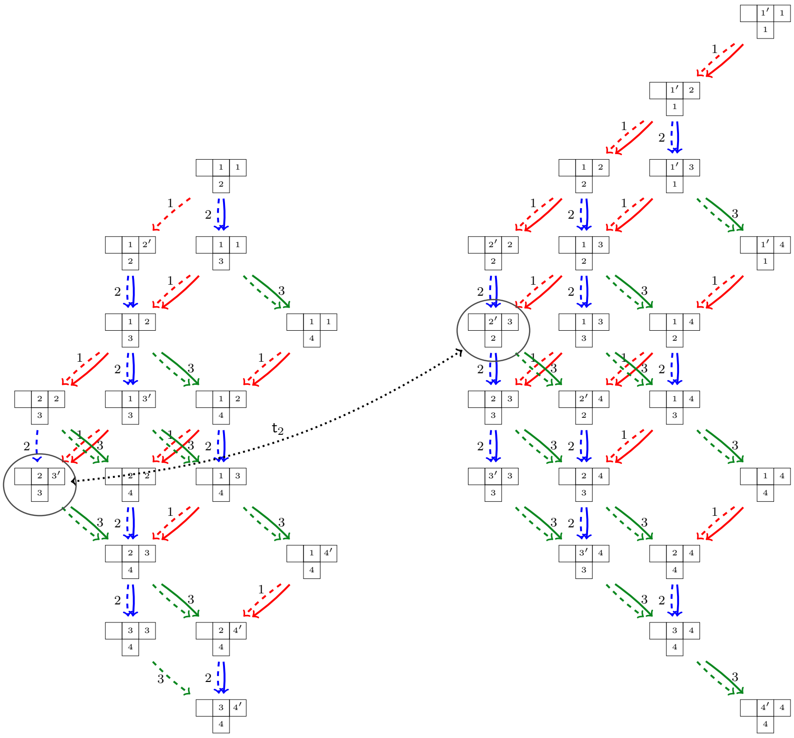


Figure 5.1: An example of the action of t_2 on a shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, 4)$, with $\lambda = (3,1)$ and $\nu = (1)$, which has two connected components.

Example 5.9. Considering T of the previous example, we have

$$T = \begin{ytableau} 1 & 1 & 1 & 2' & 2 \\ \none & 2 & 2 & 3 \\ \none & \none & 3 \end{ytableau} \equiv_k \begin{ytableau} {} & {} & 1' & 2' & 2 \\ \none & 1 & 1 & 2 & 3 \\ \none & \none & 2 & 3 \end{ytableau} = T'$$

and

$$\mathsf{t}_2(T) = \begin{ytableau} 1 & 1 & 1 & 3' & 3 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau} \not\equiv_k \begin{ytableau} {} & {} & 1' & 2 & 2 \\ \none & 1 & 1 & 3' & 3 \\ \none & \none & 3 & 3 \end{ytableau} = \mathsf{t}_2(T').$$

Moreover, note that (see Example 3.28)

$$\sigma_2(T) = \begin{ytableau} 1 & 1 & 1 & 2 & 3' \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau} \neq \mathsf{t}_2(T).$$

Like the case for type A, we can define a shifted version of the *promotion* operator due to Schützenberger [64], using the shifted Bender–Knuth involutions, and then recover the shifted evacuation and shifted crystal reflection operators for straight-shaped tableaux.

Definition 5.10. Given $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $i \in I$, we define the *shifted promotion operator* p_i as

$$\mathsf{p}_i(T) := \mathsf{t}_i \mathsf{t}_{i-1} \cdots \mathsf{t}_1(T).$$

As a result of t_i being involutions, we have $\mathsf{p}_i^{-1} = \mathsf{t}_1 \cdots \mathsf{t}_{i-1} \mathsf{t}_i$.

We will show that the promotion $\mathsf{p}_i(T)$ coincides with the shifted tableau switching on the pairs $(T^1, T^2 \sqcup \cdots \sqcup T^{i+1})$, followed by an adequate cyclic substitution of the letters. We first prove some auxiliary results.

Lemma 5.11. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $1 \leq i < j \leq n-1$. Then, $T^{i+1} \sqcup \cdots \sqcup T^j$ extends T^i, and for any $\tau \in \mathfrak{S}_n$ we have*

$$\tau \mathsf{SW}_{i|i+1,\ldots,j}(T) = \mathsf{SW}_{\tau(i)|\tau(i+1),\ldots,\tau(j)}\tau(T).$$

Proof. By Definition 5.1 and Lemma 5.3, we have

$$\begin{aligned}\tau \mathsf{SW}_{i|i+1,\ldots,j}(T) &= \tau \mathsf{SP}_{i,j}\mathsf{SP}_{i,j-i}\cdots \mathsf{SP}_{i,i+1}(T)\\ &= \mathsf{SP}_{\tau(i),\tau(j)}\mathsf{SP}_{\tau(i),\tau(j-1)}\cdots \mathsf{SP}_{\tau(i),\tau(i+1)}\tau(T)\\ &= \mathsf{SW}_{\tau(i)|\tau(i+1),\ldots,\tau(j)}\tau(T).\end{aligned}$$

□

Lemma 5.12. *Let $T \in \mathsf{ShST}(\lambda/\mu, n)$ and let $1 < i \leq n-1$. We have*

$$\zeta_i \mathsf{SW}_{i|i+1}\mathsf{SW}_{i-1|i,i+1}\cdots \mathsf{SW}_{2|3,\ldots,i+1}(T) = \mathsf{SW}_{i-1|i}\mathsf{SW}_{i-2|i-1,i}\cdots \mathsf{SW}_{1|2,\ldots,i}\zeta_i(T).$$

Proof. Applying successively Lemma 5.11, we have

$$\begin{aligned}\zeta_i \mathsf{SW}_{i|i+1}\mathsf{SW}_{i-1|i,i+1}\cdots \mathsf{SW}_{2|3,\ldots,i+1} &= \mathsf{SW}_{\zeta_i(i)|\zeta_i(i+1)}\mathsf{SW}_{\zeta_i(i-1)|\zeta_i(i),\zeta_i(i+1)}\cdots \mathsf{SW}_{\zeta_i(2)|\zeta_i(3),\ldots,\zeta_i(i+1)}\zeta_i\\ &= \mathsf{SW}_{i-1|i}\mathsf{SW}_{i-2|i-1,i}\cdots \mathsf{SW}_{1|2,\ldots,i}\zeta_i.\end{aligned}$$

□

Proposition 5.13. *Given $T \in \mathsf{ShST}(\lambda/\mu, n)$, and $i \in I$, we have*

$$\mathsf{p}_i(T) = \zeta_i \mathsf{SW}_{1|2,\ldots,i+1}(T).$$

Proof. The proof is done by induction on i. For $i = 1$, we have

$$\mathsf{p}_1(T) = \mathsf{t}_1(T) = \theta_1 \mathsf{SP}_{1,2}(T) = \zeta_1 \mathsf{SW}_{1|2}(T).$$

Assuming the result is true for some $i \geq 1$, by Definition 5.1 and Lemma 5.3, we have

$$\begin{aligned}\mathsf{p}_{i+1}(T) &= \mathsf{t}_{i+1}\mathsf{p}_i(T)\\ &= \theta_{i+1}\mathsf{SP}_{i+1,i+2}\zeta_i \mathsf{SW}_{1|2,\ldots,i+1}(T)\\ &= \theta_{i+1}\zeta_i \mathsf{SP}_{\zeta_i^{-1}(i+1),\zeta_i^{-1}(i+2)}\mathsf{SW}_{1|2,\ldots,i+1}(T)\\ &= \theta_{i+1}\zeta_i \mathsf{SP}_{1,i+2}\mathsf{SW}_{1|2,\ldots,i+1}(T)\\ &= \zeta_{i+1}\mathsf{SW}_{1|2,\ldots,i+1,i+2}(T).\end{aligned}$$

□

For $i \geq 1$, we define

$$\mathsf{q}_i := \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\cdots\mathsf{t}_1). \tag{5.1}$$

Recall that $\widetilde{\mathsf{evac}}_k$ is the operator obtained by allowing skew-shaped tableaux on the algorithm of Figure 2.6, which differs from the reversal on skew shapes. We will show that $\widetilde{\mathsf{evac}}_k$ and evac_k may be written as a composition of promotion operators. As a consequence, q_i coincides with $\widetilde{\mathsf{evac}}_{i+1}$ on skew-shaped shifted tableaux and with evac_{i+1} on straight-shapes ones. This coincidence implies that q_i are involutions, for any $i \geq 1$.

Proposition 5.14. *Given $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $i \in I$, we have*

$$\widetilde{\mathsf{evac}}_{i+1}(T) = \mathsf{q}_i(T) = \mathsf{p}_1\mathsf{p}_2\cdots\mathsf{p}_i(T) = \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)(T).$$

In particular, when $T \in \mathsf{ShST}(\nu, n)$ we have

$$\eta_{1,i+1}(T) = \mathsf{evac}_{i+1}(T) = \mathsf{q}_i(T) = \mathsf{p}_1\mathsf{p}_2\cdots\mathsf{p}_i(T) = \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)(T).$$

Proof. The proof is analogous either for straight or skew shape cases, as evac_{i+1} and $\widetilde{\mathsf{evac}}_{i+1}$ coincide on straight-shaped tableaux. By (2.8), we have $\mathsf{d}_{i+1}\mathsf{neg}_{i+1}\cdots\mathsf{neg}_1 = \theta_{1,i+1} = \zeta_1\cdots\zeta_i$. Moreover, it is clear that, for $l < k < i$,

$$\begin{aligned} \mathsf{SW}_{\text{-}k|k+1,\dots,i+1}\mathsf{neg}_k &= \mathsf{neg}_k\mathsf{SW}_{k|k+1,\dots,i+1} \\ \mathsf{SW}_{k|k+1,\dots,i+1}\mathsf{neg}_l &= \mathsf{neg}_l\mathsf{SW}_{k|k+1,\dots,i+1}. \end{aligned} \tag{5.2}$$

Then, the algorithm for $\widetilde{\mathsf{evac}}_{i+1}$ (see Figure 2.6) performed on T can be written as:

$$\begin{aligned} \widetilde{\mathsf{evac}}_{i+1}(T) &= \mathsf{d}_{i+1}\mathsf{neg}_{i+1}\mathsf{SW}_{\text{-}i|i+1}\mathsf{neg}_i\cdots\mathsf{SW}_{\text{-}2|3,\dots,i+1}\mathsf{neg}_2\mathsf{SW}_{\text{-}1|2,\dots,i+1}\mathsf{neg}_1(T) \\ &= \mathsf{d}_{i+1}\mathsf{neg}_{i+1}\mathsf{neg}_i\mathsf{SW}_{i|i+1}\cdots\mathsf{neg}_2\mathsf{SW}_{2|3,\dots,i+1}\mathsf{neg}_1\mathsf{SW}_{1|2,\dots,i+1}(T) \\ &= \mathsf{d}_{i+1}\mathsf{neg}_{i+1}\cdots\mathsf{neg}_2\mathsf{neg}_1\mathsf{SW}_{i|i+1}\cdots\mathsf{SW}_{2|3,\dots,i+1}\mathsf{SW}_{1|2,\dots,i+1}(T) \\ &= \zeta_1\cdots\zeta_i\mathsf{SW}_{i|i+1}\cdots\mathsf{SW}_{2|3,\dots,i+1}\mathsf{SW}_{1|2,\dots,i+1}(T). \end{aligned}$$

To conclude the proof, we claim that

$$\zeta_1\cdots\zeta_i\mathsf{SW}_{i|i+1}\cdots\mathsf{SW}_{2|3,\dots,i+1}\mathsf{SW}_{1|2,\dots,i+1}(T) = \zeta_1\mathsf{SW}_{1|2}\zeta_2\mathsf{SW}_{1|2,3}\cdots\zeta_i\mathsf{SW}_{1|2,\dots,i}(T) = \mathsf{p}_1\mathsf{p}_2\cdots\mathsf{p}_i(T). \tag{5.3}$$

We prove (5.3) by induction on i. The base case is trivial. For the induction step, assume the claim holds for some $i \geq 1$. Then, by Lemma 5.12 and Proposition 5.13, we have

$$\begin{aligned}
&\zeta_1 \cdots \zeta_i \zeta_{i+1} \mathsf{SW}_{i+1|i+2} \cdots \mathsf{SW}_{2|3,\ldots,i+1,i+2} \mathsf{SW}_{1|2,\ldots,i+1,i+2}(T) = \\
&\qquad = \zeta_1 \cdots \zeta_i \mathsf{SW}_{i|i+1} \cdots \mathsf{SW}_{1|2,\ldots,i+1} \zeta_{i+1} \mathsf{SW}_{1|2,\ldots,i+1,i+2} \\
&\qquad = \zeta_1 \mathsf{SW}_{1|2} \zeta_2 \mathsf{SW}_{1|2,3} \cdots \zeta_i \mathsf{SW}_{1|2,\ldots,i} \zeta_{i+1} \mathsf{SW}_{1|2,\ldots,i+1,i+2}(T) \\
&\qquad = \mathsf{p}_1 \mathsf{p}_2 \cdots \mathsf{p}_i \mathsf{p}_{i+1}(T).
\end{aligned}$$

□

Corollary 5.15. *Let $i \in I$. Then $\mathsf{q}_i^2 = 1$ and $\mathsf{wt}(\mathsf{q}_i(T)) = \theta_{1,i+1}(T)$.*

Proof. Since $\widetilde{\mathsf{evac}}_{i+1}$ is an involution, for any $i \geq 1$, then so it is q_i. From Proposition 5.7, we have $\mathsf{wt}(\mathsf{q}_i(T)) = \mathsf{wt}(\mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1) \cdots (\mathsf{t}_i\mathsf{t}_{i-1} \cdots \mathsf{t}_1)(T)) = \theta_1(\theta_2\theta_1) \cdots (\theta_i \cdots \theta_1)(T) = \theta_{1,i+1}(T)$. □

Corollary 5.16. *Given $T \in \mathsf{ShST}(\nu, n)$ and $i \in I$, we have*

$$\sigma_i(T) = \mathsf{evac}_{i+1}\mathsf{evac}_2\mathsf{evac}_{i+1}(T) = \mathsf{q}_i\mathsf{t}_1\mathsf{q}_i(T) = \mathsf{p}_1(\mathsf{p}_2 \cdots \mathsf{p}_i)^2(T).$$

Proof. By Theorem 3.30 and Corollary 4.2, we have $\sigma_i(T) = \mathsf{evac}_{i+1}\mathsf{evac}_2\mathsf{evac}_{i+1}(T)$. From Proposition 5.14, we have

$$\begin{aligned}
\mathsf{evac}_{i+1}\mathsf{evac}_2\mathsf{evac}_{i+1}(T) &= \mathsf{q}_i\mathsf{q}_1\mathsf{q}_i(T) = \mathsf{q}_i\mathsf{t}_1\mathsf{q}_i(T) \\
&= (\mathsf{p}_1\mathsf{p}_2 \cdots \mathsf{p}_i)\mathsf{t}_1(\mathsf{p}_1\mathsf{p}_2 \cdots \mathsf{p}_i)(T) \\
&= (\mathsf{p}_1\mathsf{p}_2 \cdots \mathsf{p}_i)\mathsf{t}_1(\mathsf{t}_1\mathsf{p}_2 \cdots \mathsf{p}_i)(T) \\
&= \mathsf{p}_1(\mathsf{p}_2 \cdots \mathsf{p}_i)(\mathsf{p}_2 \cdots \mathsf{p}_i)(T).
\end{aligned}$$

□

For $1 \leq i < j \leq n$, it is natural to consider the restriction of the operator $\widetilde{\mathsf{evac}}_k$ to an interval $[i,j]'$, in the same fashion as Definition 3.22. For $T \in \mathsf{ShST}(\lambda/\mu, n)$ and $1 \leq i < j \leq n$, we define

$$\widetilde{\mathsf{evac}}_{i,j}(T) := T^{1,i-1} \sqcup \widetilde{\mathsf{evac}}(T^{i,j}) \sqcup T^{j+1,n}. \tag{5.4}$$

Clearly, $\widetilde{\mathsf{evac}}_{1,k} = \widetilde{\mathsf{evac}}_k$ and $\widetilde{\mathsf{evac}}_{i,j}$ coincides with $\eta_{i,j}$, on straight-shaped shifted tableaux. However, these operators do not satisfy the relation $\widetilde{\mathsf{evac}}_{i,j} = \widetilde{\mathsf{evac}}_j \widetilde{\mathsf{evac}}_{j-i+1} \widetilde{\mathsf{evac}}_j$, for $\mu \neq \varnothing$, unlike the operators $\eta_{i,j}$ (Corollary 4.2), as shown in the next example.

Example 5.17. Considering $T = \begin{ytableau} 1 & 1 & 1 & 1 & 3' \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau}$, we have

$$\widetilde{\mathsf{evac}}_{2,3}(T) = \begin{ytableau} 1 & 1 & 1 & 1 & 2' \\ \none & 2 & 2 & 3 \\ \none & \none & 3 \end{ytableau} \neq \begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 \end{ytableau} = \widetilde{\mathsf{evac}}_3 \widetilde{\mathsf{evac}}_2 \widetilde{\mathsf{evac}}_3(T).$$

Remark 5.18. Stembridge introduced a shifted version of Bender–Knuth moves in [69, Section 6]. These are two-to-two maps acting on adjacent letters by reverting their weight. Shifted tableaux are not required to be in canonical form here, and in general, these maps are not compatible with canonical form. For instance, consider the following tableau, in canonical form:

$$T = \begin{ytableau} \none & \none & \none & \none & \none & \none & 1 & 2' \\ \none & \none & \none & \none & 2' & 2 & 2 \\ 1' & 1 & 1 & 1 & 2 \\ 1 & 2' \\ \none & 2 \end{ytableau}$$

and consider the representatives of T:

$$T_1 = \begin{ytableau} \none & \none & \none & \none & \none & \none & 1 & 2' \\ \none & \none & \none & \none & 2' & 2 & 2 \\ 1' & 1 & 1 & 1 & 2 \\ 1 & 2' \\ \none & 2 \end{ytableau} \quad T_2 = \begin{ytableau} \none & \none & \none & \none & \none & \none & 1 & 2' \\ \none & \none & \none & \none & 2' & 2 & 2 \\ 1' & 1 & 1 & 1 & 2 \\ 1 & 2' \\ \none & 2' \end{ytableau} \quad T_3 = \begin{ytableau} \none & \none & \none & \none & \none & \none & 1 & 2' \\ \none & \none & \none & \none & 2' & 2 & 2 \\ 1' & 1 & 1 & 1 & 2 \\ 1' & 2' \\ \none & 2 \end{ytableau} \quad T_4 = \begin{ytableau} \none & \none & \none & \none & \none & \none & 1 & 2' \\ \none & \none & \none & \none & 2' & 2 & 2 \\ 1' & 1 & 1 & 1 & 2 \\ 1' & 2' \\ \none & 2' \end{ytableau}.$$

Using the maps in [69, Section 6], we have:

$$\{T_1, T_2\} \longrightarrow \left\{ \begin{ytableau} \none & \none & \none & \none & \none & \none & 1' & 2 \\ \none & \none & \none & \none & 1' & 1 & 1 \\ 1' & 1 & 2 & 2 & 2 \\ 1 & 2' \\ \none & 2 \end{ytableau}, \begin{ytableau} \none & \none & \none & \none & \none & \none & 1' & 2 \\ \none & \none & \none & \none & 1' & 1 & 1 \\ 1' & 1 & 2 & 2 & 2 \\ 1 & 2' \\ \none & 2' \end{ytableau} \right\} =: \{\hat{T}_1, \hat{T}_2\}$$

$$\{T_3, T_4\} \longrightarrow \left\{ \begin{ytableau} \none & \none & \none & \none & \none & \none & 1' & 2 \\ \none & \none & \none & \none & 1' & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 \\ 1' & 2' \\ \none & 2 \end{ytableau}, \begin{ytableau} \none & \none & \none & \none & \none & \none & 1' & 2 \\ \none & \none & \none & \none & 1' & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 \\ 1' & 2' \\ \none & 2' \end{ytableau} \right\} =: \{\hat{T}_3, \hat{T}_4\}$$

The tableaux in $\{\hat{T}_1, \hat{T}_2\}$ do not have the same canonical form as the ones in $\{\hat{T}_3, \hat{T}_4\}$.

5.2 The Berenstein–Kirillov group

The Bender–Knuth involutions t_i, for $i \in I$, are involutions on semistandard Young tableaux filled in $[n]$, that act only on the letters $\{i, i+1\}$, reverting their weight [4]. They are known to coincide with the tableau switching on type A on two consecutive letters, together with a swapping of those letters [5]. The *Berenstein–Kirillov group* $\mathcal{BK}$ (or *Gelfand-Tsetlin group*) , is the free group generated by these involutions t_i, for $i > 0$, modulo the relations they satisfy on semistandard Young tableaux of any shape [8, 7, 10]. Some of the known relations to hold in $\mathcal{BK}$ [7, Corollary 1.1] are

$$t_i^2 = 1, \qquad t_it_j = t_jt_i, \text{ for } |i-j| > 1, \qquad (t_1q_i)^4 = 1, \text{ for } i > 2, \tag{5.5}$$

where $q_i := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$, for $i \geq 1$, are involutions, and

$$(t_1t_2)^6 = 1. \tag{5.6}$$

The restriction of the evacuation to the alphabet $\{1,\dots,i\}$, on straight-shaped semistandard Young tableaux, may be regarded as an element of $\mathcal{BK}$, and it is computed by q_{i-1} [7, 10, 30, 29]. We also let $q_{j,k} := q_{k-1}q_{k-j}q_{k-1}$, for $j < k$. In particular, $q_i = q_{1,i+1}$ and $q_{j,k}$ computes the restriction of the evacuation to the alphabet $\{j,\dots,k\}$, as an element of $\mathcal{BK}$. Chmutov, Glick and Pylyavskyy found another relation[10, Theorem 1.6].

$$(t_iq_{j,k})^2 = 1, \text{ for } i+1 < j < k. \tag{5.7}$$

The relation (5.7) does not follow from the previous known relations (5.5) and (5.6) in $\mathcal{BK}$, but is instead a consequence from the cactus relations satisfied by the operators $q_{i,j}$ in $\mathcal{BK}$, studied by Halacheva [30, 29] and Chmutov, Glick and Pylyavskyy [10]. We remark that (5.7) generalizes the relation $(t_1q_i)^4 = 1$, since

$$(t_1q_i)^4 = (t_1q_it_1q_i)^2 = (t_1q_iq_1q_i)^2 = (t_1q_{i,i+1})^2.$$

Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1,\dots,t_{n-1}$. The involutions q_i, for $i \in I$, provide another set of generators for $\mathcal{BK}_n$, and their action on straight-shaped Young tableaux coincide with the one of the restriction of the Schützenberger involution (or evacuation) to $[i+1]$ [7, Remark 1.3]. It was shown in [10], using semistandard growth diagrams, that $\mathcal{BK}_n$ is isomorphic to a quotient of the cactus group. This result could also be derived by noting the coincidence of the actions of J_n [29] and $\mathcal{BK}_n$ on a straight-shaped semistandard Young tableau crystal $\mathsf{SSYT}(\nu, n)$, as noted in [30, Remark 3.9].

Theorem 5.19. *The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n, as a result of the following being group epimorphisms from J_n to $\mathcal{BK}_n$:*

1. *$s_{i,j} \mapsto q_{i,j}$ [10, Theorem 1.4].*

2. *$s_{1,j} \mapsto q_{j-1}$ [7, Remark 1.3], [29, Section 10.2], [30, Remark 3.9].*

Chmutov, Glick and Pylyavskyy established in [10, Theorem 1.8] an equivalence between the relations (5.5) and (5.7) that are satisfied in $\mathcal{BK}_n$ and the ones of the cactus group J_n (see Definition 1.1), thus obtaining an alternative presentation for the latter via the Bender–Knuth moves. More precisely, they consider the free group generated by t_i, for $i \in \mathbb{Z}_{>0}$, and consider another free group generated by $q_{i,j}$, $1 \leq i \leq j$.

Theorem 5.20 ([10, Theorem 1.8]). *The relations*

$$t_i^2 = 1, \qquad t_i t_j = t_j t_i, \text{ for } |i-j| > 1, \qquad (t_i q_{k-1} q_{k-j} q_{k-1})^2 = 1, \text{ for } i+1 < j < k \quad (5.8)$$

where $q_i := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$*, are equivalent to the relations*

$$q_{i,j}^2 = 1, \qquad q_{i,j}q_{k,l} = q_{i+j-l,i+j-k}q_{i,j}, \text{ for } i \leq k < l \leq j, \qquad q_{i,j}q_{k,l} = q_{k,l}q_{i,j}, \text{ for } j < k. \quad (5.9)$$

As a consequence, we have the following group isomorphism

$$\langle t_i, i \in I |\ \text{relations in } (5.8)\rangle \simeq \langle q_{i,j}, 1 \leq i < j \leq n |\ \text{relations in } (5.9)\rangle = J_n.$$

Remark 5.21. In type A crystals, the crystal reflection operators ς_i (see [9, 44]) acting on straight-shaped Young tableaux are elements of the group $\mathcal{BK}_n$, since they can be written as $\varsigma_i := q_i t_1 q_i$, for $i \in I$. Moreover, they satisfy the relation [7, Proposition 1.4]

$$(\varsigma_i\varsigma_{i+1})^3 = q_i t_1 p_{i+1} t_1 (t_1t_2)^6 t_1 p_{i+1}^{-1} t_1 q_i \quad (5.10)$$

for $i \in [n-2]$, where $p_i := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$. Thus, the relation $(t_1t_2)^6 = 1$ is equivalent to the braid relation $(\varsigma_i\varsigma_{i+1})^3 = 1$, for all $1 \leq i \leq n-2$. It is known that the operators ς_i define an action of the symmetric group on a type A crystal (for instance, see [9, Theorem 11.14]). We shall see in Proposition 5.22 that the shifted crystal reflection operators σ_i satisfy a similar identity, but since the braid relations do not need to be satisfied by σ_i (see Example 3.31), then the relation $(\mathtt{t}_1\mathtt{t}_2)^6 = 1$ does not need to hold as well (see Example 5.23).

5.3 A shifted Berenstein–Kirillov group and a cactus group action

Motivated by the definition of the Berenstein–Kirillov group, we consider $\mathcal{SBK}$ to be the free group generated by the shifted Bender–Knuth involutions $\mathtt{t}_i$, for $i > 0$, modulo the relations they satisfy when acting on shifted semistandard tableaux of any shape. We call it the *shifted Berenstein–Kirillov group*, and consider its subgroup $\mathcal{SBK}_n$ generated by $\mathsf{t}_1, \dots, \mathsf{t}_{n-1}$. From Proposition 5.7, we know that the relations $\mathsf{t}_i^2 = 1$ and $\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i$, for $|i-j| > 1$, hold in $\mathcal{SBK}$. Recall from (5.1), that

$$\mathsf{q}_i := \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)$$

for $i \geq 1$. From Proposition 5.14, the shifted evacuation restricted to the primed interval $[1, i+1]'$, on straight-shaped shifted tableaux, is an element of $\mathcal{SBK}$, being computed by q_i. In particular, the operators q_i are involutions. We will show in Proposition 5.26 that the relation $(\mathsf{t}_i\mathsf{q}_{j,k})^2 = 1$, for $2 \leq i+1 < j < k \leq n$, which is the shifted version of (5.7) (see [10, Theorem 1.6]), also holds in $\mathcal{SBK}$.

Recall from Definition 5.10 that $\mathsf{p}_i = \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)$ and the promotion operators p_i are elements of $\mathcal{SBK}$. By Corollary 5.16, the shifted crystal reflection operators σ_i are also elements of $\mathcal{SBK}$, for $i \geq 1$, as they can be written as $\sigma_i = \mathsf{q}_i\mathsf{t}_1\mathsf{q}_i$. Following a similar computation in [7, Proposition 1.4], we show that they satisfy the following identity.

Proposition 5.22. *Let $i \in [n-2]$ and $m \in \mathbb{N}$. Then, writing $\sigma_i = \mathsf{q}_i\mathsf{t}_1\mathsf{q}_i$, we have*

$$(\sigma_i\sigma_{i+1})^m = \mathsf{q}_i\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1(\mathsf{t}_1\mathsf{t}_2)^{2m}\mathsf{t}_1\mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{q}_i. \tag{5.11}$$

Thus, in particular we have

$$(\sigma_i\sigma_{i+1})^3 = \mathsf{q}_i\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1(\mathsf{t}_1\mathsf{t}_2)^{6}\mathsf{t}_1\mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{q}_i. \tag{5.12}$$

Proof. By Corollary 5.16 and the fact that q_i is an involution, we have

$$\begin{aligned}
(\sigma_i\sigma_{i+1})^m &= (\mathsf{q}_i\mathsf{t}_1\mathsf{q}_i\mathsf{q}_{i+1}\mathsf{t}_1\mathsf{q}_{i+1})^m \\
&= (\mathsf{q}_i\mathsf{t}_1\mathsf{q}_i\mathsf{q}_i\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{q}_{i+1})^m \\
&= (\mathsf{q}_i\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{q}_{i+1})^m \\
&= (\mathsf{q}_i\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{p}_{i+1}^{-1}\mathsf{q}_i)^m \\
&= \mathsf{q}_i(\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{p}_{i+1}^{-1})^m\mathsf{q}_i \\
&= \mathsf{q}_i(\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{p}_{i+1}^{-1})^m(\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1\mathsf{t}_1\mathsf{p}_{i+1}^{-1}\mathsf{t}_1)\mathsf{q}_i \\
&= \mathsf{q}_i\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1(\mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1)^m\mathsf{t}_1\mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{q}_i.
\end{aligned}$$

To conclude the proof, we claim that $\mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{p}_{i+1}\mathsf{t}_1 = (\mathsf{t}_1\mathsf{t}_2)^2$, for any $i \geq 1$. The proof is done by induction. For $i = 1$, we have

$$\mathsf{p}_2^{-1}\mathsf{t}_1\mathsf{p}_2\mathsf{t}_1 = (\mathsf{t}_1\mathsf{t}_2)\mathsf{t}_1(\mathsf{t}_2\mathsf{t}_1)\mathsf{t}_1 = (\mathsf{t}_1\mathsf{t}_2)^2.$$

For the induction step, assume the claim holds for some $i \geq 1$. Then, due to Proposition 5.7, as $|(i+2)-1| > 1$, we have

$$\begin{aligned}
\mathsf{p}_{i+2}^{-1}\mathsf{t}_1\mathsf{p}_{i+2}\mathsf{t}_1 &= (\mathsf{t}_1 \cdots \mathsf{t}_{i+1}\mathsf{t}_{i+2})\mathsf{t}_1(\mathsf{t}_{i+2}\mathsf{t}_{i+1}\cdots \mathsf{t}_1)\mathsf{t}_1 \\
&= \mathsf{p}_{i+1}^{-1}\mathsf{t}_{i+2}\mathsf{t}_1\mathsf{t}_{i+2}\mathsf{p}_{i+1}\mathsf{t}_1 \\
&= \mathsf{p}_{i+1}^{-1}\mathsf{t}_1(\mathsf{t}_{i+2})^2\mathsf{p}_{i+1}\mathsf{t}_1 \\
&= \mathsf{p}_{i+1}^{-1}\mathsf{t}_1\mathsf{p}_{i+1} = (\mathsf{t}_1\mathsf{t}_2)^2.
\end{aligned}$$

□

Recall that the braid relations $(\sigma_i\sigma_{i+1})^3 = 1$, for $1 \leq i \leq n-2$, for the shifted crystal reflection operators do not need to hold (see Example 3.31). Thus, (5.12) ensures that the relation $(\mathsf{t}_1\mathsf{t}_2)^6 = 1$ does not need to hold either, as illustrated in Example 5.23. This will have no effects on our results, as none of the cactus group relations is equivalent to this one [10, Remark 1.9].

Example 5.23. Let $T = \begin{array}{|c|c|c|c|c|} \hline 1&1&2'&2&3 \\ \hline \end{array}$ with rows $\begin{array}{ccccc} 1&1&2'&2&3\\ &2&3'&3& \\ &&3&& \end{array}$. Then, we have

$$\begin{aligned}
T = \begin{array}{ccccc} 1&1&2'&2&3\\ &2&3'&3& \\ &&3&& \end{array} &\xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&2&2&2\\ &2&3'&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&1&1&2'\\ &2&3'&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&1&1&3'\\ &2&2&2& \\ &&3&& \end{array} \\
&\xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&1&2'&3'\\ &2&2&2& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&1&2'&3'\\ &2&3'&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&2&2&3'\\ &2&3'&3& \\ &&3&& \end{array} \\
&\xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&2'&2&3\\ &2&2&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&1&1&3\\ &2&2&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&1&1&2\\ &2&2&3'& \\ &&3&& \end{array} \\
&\xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&1&2'&2\\ &2&2&3'& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_2} \begin{array}{ccccc} 1&1&1&3'&3\\ &2&2&3& \\ &&3&& \end{array} \xrightarrow{\mathsf{t}_1} \begin{array}{ccccc} 1&1&2'&3'&3\\ &2&2&3& \\ &&3&& \end{array} = (\mathsf{t}_1\mathsf{t}_2)^6(T) \neq T.
\end{aligned}$$

Proposition 5.24. *As elements of $\mathcal{SBK}$, we have*

$$\mathsf{t}_1 = \mathsf{q}_1, \qquad \mathsf{t}_i = \mathsf{q}_{i-1}\mathsf{q}_i\mathsf{q}_{i-1}\mathsf{q}_{i-2}, \textit{ for } i \geq 2,$$

considering $\mathsf{q}_0 := 1$. Consequently, the elements $\mathsf{q}_1, \ldots, \mathsf{q}_{n-1}$ are generators of $\mathcal{SBK}_n$.

Proof. The first identity is a direct consequence of the definition of q_1. For the second one, we note that by definition of the promotion operators, we have $\mathsf{p}_i = \mathsf{t}_i\mathsf{p}_{i-1}$, for $i \geq 2$, and thus $\mathsf{t}_i = \mathsf{p}_i\mathsf{p}_{i-1}^{-1}$. It also follows from the definition that, for $i \geq 2$, $\mathsf{q}_i = \mathsf{q}_{i-1}\mathsf{p}_i$, which is equivalent to $\mathsf{p}_i = \mathsf{q}_{i-1}\mathsf{q}_i$, as q_j are involutions, for any $j \geq 1$. Then, we have

$$\mathsf{t}_i = \mathsf{p}_i\mathsf{p}_{i-1}^{-1} = \mathsf{q}_{i-1}\mathsf{q}_i(\mathsf{q}_{i-2}\mathsf{q}_{i-1})^{-1} = \mathsf{q}_{i-1}\mathsf{q}_i\mathsf{q}_{i-1}\mathsf{q}_{i-2}.$$

□

We denote, for $1 \leq i < j \leq n$,

$$\mathsf{q}_{i,j} := \mathsf{q}_{j-1}\mathsf{q}_{j-i}\mathsf{q}_{j-1}. \tag{5.13}$$

In particular, we have $\mathsf{q}_i = \mathsf{q}_{1,i+1}$. Corollary 4.2 ensures that $\mathsf{q}_{i,j}$ is realized by

$$\eta_{i,j} = \mathsf{evac}_j \mathsf{evac}_{j-i+1} \mathsf{evac}_j$$

when acting on straight-shaped shifted tableaux. As an element of the $\mathcal{SBK}$ group, the shifted Schützenberger involution restricted to the alphabet $[i, j]'$, on straight-shaped shifted tableaux, is computed by $\mathsf{q}_{i,j}$, for $1 \leq i < j \leq n$. In general, $\mathsf{q}_{i,j}$ is not realized by $\eta_{i,j}$ when acting on skew shapes (see Example 5.17).

As a consequence of the internal action of the cactus group in $\mathsf{ShST}(\nu, n)$ (Theorem 4.3), we have the following result.

Theorem 5.25. *The following map is an epimorphism, for $1 \leq i < j \leq n$.*

$$\psi : J_n \longrightarrow \mathcal{SBK}_n$$
$$s_{i,j} \longmapsto \mathsf{q}_{i,j}.$$

Hence $\mathcal{SBK}_n$ is isomorphic to $J_n / \ker \psi$.

Proof. From Proposition 5.24, $\mathcal{SBK}_n$ is generated by q_i, for $i \in I$. Then, considering that $\mathsf{q}_i = \mathsf{q}_{1,i}$ we have $\mathsf{q}_i = \psi(s_{1,i})$, and thus ψ is a surjection. Since $\mathsf{q}_i = \mathsf{evac}_{i+1}$ for straight-shaped tableaux, Theorem 4.3 then ensures that ψ is a homomorphism. Thus, $\mathcal{SBK}_n$ is isomorphic to the quotient of J_n by $\ker \psi$. $\square$

As a consequence, we are able to recover the relation (5.7) for the shifted operators. The known relations that are satisfied in $\mathcal{SBK}$ are listed below.

Proposition 5.26. *The following relations hold in $\mathcal{SBK}_n$:*

1. $\mathsf{t}_i^2 = 1$, *for* $i \in I$.

2. $\mathsf{t}_i \mathsf{t}_j = \mathsf{t}_j \mathsf{t}_i$, *for* $|i - j| > 1$.

3. $(\mathsf{t}_i \mathsf{q}_{j,k})^2 = 1$, *for* $2 \leq i + 1 < j < k \leq n$.

Proof. The first two relations correspond to Proposition 5.7. By Theorem 5.25, the action of the operator $\mathsf{q}_{j,k}$ on straight-shaped shifted tableaux defines an action of the cactus group. Thus, since $[1, 2] \cap [j, k] = \varnothing$, we have $(\mathsf{t}_i \mathsf{q}_{j,k})^2 = (\mathsf{q}_{1,2} \mathsf{q}_{j,k})^2 = 1$. $\square$

Theorem 5.20 is stated and proved in terms of group generators satisfying the relations in Proposition 5.26, and do not depend on specific operators. This ensures that the relations in Proposition 5.26 are equivalent to

$$\mathsf{q}_{i,j}^2 = 1, \qquad \mathsf{q}_{i,j}\mathsf{q}_{k,l}\mathsf{q}_{i,j} = \mathsf{q}_{i+j-l,i+j-k}, \text{ for } i \le k < l \le j, \qquad \mathsf{q}_{i,j}\mathsf{q}_{k,l} = \mathsf{q}_{k,l}\mathsf{q}_{i,j}, \text{ for } j < k.$$

Then, we have the following alternative presentation for the cactus group, via the shifted Bender–Knuth moves:

$$\begin{aligned} J_n = \langle \mathsf{t}_i,\ i \in I \mid \mathsf{t}_i^2 = 1, \mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i,\ \text{if } |i-j| > 1, \\ (\mathsf{t}_i\mathsf{q}_{k-1}\mathsf{q}_{k-j}\mathsf{q}_{k-1})^2 = 1,\ \text{for } i+1 < j < k\rangle. \end{aligned} \tag{5.14}$$

CHAPTER 6

SHIFTED GROWTH DIAGRAMS

In this chapter we recall the notion of growth diagrams for shifted standard tableaux due to Thomas and Yong [71]. We give alternative formulations for some of the algorithms presented before in the same fashion as [10], namely, the shifted *jeu de taquin*, tableau switching, evacuation and its restrictions. Using the semistandardization process of Pechenik and Yong [52], these algorithms may be applied to shifted semistandard tableaux.

Using growth diagrams, we provide an alternative proof that the cactus group J_n acts on a shifted tableau crystal $\mathsf{ShST}(\lambda/\mu, n)$ (Theorem 4.1, [54, Theorem 5.7]). This proof relies on the algorithmic description of partial Schützenberger involutions as the restrictions of the shifted reversal to primed intervals, while the one in [54, Theorem 5.7] uses the description in terms of the Schützenberger–Lusztig involutions using the shifted tableau crystal operators (see [54, Lemma 5.4]).

We remark that, unlike the case for semistandard growth diagrams for Young tableaux introduced by Chmutov, Glick and Pylyavskyy [10, Section 3], shifted semistandard tableaux, filled in a primed alphabet, are not encoded by a sequence of shape chains, as both each entry i and i' contribute the same to the weight.

6.1 Shifted *jeu de taquin* and infusion

Definition 6.1. Let T be a shifted standard tableau of shape λ/μ. Its *shape chain* is the saturated chain of strict partitions

$$\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(k)} = \lambda$$

where $k = |\lambda| - |\mu|$ and $\lambda^{(i)}$ is the shape of $T^1 \sqcup \cdots \sqcup T^i$, for $i \geq 1$. Since T is standard, each shape $\lambda^{(i)}$ has exactly one more box than $\lambda^{(i-1)}$.

The shape chain uniquely represents T. Since T is standard, $\lambda^{(i)}$ differs from $\lambda^{(i-1)}$ by exactly one box. If T is straight-shaped, then the chain starts with $\mu = \varnothing$. Moreover, the sub-chain

$$\lambda^{(i-1)} \subseteq \lambda^{(i)} \subseteq \cdots \subseteq \lambda^{(j)},$$

for $i \geq j$, encodes the tableau $T^{i,j}$. More precisely, it encodes the shifted standard tableau with the same shape as $T^{i,j}$, filled by the letters $\{1, \ldots, i-j+1\}$, but one may consider a relabelling of those letters, in order to have $T^{i,j}$.

Example 6.2. Consider the following shifted standard tableau of shape $(5,3,1)/(3,1)$,

$$T = \begin{ytableau} \; & \; & \; & 1 & 3 \\ \none & \; & 2 & 5 \\ \none & \none & 4 \end{ytableau}$$

which is represented by

$$(3,1) \subseteq (4,1) \subseteq (4,2) \subseteq (5,2) \subseteq (5,2,1) \subseteq (5,3,1).$$

Given a skew-shaped standard tableau of shape λ/μ, a sequence of slides to rectify it may be encoded by a straight-shaped standard tableau of shape μ, where the slides are performed starting on the inner corner corresponding to the largest entry.

Example 6.3. Considering the tableau of the previous example, we have the following rectification sequences (corresponding to the straight-shaped tableaux in the inner shape of T, with gray letters):

$$T = \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & \textcolor{gray}{3} & 1 & 3 \\ \none & \textcolor{gray}{4} & 2 & 5 \\ \none & \none & 4 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & \textcolor{gray}{3} & 1 & 3 \\ \none & 2 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & 1 & 3 \\ \none & 2 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & 1 & 3 & 5 \\ \none & 2 & 4 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 3 & 5 \\ \none & 4 \end{ytableau} = \mathsf{rect}(T)$$

$$T = \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & \textcolor{gray}{4} & 1 & 3 \\ \none & \textcolor{gray}{3} & 2 & 5 \\ \none & \none & 4 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & 1 & 3 \\ \none & \textcolor{gray}{3} & 2 & 5 \\ \none & \none & 4 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & \textcolor{gray}{2} & 1 & 3 \\ \none & 2 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \textcolor{gray}{1} & 1 & 3 & 5 \\ \none & 2 & 4 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 3 & 5 \\ \none & 4 \end{ytableau} = \mathsf{rect}(T).$$

The order in which the shifted *jeu de taquin* slides must be performed in these two cases is encoded by the following shape chains, respectively,

$$\varnothing \subseteq (1) \subseteq (2) \subseteq (3) \subseteq (3,1),$$

$$\varnothing \subseteq (1) \subseteq (2) \subseteq (2,1) \subseteq (3,1).$$

Each of the tableaux that appear in the intermediate steps of the rectification process may be encoded as well, thus we have the following definition.

Definition 6.4 ([71, Section 2.1])**.** A *shifted rectification growth diagram* for T a standard tableau of shape λ/μ is a table with $|\mu|$ rows and $|\lambda| - |\mu|$ columns, where the leftmost column is filled with the chain encoding a fixed rectification sequence, the top row is filled with the chain encoding T, and the subsequent rows are filled with the chain encoding the intermediate tableaux corresponding to the said rectification sequence. In particular, the bottom row will encode $\mathsf{rect}(T)$ and the rightmost column encodes the order in which the boxes were vacated during the rectification process.

The following table is a shifted rectification growth diagram for the tableau T of Example 6.2, fixing the first rectification sequence of Example 6.3. It is also convenient to display these diagrams under a rotation, as depicted in Figure 6.1.

$(3,1)$	$(4,1)$	$(4,2)$	$(5,2)$	$(5,2,1)$	$(5,3,1)$
(3)	(4)	$(4,1)$	$(5,1)$	$(5,2)$	$(5,3)$
(2)	(3)	$(3,1)$	$(4,1)$	$(4,2)$	$(4,3)$
(1)	(2)	$(2,1)$	$(3,1)$	$(3,2)$	$(4,2)$
$\varnothing$	(1)	(2)	(3)	$(3,1)$	$(4,1)$

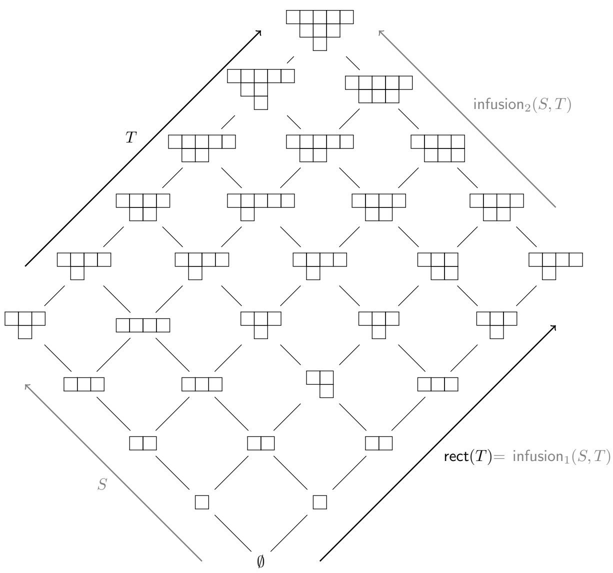


Figure 6.1: A growth diagram depicting rectification of T, according to a rectification sequence encoded by S. This may also be used to compute the type C infusion on a pair of shifted standard tableaux (S,T).

We have seen in Lemma 2.48 that the shifted tableau switching and the type C infusion maps agree on shifted standard tableaux, and both can be regarded as a sequence of shifted *jeu* *de taquin* slides. Thus, given (S, T) a pair of shifted standard tableaux, where S is a straight-shaped shifted tableau extended by T, we may place S and T on the southwestermost and northwesternmost sides of a shifted rectification growth diagram, respectively, and then the southeasternmost and northeasternmost sides will encode $\mathsf{infusion}_1(S, T)$ and $\mathsf{infusion}_2(S, T)$, respectively. Thus, the diagram in Figure 6.1 is also referred to as a *shifted infusion growth diagram*.

Example 6.5. Consider the following pair of shifted standard tableaux (these correspond to T and the first rectification sequence, as in Example 6.3):

$$(S, T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 1 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 2 & 5 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}\ .$$

This pair is encoded in the southwestern and northwestern edges of the diagram of Figure 6.1. Thus, we have

$$\mathsf{infusion}(S, T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}\ .$$

The obtained pair is encoded in the southeastern and northeastern edges of the said diagram.

Similar to the growth diagrams for standard Young tableaux, which are characterized by *local rules*, due to Fomin [67, Proposition A1.2.7], the shifted growth diagrams may also be described by similar rules.

Theorem 6.6 ([71, Theorem 2.1]). *An array of straight shapes is a shifted growth diagram if and only if for any subgrid of the form*

$$\begin{array}{ccccc} & & \lambda & & \\ & / & & \backslash & \\ \mu & & & & \mu' \\ & \backslash & & / & \\ & & \nu & & \end{array}$$

where $\nu \subseteq \mu \subseteq \lambda$ and $\nu \subseteq \mu' \subseteq \lambda$, the Fomin growth conditions hold:

1. *λ/μ, λ/μ', μ/ν and μ'/ν consist of a single box.*

2. *If μ is the unique shape that is contained in λ and contains ν, then $\mu' = \mu$.*

3. *Otherwise, there exists exactly one strict partition in the same conditions other than μ, which is μ'.*

These growth conditions exhibit a symmetry under a vertical reflection. Thus, vertically reflecting the diagram of Figure 6.1, we obtain

$$S = \mathsf{infusion}_1(\mathsf{infusion}_1(S,T), \mathsf{infusion}_2(S,T))$$
$$T = \mathsf{infusion}_2(\mathsf{infusion}_1(S,T), \mathsf{infusion}_2(S,T))$$

which explains that the infusion is an involution.

Corollary 6.7 ([71, Lemma 2.2])**.** *Let (S,T) be a pair of shifted standard tableaux, with T extending S. Then,* $\mathsf{infusion}(\mathsf{infusion}(S,T)) = (S,T)$.

6.2 Evacuation and reversal

We may obtain growth diagrams for the shifted evacuation and reversal (Section 2.4), by combining the previous diagrams and local rules. As in the previous sections, most results will be stated for shifted standard tableaux, and may be extended to the semistandard case using the semistandardization process [52]. Throughout the next sections, unless otherwise stated, we consider any standard shifted tableau to have n boxes, filled with the letters in $[n]$.

Proposition 6.8. *Let T be straight-shaped shifted standard tableau. Consider an equilateral triangular array such that the shape chain encoding T is placed on the northwestern edge and each vertex of the bottom edge is filled with $\varnothing$ and apply the local growth rules from left to right. Then, the shape chain on the northeastern edge corresponds to* $\mathsf{evac}(T)$.

Proof. Proposition 5.14 states that the evacuation of T may be obtained by applying sequentially the promotion operators $\mathsf{p}_{n-1}, \mathsf{p}_{n-2}, \dots, \mathsf{p}_1$ to T, where we recall that we are assuming that T has n boxes, filled in $[n]$. By Proposition 5.13,

$$\mathsf{p}_i(T) = \zeta_i \mathsf{SW}_{1|2,\dots,i+1}(T) = \zeta_i(\mathsf{infusion}_1(T^1, T^{2,i+1}) \sqcup \mathsf{infusion}_2(T^1, T^{2,i+1})),$$

and then each of promotion operator p_i, acting on standard tableaux, may be computed using a shifted infusion growth diagram with the southwestern edge having length 1, and northwestern edge having length i. Then, the diagram in Figure 6.3 corresponds to sequentially concatenate, from left to right, the growth diagrams of promotion operators $\mathsf{p}_{n-1}, \mathsf{p}_{n-2}, \dots, \mathsf{p}_1$, thus coinciding with $\mathsf{evac}(T)$. □

The symmetry of the local growth rules ensures that the diagram is symmetric under a vertical reflection. Thus, taking a shifted evacuation growth diagram on input $\mathsf{evac}(T)$, we obtain $\mathsf{evac}(\mathsf{evac}(T)) = T$, thus exhibiting the fact that the shifted evacuation is an involution. Since $\mathsf{evac}_i(T) := \mathsf{evac}(T^{1,i}) \sqcup T^{i+1,n}$, we have the following result.

Corollary 6.9. *Let T be a straight-shaped standard shifted tableau and let $i \in [n]$. Consider the shifted evacuation growth diagram having T as input on the northwestern edge and* $\mathsf{evac}(T)$ *on the northeastern one. Then:*

1. *Removing the $n - i$ rightmost northeastern edges of the diagram yields the shifted evacuation growth diagram on input $T^{1,i}$.*

2. *Removing the $n - j$ leftmost northwestern edges of the diagram yields the shifted evacuation growth diagram computing on input* $\mathsf{rect}(T^{n-j+1,n})$.

3. *Removing simultaneously the $n - i$ rightmost northeastern edges and the $n - j$ leftmost northwestern edges of the diagram, for $i \geq j$, yields the shifted evacuation growth diagram on input* $\mathsf{rect}(T^{i-j+1,i})$.

Example 6.10. Consider the following shifted standard tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

Then, the left side of the triangular array in Figure 6.2 corresponds to the shape chain of T, while the right side corresponds to $\mathsf{evac}(T)$. Then, we have

$$\mathsf{evac}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

Using the same diagram we also obtain restrictions of evac. For instance, removing the rightmost 3 northeastern edges (see the gray area in Figure 6.2), we have

$$\mathsf{evac}_4(T) = \mathsf{evac}(T^{1,4}) \sqcup T^{5,7} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & \ \\ \hline \multicolumn{1}{c|}{} & 3 & \ & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & \ & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 7 \\ \hline \multicolumn{1}{c|}{} & \ & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

The shifted *jeu de taquin* and shifted tableau switching are compatible with standardization. Thus, the previous characterizations with growth diagrams may be applied to a shifted semistandard tableau T, by first standardizing it, then apply the standard growth diagrams, and then compute the adequate semistandardization of the obtained tableau.

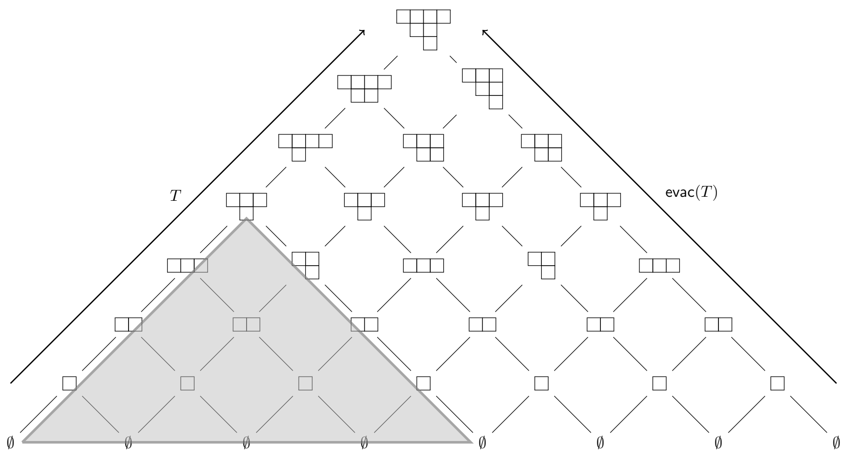


Figure 6.2: A shifted evacuation growth diagram. The smaller gray diagram computes the restriction evac_4, on $T^{1,4}$.

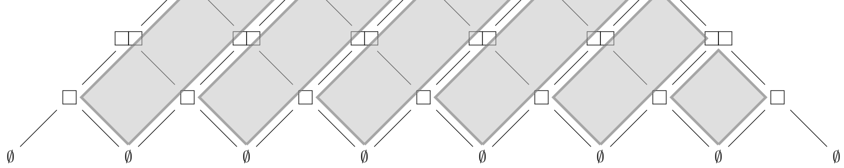


Figure 6.3: Illustration of the shifted evacuation as a composition of promotion operators, corresponding to the gray rectangles.

Example 6.11. Consider the following shifted semistandard tableau of weight $\nu = (2,2,3)$,

$$T = \begin{array}{cccc} 1 & 1 & 2' & 3' \\ & 2 & 3' & \\ & & 3 & \end{array}.$$

To compute $\mathsf{evac}(T)$ using growth diagrams, we consider its standardization and compute the growth diagram (see Example 6.10) and then apply the semistandardization with respect to $\nu' = \theta_{1,3}(\nu) = (3,2,2)$:

$$\mathsf{std}(T) = \begin{array}{cccc} 1 & 2 & 3 & 5 \\ & 4 & 6 & \\ & & 7 & \end{array} \xrightarrow{\mathsf{evac}} \begin{array}{cccc} 1 & 2 & 3 & 7 \\ & 4 & 5 & \\ & & 6 & \end{array} \xrightarrow{\mathsf{sstd}_{\nu'}} \begin{array}{cccc} 1 & 1 & 1 & 3 \\ & 2 & 2 & \\ & & 3 & \end{array} = \mathsf{evac}(T).$$

Given T a skew-shaped shifted standard tableau, Proposition 2.45 says that the reversal T^e may be computed by filling the diagram of μ with a standard tableau U, applying the shifted infusion (or shifted tableau switching) to the pair (S,T) obtaining $\mathsf{infusion}(S,T) = \big(\mathsf{rect}(T), \mathsf{infusion}_2(S,T)\big)$, applying the evacuation to $\mathsf{rect}(T)$, and then the shifted tableau switching again to the pair $\big(\mathsf{evac}(\mathsf{rect}(T)), \mathsf{infusion}_2(S,T)\big)$. Then,

$$T^e = \mathsf{infusion}_2\big(\mathsf{evac}(\mathsf{rect}(T)), \mathsf{infusion}_2(U,T)\big) \tag{6.1}$$

Thus, we have the following.

Proposition 6.12. *Let T be a shifted standard tableau of shape λ/μ. Consider a diagram as in Figure 6.4, with T on the segment $[bc]$ and any standard tableau S of shape μ on the segment $[ab]$[1], and such that $[dc] = [df]$. Then, the segment $[gf]$ encodes T^e.*

Proof. The diagram $[abcd]$ computes the shifted tableau switching on the pair (S,T), thus $[ad]$ encodes $\mathsf{infusion}_1(S,T) = \mathsf{rect}(T)$ and $[dc]$ encodes $\mathsf{infusion}_2(S,T)$. By Proposition 6.8, the diagram $[ade]$ computes the evacuation with input $[ad]$, thus the segment $[ed]$ corresponds to $\mathsf{evac}(\mathsf{rect}(T))$. Finally, since $[df] = [dc]$, the diagram $[edfg]$ computes the shifted tableau switching on the pair $\big(\mathsf{evac}(\mathsf{rect}(T)), \mathsf{infusion}_2(S,T)\big)$. It then follows from (6.1) that $[gf]$ corresponds to T^e. $\square$

Proposition 6.13. *Given T a shifted semistandard tableau of shape λ/μ and weight ν, its reversal T^e may be obtained in the following way:*

1. *Standardize T and fill the diagram of μ with a standard tableau S and add $|\lambda| - |\mu|$ to each entry.*

[1] We consider a segment $[ab]$ to be directed, from a to b. In the growth diagrams, segments are read from bottom to top.

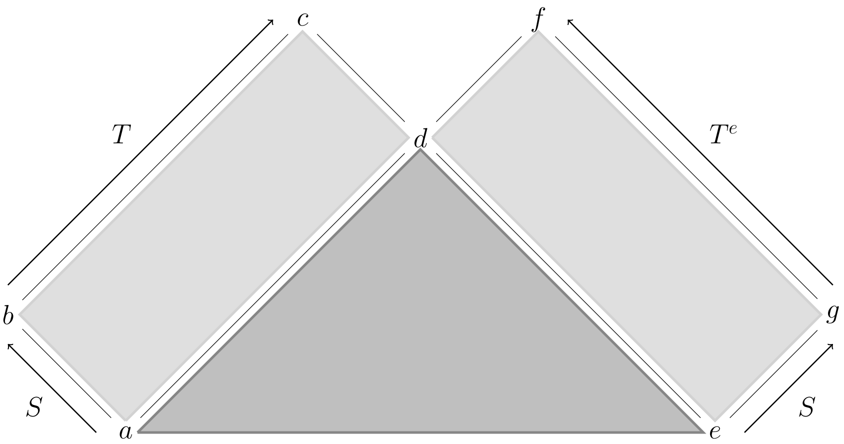


Figure 6.4: Growth diagram to compute the shifted reversal on skew shapes. By construction, we put $[dc] = [df]$.

2. *Perform the shifted infusion on the pair* (S, T).

3. *Reflect the obtained tableau along the anti-diagonal of the ambient triangle* $\delta = (\lambda_1, \lambda_1 - 1, \ldots, 1)$, *while complementing in* $\{1, \ldots, |\lambda|\}$.

4. *Apply rectification.*

5. *Let* T' *be the tableau corresponding to the boxes filled in* $\{|\mu| + 1, |\mu| + 2, \ldots, |\lambda|\}$, *and subtract* $|\mu|$ *to each entry. Then, putting* $\nu' = \theta_{1,n}(\nu)$, *we have* $T^e = \mathsf{sstd}_{\nu'}(T')$.

Example 6.14. Consider the same skew-shaped tableau of Example 2.34, of shape λ/μ, with $\lambda = (5, 3, 2, 1)$ and $\mu = (3, 1)$, and weight $\nu = (4, 2, 1)$:

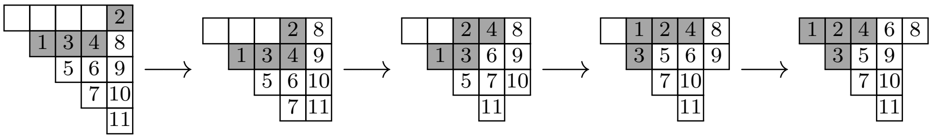
$$T = \begin{array}{ccccc} \square & \square & \square & 1' & 1 \\ & \square & 1 & 1 & \\ & & 2 & 2 & \\ & & & 3 & \end{array} \longrightarrow \begin{array}{ccccc} \square & \square & \square & 1 & 4 \\ & \square & 2 & 3 & \\ & & 5 & 6 & \\ & & & 7 & \end{array} = \mathsf{std}(T).$$

Following the procedure on Proposition 6.13, we have

$$\begin{array}{ccccc} \square & \square & \square & 1 & 4 \\ & \square & 2 & 3 & \\ & & 5 & 6 & \\ & & & 7 & \end{array} \longrightarrow \begin{array}{ccccc} 8 & 9 & 10 & 1 & 4 \\ & 11 & 2 & 3 & \\ & & 5 & 6 & \\ & & & 7 & \end{array} \longrightarrow \begin{array}{ccccc} 8 & 9 & 10 & 1 & 4 \\ & 2 & 3 & 6 & \\ & & 5 & 7 & \\ & & & 11 & \end{array} \longrightarrow \begin{array}{ccccc} 8 & 9 & 1 & 4 & 10 \\ & 2 & 3 & 6 & \\ & & 5 & 7 & \\ & & & 11 & \end{array}$$

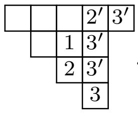
$$\longrightarrow \begin{array}{ccccc} 8 & 1 & 3 & 4 & 10 \\ & 2 & 5 & 6 & \\ & & 7 & 9 & \\ & & & 11 & \end{array} \longrightarrow \begin{array}{ccccc} 1 & 2 & 3 & 4 & 10 \\ & 5 & 6 & 8 & \\ & & 7 & 9 & \\ & & & 11 & \end{array}$$

Reflecting and complementing in the ambient triangle, and then rectifying, we obtain T' (in white boxes).



$= T'$

Finally, semistandardizing T' with respect to $\nu' = (1, 2, 4)$, we have

$T^e =$  .

Proof of Proposition 6.13. Without loss of generality, we consider T a shifted standard tableau. Consider the diagram in Figure 6.5, where the segment $[ab]$ corresponds to S and $[bc]$ to T. Then, the segment $[ad]$ encodes $\mathsf{infusion}_1(S, T) = \mathsf{rect}(T)$. Then, place $[dc]$, which corresponds to $\mathsf{infusion}_2(S, T)$, in $[df]$, and put $[fv]$ to encode any standard tableau U of shape $\lambda^\vee$ (this tableau will encode a rectification sequence, to apply after reflection). We remark that $[av]$ encodes a tableau of shape δ (the ambient staircase triangle), thus its reflection coincides with the evacuation, which is then encoded in $[pv]$. Finally, rectification is achieved by applying *jeu de taquin* growth diagram with rectification sequence determined by letters corresponding to U (on the reflected tableau) on $[ws] := [wv]$, thus obtaining a standard tableau encoded by $[pqt]$. Considering only the letters corresponding to T, the obtained tableau (before semistandardization) is encoded in $[qt]$. By Proposition 6.12, T^e is encoded by $[gf]$. Recall that the rectification process is independent from the rectification sequence. Thus, since we put $[wv] = [ws]$, we have $[hf] = [pt]$. Similarly, $[wr] = [wu]$ implies that $[hg] = [pq]$. Hence, $[gf] = [qt]$, which concludes the proof. $\square$

6.3 Partial Schützenberger involutions

Following the same approach as in [10, Section 4.1], we may use the shifted growth diagrams for rectification and evacuation to construct an array that computes $\eta_{i,j}$ for straight-shaped shifted tableaux. From (5.13) and Proposition 5.14, and since $\eta_{i,j}$ is computed by $\mathsf{q}_{i,j}$ when acting on straight shapes, we have $\eta_{i,j}(T) = \mathsf{q}_{i,j}(T)$, for $T \in \mathsf{ShST}(\nu, n)$, thus the next growth diagram computes $\mathsf{q}_{i,j}$ as well. From Definition 3.22, given $T \in \mathsf{ShST}(\lambda/\mu, n)$, then $\eta_{i,j}(T) = T^{1,i-1} \sqcup \eta(T^{i,j}) \sqcup T^{j+1,n} = \eta_{i,j}(T^{1,j}) \sqcup T^{j+1,n}$.

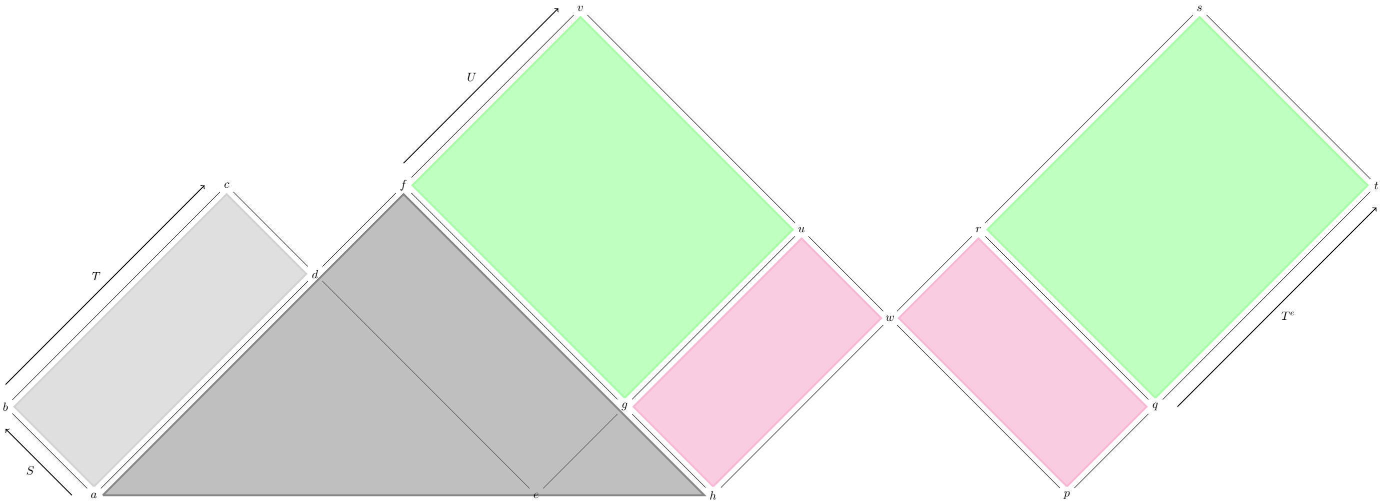


Figure 6.5: Another method to compute the reversal of a shifted standard tableau T, of shape λ/μ. The tableaux S and U are any standard tableaux of shapes μ and $\lambda^\vee$, respectively. If T is straight-shaped, the diagram consists only of the dark gray triangle $[ade]$, together with the green rectangles, with the segment $[fg]$ now being adjacent to $[de]$ and $u = r$.

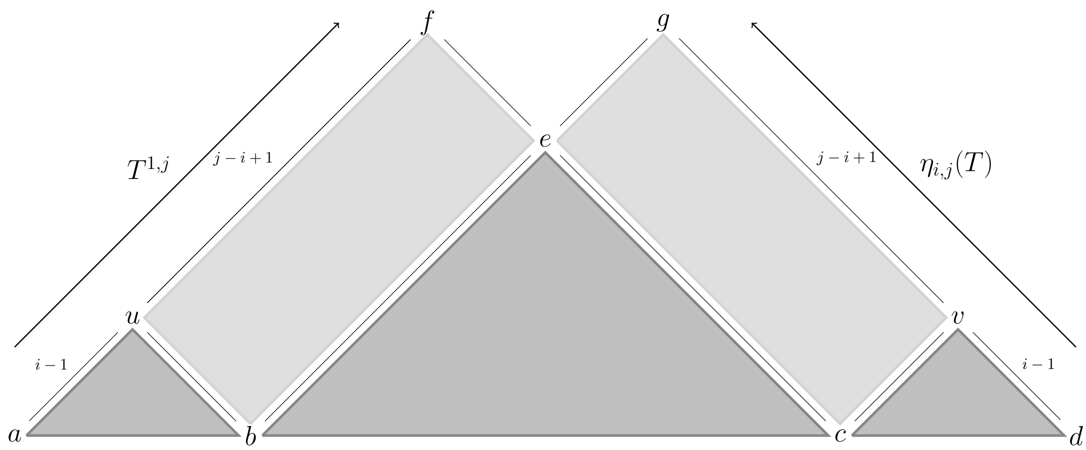


Figure 6.6: The growth diagram to compute $\eta_{i,j}$ or $\mathsf{q}_{i,j}$ on straight-shaped tableaux [10, Figure 6]. By construction, $[ef] = [eg]$.

Proposition 6.15. *Let $1 \leq i < j \leq n$ and T be a straight-shaped shifted standard tableau filled in $[n]$. Consider the diagram in Figure 6.6, which consists, from left to right, in the growth diagrams of evac_{i-1}, infusion, evac_{j-i+1}, infusion, and evac_{i-1}, and such that the segments $[ef]$ and $[eg]$ coincide. Then, if the segment $[af]$ encodes $T^{1,j}$, then the segment $[dg]$ encodes $\eta_{i,j}(T^{1,j})$.*

Proof. We will show that $[dv] = T^{1,i-1} \sqcup \eta(T^{i,j})$. We have $[af] = T^{1,j}$, $[au] = T^{1,i-1}$ and $[uf] = T^{i,j}$, thus, by Proposition 6.8, $[bu] = \mathsf{evac}(T^{1,i-1}) =: S$. Applying the shifted infusion growth diagram on inputs $[bu]$ and $[uf]$, we have

$$\begin{aligned} [be] &= \mathsf{infusion}_1(S, T^{i,j}) = \mathsf{rect}(T^{i,j}) \\ [ef] &= \mathsf{infusion}_2(S, T^{i,j}) = [eg], \end{aligned} \tag{6.2}$$

and by Corollary 6.9, applying the shifted evacuation growth diagram, we have

$$[ce] = \mathsf{evac}(\mathsf{rect}(T^{i,j})). \tag{6.3}$$

Then, applying the shifted infusion growth diagram on inputs $[eg]$ (6.2) and $[ce]$ (6.3), we obtain

$$\begin{aligned} [cv] &= \mathsf{infusion}_1\big(\mathsf{evac}(\mathsf{rect}(T^{i,j})), \mathsf{infusion}_2(S, T^{i,j})\big) \\ [vg] &= \mathsf{infusion}_2\big(\mathsf{evac}(\mathsf{rect}(T^{i,j})), \mathsf{infusion}_2(S, T^{i,j})\big). \end{aligned} \tag{6.4}$$

By (6.1), we have

$$[vg] = \eta(T^{i,j}). \tag{6.5}$$

We recall that $\mathsf{infusion}_1(S, T) = \mathsf{rect}(T)$, for any standard straight-shaped tableau S extended by T. Considering that rectification does not depend on the chosen rectification sequence, from (6.4), we have

$$\begin{aligned} [cv] &= \mathsf{infusion}_1\big(\mathsf{evac}(\mathsf{rect}(T^{i,j})), \mathsf{infusion}_2(S, T^{i,j})\big) \\ &= \mathsf{infusion}_1\big(\mathsf{evac}(\mathsf{infusion}_1(S, T^{i,j})), \mathsf{infusion}_2(S, T^{i,j})\big) \\ &= \mathsf{rect}\big(\mathsf{infusion}_2(S, T^{i,j})\big) \\ &= \mathsf{infusion}_1\big(\mathsf{infusion}_1(S, T^{i,j}), \mathsf{infusion}_2(S, T^{i,j})\big) \\ &= \mathsf{infusion}_1\big(\mathsf{infusion}(S, T^{i,j})\big) = S. \end{aligned}$$

Finally, the shifted evacuation growth diagram ensures that

$$[dv] = \mathsf{evac}(S) = \mathsf{evac}^2(T^{1,i-1}) = T^{1,i-1}. \tag{6.6}$$

Thus, by (6.4) and (6.6), we have

$$[dg] = T^{1,i-1} \sqcup \eta(T^{i,j}) = \eta_{i,j}(T^{1,j}). \tag{6.7}$$

$\square$

Using Proposition 6.12, and considering that $\eta_{i,j}$ commutes with the shifted *jeu de taquin*, we may generalize the previous growth diagram for skew-shaped tableaux. We remark this generalization is not valid for $\mathsf{q}_{i,j}$, as it does not commute with the shifted *jeu de taquin*.

Corollary 6.16. *Let $1 \le i < j \le n$ and let T be a skew-shaped shifted standard tableau of shape λ/μ. Consider the diagram on Figure 6.7, where the segment $[pr]$ encodes $T^{1,j}$, S is any standard tableau of shape μ, being encoded by $[ap]$, and the segments $[er]$ and $[es]$ coincide. Then, $\eta_{i,j}(T^{1,j})$ is encoded by segment $[ws]$.*

Proof. Since $[pr] = T^{1,j}$ and $[ap] = S$, then

$$\begin{aligned} [fr] &= \mathsf{infusion}_2(S, T^{1,j}) \\ [af] &= \mathsf{infusion}_1(S, T^{1,j}) = \mathsf{rect}(T^{1,j}) = (\mathsf{rect}(T))^{1,j}. \end{aligned} \tag{6.8}$$

By Proposition 6.15, the segment $[dg]$ encodes $\eta_{i,j}((\mathsf{rect}(T))^{1,j})$. By construction, $[er] = [es]$, and thus $[gs] = [fr] = \mathsf{infusion}_2(S, T^{1,j})$. Then, considering the shifted infusion growth diagram on inputs $[ap]$ and $[pr]$,

$$\begin{aligned} [ws] &= \mathsf{infusion}_2(\eta_{i,j}((\mathsf{rect}(T))^{1,j}), \mathsf{infusion}_2(S, T^{1,j})) \\ [dw] &= \mathsf{infusion}_1(\eta_{i,j}((\mathsf{rect}(T))^{1,j}), \mathsf{infusion}_2(S, T^{1,j})) \end{aligned} \tag{6.9}$$

Since $\eta_{i,j}$ commutes with the shifted *jeu de taquin*, in particular we have

$$\eta_{i,j}((\mathsf{rect}(T))^{1,j}) = \eta_{i,j}(\mathsf{rect}(T^{1,j})) = \mathsf{rect}(\eta_{i,j}(T^{1,j})) \tag{6.10}$$

Moreover, the operator $\eta_{i,j}$ preserves shifted dual equivalence, and thus $T^{1,j}$ and $\eta_{i,j}(T^{1,j})$ are in the same shifted dual equivalence class. Then, by Proposition 2.41,

$$\mathsf{infusion}_2(S, T^{1,j}) = \mathsf{infusion}_2(S, \eta_{i,j}(T^{1,j})). \tag{6.11}$$

Then, by (6.9), (6.10) and (6.11), and since infusion is an involution, we have

$$\begin{aligned} [ws] &= \mathsf{infusion}_2(\eta_{i,j}((\mathsf{rect}(T))^{1,j}), \mathsf{infusion}_2(S, T^{1,j})) \\ &= \mathsf{infusion}_2(\eta_{i,j}((\mathsf{rect}(T))^{1,j}), \mathsf{infusion}_2(S, \eta_{i,j}(T^{1,j}))) \\ &= \mathsf{infusion}_2(\mathsf{rect}(\eta_{i,j}(T^{1,j})), \mathsf{infusion}_2(S, \eta_{i,j}(T^{1,j}))) \\ &= \mathsf{infusion}_2(\mathsf{infusion}_1(S, \eta_{i,j}(T^{1,j})), \mathsf{infusion}_2(S, \eta_{i,j}(T^{1,j}))) \\ &= \eta_{i,j}(T^{1,j}). \end{aligned}$$

□

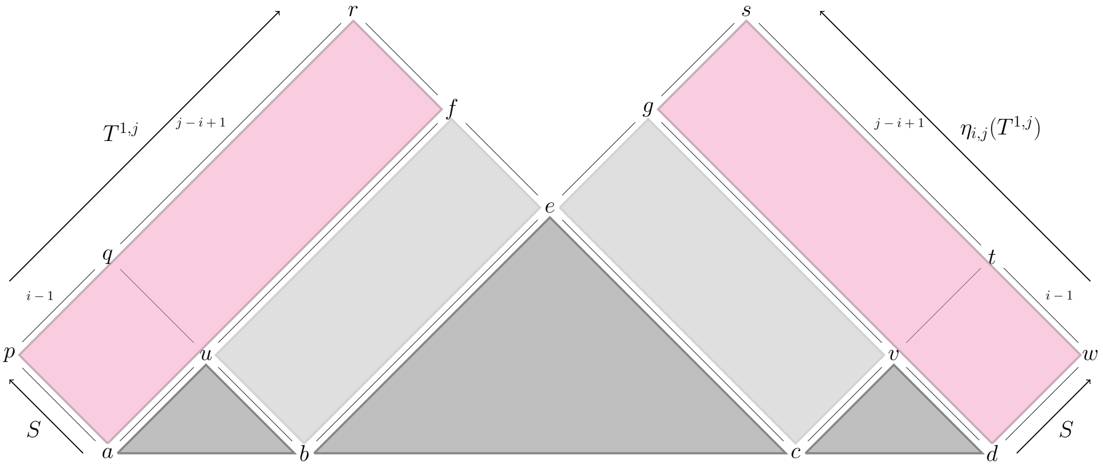


Figure 6.7: A growth diagram to compute $\eta_{i,j}$ on shifted standard tableaux of shape λ/μ, with S being any standard tableau of shape μ. By construction, $[er] = [es]$. A diagram to compute $\mathsf{q}_{i,j}$ on straight-shaped shifted standard tableaux is obtained by removing the pink sections.

As before, the growth diagrams for $\eta_{i,j}$ may be used on a shifted semistandard tableau T, with weight ν. Since we have

$$\mathsf{std}(\eta_{i,j}(T)) = \eta_{k,l}(\mathsf{std}(T)), \tag{6.12}$$

where $k := \min \mathcal{P}_i(\nu)$ and $l := \max \mathcal{P}_j(\nu)$, we may standardize T, apply $\eta_{k,l}$, and then apply the semistandardization (see Definition 2.11) with respect to ν' to the obtained tableau, with $\nu' = \theta_{i,j}(\nu)$, that is,

$$\eta_{i,j}(T) = \mathsf{sstd}_{\nu'}\big(\eta_{k,l}(\mathsf{std}(T))\big). \tag{6.13}$$

Example 6.17. Consider the following shifted semistandard tableau of weight $\nu = (2, 2, 3)$,

$$T = \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 3' \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 3 \\ \cline{3-4} \end{array}.$$

To compute $\eta_{2,3}(T)$, we use the growth diagram in Figure 6.8 on the standardization of T, followed by rectification, using the rectification sequence encoded by $S = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$.

$$(S,T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 3' \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 3 \\ \cline{3-4} \end{array} \xrightarrow{\mathsf{std}} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & 3 & 5 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & 7 \\ \cline{3-4} \end{array} \xrightarrow{\mathsf{infusion}} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 1 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & 2 \\ \cline{3-4} \end{array} = (\mathsf{rect}(\mathsf{std}(T)), S')$$

where $S' := \mathsf{infusion}_2(S,T)$. In the Figure 6.7, $\mathsf{rect}(\mathsf{std}(T))$ corresponds to the segment $[af]$ and S' to $[fr]$. Then, by (2.3), we have

$$\mathcal{P}_2(\nu) = \{3, 4\} \qquad \mathcal{P}_3(\nu) = \{5, 6, 7\}.$$

Thus, to obtain $\eta_{2,3}(T)$, we must apply $\eta_{3,7}$ to $\mathsf{rect}(\mathsf{std}(T))$. Note that the $\eta_{3,7}\big(\mathsf{rect}(\mathsf{std}(T))\big)$ is encoded in the segment corresponding to $[dg]$ in Figure 6.8.

$$\mathsf{rect}(\mathsf{std}(T)) = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ & 3 & 5 & \\ & & 6 & \end{array} \xrightarrow{\eta_{3,7}} \begin{array}{cccc} 1 & 2 & 3 & 6 \\ & 4 & 5 & \\ & & 7 & \end{array} = \eta_{3,7}\big(\mathsf{rect}(\mathsf{std}(T))\big) =: T'.$$

Then, we apply the shifted infusion growth diagram (the rightmost pink region, in Figure 6.8), to recover the skew shape before the rectification:

$$(T', S') = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ & 4 & 5 & \boxed{1} \\ & & 7 & \boxed{2} \end{array} \xrightarrow{\mathsf{infusion}} \begin{array}{cccc} \boxed{1} & \boxed{2} & 2 & 3 \\ & 1 & 4 & 6 \\ & & 5 & 7 \end{array} = \mathsf{infusion}(T', S').$$

This corresponds to the tableau of the segment $[ws]$. Finally, we apply the semistandardization with respect to ν', where $\nu' = \theta_{2,3}(2,2,3) = (2,3,2)$:

$$\mathsf{infusion}_2(T', S') = \begin{array}{cccc} \square & \square & 2 & 3 \\ & 1 & 4 & 6 \\ & & 5 & 7 \end{array} \xrightarrow{\mathsf{sstd}_{\nu'}} \begin{array}{cccc} \square & \square & 1 & 2' \\ & 1 & 2' & 3' \\ & & 2 & 3 \end{array} = \eta_{2,3}(T).$$

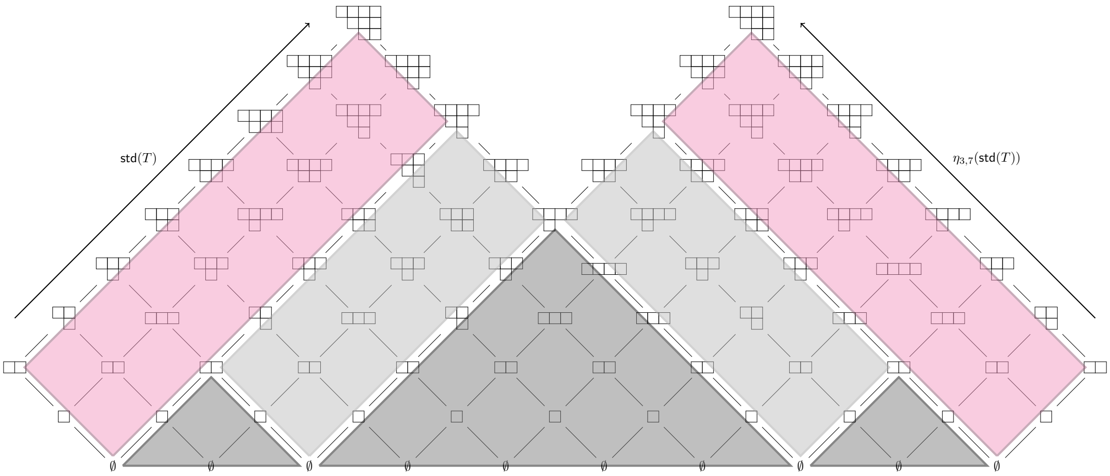


Figure 6.8: A growth diagram to compute $\eta_{3,7}$ on a skew-shaped tableau. A diagram to compute $\mathsf{q}_{3,7}$ on straight-shaped shifted standard tableaux is obtained by removing the pink sections.

6.3.1 Another proof of Theorem 4.1

.

The shifted growth diagrams may be used to obtain an alternative proof to Theorem 4.1, which then implies Theorem 4.3 and Theorem 5.25, similarly to the one presented by Chmutov, Glick and Pylyavskyy [10, Theorem 1.4]. The proof is done for shifted standard tableaux, and may be generalized for the semistandard case using (6.12). More precisely, we will consider

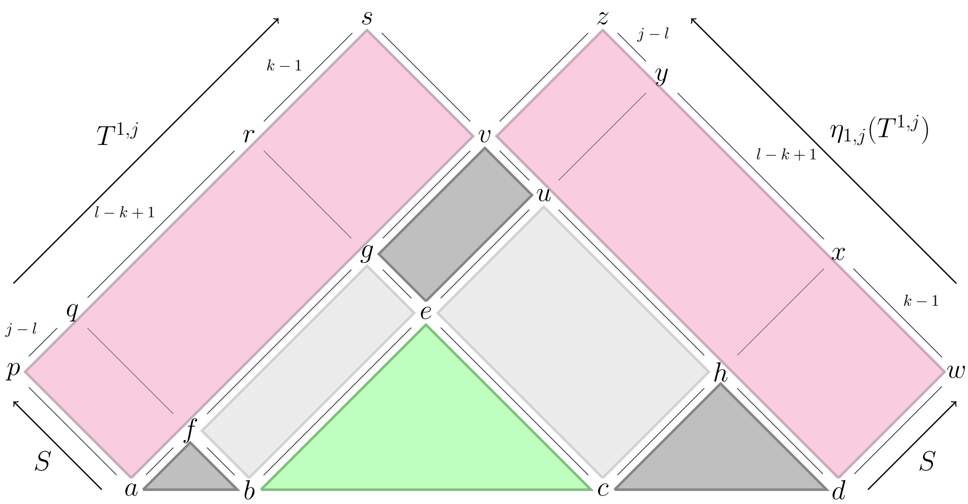


Figure 6.9: A growth diagram with input $T^{1,j}$, a shifted standard tableau of shape λ/μ, which is encoded on the segment $[ps]$, and having $\eta_{1,j}(T^{1,j})$ encoded on segment $[wz]$, with S being any standard tableau of shape μ. By construction, $[vs] = [vz]$. The corresponding diagram with primed vertices has $\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j})$ on the segment $[p's']$ and $\eta_{k,l}\eta_{1,j}(T^{1,j})$ on $[w'z']$.

the diagram in Figure 6.9, to prove that the partial Schützenberger involutions satisfy the third cactus relation (recall Definition 1.1),

$$\eta_{i,j}\eta_{k,l} = \eta_{i+j-l,i+j-k}\eta_{i,j}, \quad \text{for } [k,l] \subseteq [i,j],$$

when acting on shifted standard tableaux.

Proof of Theorem 4.1 [54, Theorem 5.7]. The relations $\eta_{i,j}^2 = 1$ and $\eta_{i,j}\eta_{k,l} = \eta_{k,l}\eta_{i,j}$, for $[k,l] \cap [i,j] = \varnothing$, are trivial, thus it remains to show that $\eta_{i,j}\eta_{k,l} = \eta_{i+j-l,i+j-k}\eta_{i,j}$, for $[k,l] \subseteq [i,j]$. By Lemma 4.4, it suffices to show that

$$\eta_{1,j}\eta_{k,l} = \eta_{j-l+1,j-k+1}\eta_{1,j},$$

for any $[k,l] \subseteq [1,j]$. We will now prove this relation, using growth diagrams. Let T be a standard tableau of shape λ/μ, and consider the diagram in Figure 6.9, where the segment $[ap]$ encodes a fixed standard tableau S of shape μ, $[ps]$ encodes $T^{1,j}$, $[av]$ encodes $\mathsf{rect}(T^{1,j}) = (\mathsf{rect}(T))^{1,j}$, $[dv]$ encodes $\eta_{1,j}(\mathsf{rect}(T^{1,j}))$ and $[wz]$ encodes $\eta_{1,j}(T^{1,j})$. Consider also another growth diagram similar to this one, with the vertices labelled as $\{a', b', c', \ldots\}$, with the segment $[a'p']$ encoding the same S as before, $[p's']$ encoding $T' := \eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j})$ and $[w'z']$ encoding $\eta_{1,j}(T') = \eta_{k,l}\eta_{i,j}(T^{1,j})$. The proof then mimics the one in [10, Theorem 1.4]. Since $[ps]$ and

$[p's']$ encode $T^{1,j}$ and $\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j})$, respectively, we have

$$\begin{aligned} [av] &= \mathsf{rect}(T^{1,j}) \\ [a'v'] &= \mathsf{rect}(\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j})). \end{aligned} \tag{6.14}$$

Taking the shifted evacuation growth diagrams, for $\eta_{1,j}$, with the inputs in (6.14), which correspond to $\eta_{1,j}$, and considering that the operators $\eta_{i,j}$ are coplactic, we have

$$\begin{aligned} [dv] &= \eta_{1,j}(\mathsf{rect}(T^{1,j})) \\ [d'v'] &= \eta_{1,j}(\mathsf{rect}(\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j}))) = \eta_{k,l}\eta_{1,j}(\mathsf{rect}(T^{1,j})). \end{aligned} \tag{6.15}$$

Thus, in particular, $[d'v'] = \eta_{k,l}[dv]$. Since $[dv] = [dh] \sqcup [hu] \sqcup [uv]$ and $[d'v'] = [d'h'] \sqcup [h'u'] \sqcup [u'v']$, by definition of $\eta_{k,l}$ we have

$$\eta_{k,l}([dv]) = [dh] \sqcup \eta([hu]) \sqcup [uv] = [d'v'],$$

and consequently

$$\begin{aligned} &[dh] = [d'h'], [uv] = [u'v'], \\ &[hu] = \eta([h'u']) \end{aligned} \tag{6.16}$$

Since $[dh] = [d'h']$, taking the shifted evacuation growth diagrams on those inputs yield

$$[ch] = [c'h']. \tag{6.17}$$

From (6.16) and (6.17), considering shifted infusion growth diagrams, we have

$$[ce] = \mathsf{infusion}_1([ch], [hu]) = \mathsf{infusion}_1([c'h'], \eta([h'u']))$$

and by Corollary 2.42,

$$\mathsf{infusion}_1([c'h'], \eta([h'u'])) = \eta(\mathsf{infusion}_1([c'h'], [h'u'])) = \eta([c'e'])$$

and thus

$$[ce] = \eta([c'e']). \tag{6.18}$$

Considering the same shifted infusion growth diagrams, we have

$$[eu] = \mathsf{infusion}_2([ch], [hu]) = \mathsf{infusion}_2([c'h'], \eta([h'u'])),$$

and by Proposition 2.41, as $[h'u']$ is shifted dual equivalent to $\eta([h'u'])$, we have

$$\mathsf{infusion}_2([c'h'], \eta([h'u'])) = \mathsf{infusion}_2([c'h'], [h'u']) = [e'u'],$$

and thus

$$[eu] = [e'u']. \tag{6.19}$$

Considering now the shifted infusion growth diagrams on inputs $[eu]$ and $[uv]$, and on inputs $[e'u']$ and $[u'v']$, respectively, from (6.16) and (6.19), we have

$$[eg] = [e'g'], [gv] = [g'v']. \tag{6.20}$$

Then, considering the shifted evacuation growth diagrams, on inputs $[be]$ and $[b'e']$, respectively, we have, from (6.18),

$$\eta([be]) = [ce] = \eta([c'e']) = [b'e'],$$

and thus

$$\eta([be]) = [b'e']. \tag{6.21}$$

We now consider the shifted infusion growth diagrams on inputs $[be]$ and $[eg]$, and on inputs $[b'e']$ and $[e'g']$, respectively. Then, by Proposition 2.41, since $\eta([be])$ is shifted dual equivalent to $[be]$, we have

$$[bf] = \mathsf{infusion}_1([be], [eg]) = \mathsf{infusion}_1(\eta([be]), [eg]),$$

and by (6.20) and (6.21),

$$\mathsf{infusion}_1(\eta([be]), [eg]) = \mathsf{infusion}_1([b'e'], [e'g']) = [b'f'],$$

and consequently

$$[bf] = [b'f']. \tag{6.22}$$

Finally, taking the shifted evacuation growth diagram with inputs in (6.22), we get

$$[af] = [a'f']. \tag{6.23}$$

By (6.20) and (6.22), we have $[gv] = [g'v']$ and $[af] = [a'f']$. Thus, $\mathsf{rect}(T^{1,j})$ agrees with $\mathsf{rect}(\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j}))$ on the letters outside of $[j-l+1, j-k+1]$ and may differ on the segments $[fg]$ and $[f'g']$. Considering the shifted infusion diagram on inputs $[be]$ and $[eg]$, and on inputs $[b'e']$ and $[e'g']$, respectively, by (6.20) and (6.21), we have

$$[f'g'] = \mathsf{infusion}_2([b'e'], [e'g']) = \mathsf{infusion}_2(\eta([be]), [eg]),$$

and by Corollary 2.42, we have

$$\mathsf{infusion}_2(\eta([be]), [eg]) = \eta(\mathsf{infusion}_2([be], [eg])) = \eta([fg]),$$

and thus,

$$\eta([fg]) = [f'g']. \tag{6.24}$$

Then, from the definition of $\eta_{j-l+1,j-k+1}$ and the fact that it is coplactic, we have

$$\begin{aligned}
\mathsf{rect}(\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j})) &= [a'v'] = [a'f'] \sqcup [f'g'] \sqcup [g'v'] \\
&= [af] \sqcup \eta([fg]) \sqcup [gv] && \text{by (6.20), (6.23) and (6.24)} \\
&= \eta_{j-l+1,j-k+1}([av]) \\
&= \eta_{j-l+1,j-k+1}(\mathsf{rect}(T^{1,j})) \\
&= \mathsf{rect}(\eta_{j-l+1,j-k+1}(T^{1,j})).
\end{aligned}$$

It remains to show that the segments $[ps]$ and $[p's']$ differ only on $[qr]$ and $[q'r']$, and that $\eta([qr]) = [q'r']$. We have $[pq] = T^{1,j-l}$. By the definition $\eta_{1,j}\eta_{k,l}\eta_{1,j}$ and since $j - l \leq j$, we have

$$\begin{aligned}
[p'q'] &= (\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j}))^{1,j-l} \\
&= \eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j-l}) \\
&= \eta_{1,j}\eta_{k,l}\eta_{1,j}([pq]).
\end{aligned}$$

We recall that, by construction, $[ap] = [a'p'] = S$. Since the partial Schützenberger involutions preserve shifted dual equivalence, $[pq]$ is shifted dual equivalent to $[p'q']$, and thus, by Proposition 2.41, we have

$$[fq] = \mathsf{infusion}_2([ap], [pq]) = \mathsf{infusion}_2([a'p'], [p'q']) = [f'q'],$$

that is,

$$[fq] = [f'q']. \tag{6.25}$$

Then, by (6.23) and (6.25),

$$[pq] = \mathsf{infusion}_2([af], [fq]) = \mathsf{infusion}_2([a'f'], [f'q']) = [p'q'],$$

and thus

$$[pq] = [p'q']. \tag{6.26}$$

Since $[p's'] = \eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j}) = \eta_{1,j}\eta_{k,l}\eta_{1,j}([ps])$, and the partial Schützenberger involutions preserve shifted dual equivalence, then $[ps]$ is shifted dual equivalent to $[p's']$. Then, Proposition 2.41 and the fact that $[ap] = [a'p']$ ensure that

$$[vs] = \mathsf{infusion}_2([ap], [ps]) = \mathsf{infusion}_2([a'p'], [p's']) = [v's'],$$

that is,

$$[vs] = [v's']. \tag{6.27}$$

Then, by (6.20) and (6.27),

$$[rs] = \mathsf{infusion}_2([gv], [vs]) = \mathsf{infusion}_2([g'v'], [v's']) = [r's']$$

and then,

$$[rs] = [r's']. \tag{6.28}$$

From (6.20) and (6.27) we also conclude that

$$[gr] = \mathsf{infusion}_1([gv], [vs]) = \mathsf{infusion}_1([g'v'], [v's']) = [g'r']$$

and thus, by (6.20), we have

$$[er] = [eq] \sqcup [gr] = [e'q'] \sqcup [g'r'] = [e'r']. \tag{6.29}$$

Then, by (6.21) and (6.29), we have

$$[q'r'] = \mathsf{infusion}_2([b'e'], [e'r']) = \mathsf{infusion}_2(\eta([be]), [er])$$

and by Corollary 2.42,

$$\mathsf{infusion}_2(\eta([be]), [er]) = \eta(\mathsf{infusion}_2([be], [er])) = \eta([qr]),$$

and then

$$\eta([qr]) = [q'r']. \tag{6.30}$$

To conclude the proof, we remark that by the definition of $\eta_{j-l+1,j-k+1}$, we have

$$\begin{aligned}
\eta_{1,j}\eta_{k,l}\eta_{1,j}(T^{1,j}) &= [p's'] = [p'q'] \sqcup [q'r'] \sqcup [r's'] \\
&= [pq] \sqcup \eta([qr]) \sqcup [rs] && \text{by (6.26), (6.28) and (6.30)} \\
&= \eta_{j-l+1,j-k+1}([ps]) \\
&= \eta_{j-l+1,j-k+1}(T^{1,j}).
\end{aligned}$$

□

FINAL REMARKS

Shifted crystal reflection operators. We have seen that the shifted crystal reflection operators σ_i do not need to satisfy the braid relations of $\mathfrak{S}_n$, $(\sigma_i \sigma_{i+1})^3 = 1$. As a consequence, the action ϕ of the cactus group on $\mathsf{ShST}(\lambda/\mu, n)$ (Theorem 4.1) does not factor through the braid relations, i.e., the subgroup $\{(s_{i,i+1}, s_{i+1,i+2})^{3m}, i \in [n-2], m \in \mathbb{Z}\}$ is not contained in $\ker \phi$. As stated in Remark 3.33, the group $G_n := \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ is not isomorphic to $\mathfrak{S}_n$. In the future, it would be interesting to determine if other relations are satisfied here, besides the ones listed in Proposition 3.29. It is also an open question in type A whether $\langle \varsigma_1, \dots, \varsigma_{n-1} \rangle$ is isomorphic to $\mathfrak{S}_n$ or whether there are other relations satisfied by the crystal reflection operators.

Proposition 5.22 shows that a possible relation of the form $(\sigma_i \sigma_{i+1})^m = 1$, for $m > 3$, is equivalent to a relation $(\mathsf{t}_1 \mathsf{t}_2)^{2m} = 1$ satisfied by the shifted Bender–Knuth operators.

Shifted Berenstein–Kirillov group. Berenstein and Kirillov have also showed in [8] that the Berenstein–Kirillov group is isomorphic to a quotient of the cactus group. This was done independently of the work of Chmutov, Glick and Pylyavskyy [10]. Moreover, the Berenstein–Kirillov group $\widetilde{\mathcal{BK}_n}$ considered in [8] differs from the one considered here, being defined as the free group generated by $t_1, \dots, t_{n-1}$, subject only to the relations

1. $(t_i)^2 = 1$,
2. $(t_1 t_2)^6 = 1$,
3. $t_i t_j = t_j t_i$, for $|i - j| > 1$,
4. $(t_1 q_i)^4 = 1$, for $i > 2$.

Thus, besides concluding that $\widetilde{\mathcal{BK}_n}$ is isomorphic to a quotient $J_n / \ker \tilde{\phi}$ of the cactus group, where $\tilde{\phi}$ is an epimorphism from J_n to $\widetilde{\mathcal{BK}_n}$, this quotient is completely described as $\ker \tilde{\phi}$ is

$\{(s_{1,2}s_{1,3})^{6m}, m \in \mathbb{Z}\}$, the normal subgroup of J_n generated by $(s_{1,2}s_{1,3})^6$. For the Berenstein–Kirillov group presented in [10], considering $\psi : s_{i,j} \mapsto q_{i,j}$, one concludes that $\ker\psi$, must contain $\{(s_{1,2}s_{1,3})^{6m}, m \in \mathbb{Z}\}$, as it follows from a relation holding on $\mathcal{BK}_n$ that is not equivalent to any relation of the cactus group. But since a comprehensive set of relations for $\mathcal{BK}_n$ is not known, it could be the case that there would be other relations not following from the cactus group.

For the shifted case, we have seen that the relation $(\mathsf{t}_1\mathsf{t}_2)^6 = 1$ does not need to hold. However, fixing a shifted tableau crystal $\mathsf{ShST}(\nu, n)$, which is finite, there must exist some $m > 6$ such that $(\mathsf{t}_1\mathsf{t}_2)^m(T) = T$, for all $T \in \mathsf{ShST}(\nu, n)$. Previous computations suggested that if there exists $r \in \mathbb{Z}_{>0}$ such that $(\sigma_1\sigma_2)^r = 1$, then $r \geq 90$ [54, Appendix A]. Thus, if there exists m such that $(\mathsf{t}_1\mathsf{t}_1)^m = 1$, for any shape ν, Proposition 5.22 implies that $m \geq 180$. We do not know if there exists such m valid for any shifted tableau crystal.

Thus, considering the epimorphism ψ between J_n and $\mathcal{SBK}_n$ of Theorem 5.25, an explicit element of the kernel $\ker\psi$ is not known, although we can state that the kernel does not contain $\{(s_{1,2}s_{1,3})^{6m}, m \in \mathbb{Z}\}$. Proposition 5.22 shows that the study of $\ker\psi$ is closely related to the study of the action of the shifted crystal reflection operators σ_i on $\mathsf{ShST}(\nu, n)$. For future work, it would also be interesting to find whether there are other relations that are satisfied in $\mathcal{SBK}_n$ that do not follow from the cactus group relations. We refer to [6, Problem 1.7] for similar problems.

APPENDIX A

ADDITIONAL EXAMPLES

In this appendix we present some additional examples of the shifted tableau crystal $\mathsf{ShST}(\nu, n)$, for $n = 3$. Shifted tableaux will be enumerated according to their list enumeration in SageMath (with a slight modification so that the enumeration starts with 1). We remark the shifted tableaux used in SageMath are the ones generating Schur P-functions [1], not the ones in [23]. We recall that these shifted tableaux are not required to be in canonical form and there are no primed entries on the main diagonal. However, we remark that both definitions coincide for shifted tableaux of straight shape ν that are filled with $[n]'$, where $n := \ell(\nu)$, as this ensures that the first occurrence of each letter i or i' appears on the main diagonal, thus being unprimed in canonical form. For instance, the tableau $T \in \mathsf{ShST}(\nu, 3)$, with $\nu = (5, 3, 1)$ (see Figure A.2), in Example 3.31 is T_{61}.

We have seen in Example 3.31 that $(\sigma_1\sigma_2)^3(T_{61}) \neq T_{61}$. However, we have that $(\sigma_1\sigma_2)^9(T_{61}) = T_{61}$. If we set $m_i := \min\{m : (\sigma_1\sigma_2)^m(T_i) = T_i\}$, for $i \in [|\mathsf{ShST}(\nu, 3)|] = [64]$, then, we have the following in $\mathsf{ShST}(\nu, n)$:

$$m_i = \begin{cases} 3 & \text{for } i \in \{1, 2, 3, 9, 10, 11, 15, 18, 19, 22, 23, 27, 28, 31, 37, 38, 44, 47\} \\ 5 & \text{for } i \in \{6, 13, 25, 33, 42, 50, 54, 58, 62, 64\} \\ 9 & \text{otherwise} \end{cases}$$

Therefore, taking $m = \operatorname{lcm}(3, 5, 9) = 45$, we have that $(\sigma_1\sigma_2)^m(T) = T$, for all $T \in \mathsf{ShST}(\nu_1, 3)$.

Similarly, for $\nu_2 = (5, 2, 1)$ (see Figure A.1), we have that, for all $i \in [|\mathsf{ShST}(\nu_2, 3)|] = [48]$,

$$m_i = \begin{cases} 3 & \text{for } i \in \{1,2,3,4,5,7,10,11,12,13,14,15,16,19,20, \\ & \qquad 21,23,24,25,26,30,31,32,33,35,36,39,40,42,45\} \\ 9 & \text{otherwise} \end{cases}$$

Hence, putting $m = \mathrm{lcm}(3,9) = 9$, we have that $(\sigma_1\sigma_2)^m(T) = T$ for all $T \in \mathsf{ShST}(\nu_2, 3)$. The following table summarizes these and other computations we did. We remark that Lemma 3.38 ensures that the effect of σ_i on $\mathsf{rect}(T^{i,i+1})$ (which has, at most, two rows) does not depend on the first diagonals, except for one, with two elements. Thus, it suffices to check strict partitions whose last part is equal to one. This means that the results obtained for $(3,2,1)$ are the same for $(3+k, 2+k, 1+k)$, for $k \geq 1$.

ν	$\lvert\mathsf{ShST}(\nu,3)\rvert$	least m such that $(\sigma_1\sigma_2)^m(T) = T$ for all $T \in \mathsf{ShST}(\nu,3)$
$(3,2,1)$	8	3
$(4,2,1)$	24	3
$(4,3,1)$	24	3
$(5,2,1)$	48	9
$(5,3,1)$	64	45
$(5,4,1)$	48	9
$(6,2,1)$	80	18
$(6,3,1)$	120	18
$(6,4,1)$	120	18

Given ν a strict partition, since $\mathsf{ShST}(\nu,3)$ is finite, there exists a $m > 3$ such that

$$(\sigma_1\sigma_2)^m(T) = T,$$

for all $T \in \mathsf{ShST}(\nu,3)$. These computations show that, if there exits an m such that $(\sigma_1\sigma_2)^m = 1$, for any ν, then it should be greater or equal to $\mathrm{lcm}(3,9,18,45) = 90$. However, an upper bound for any ν is not known.

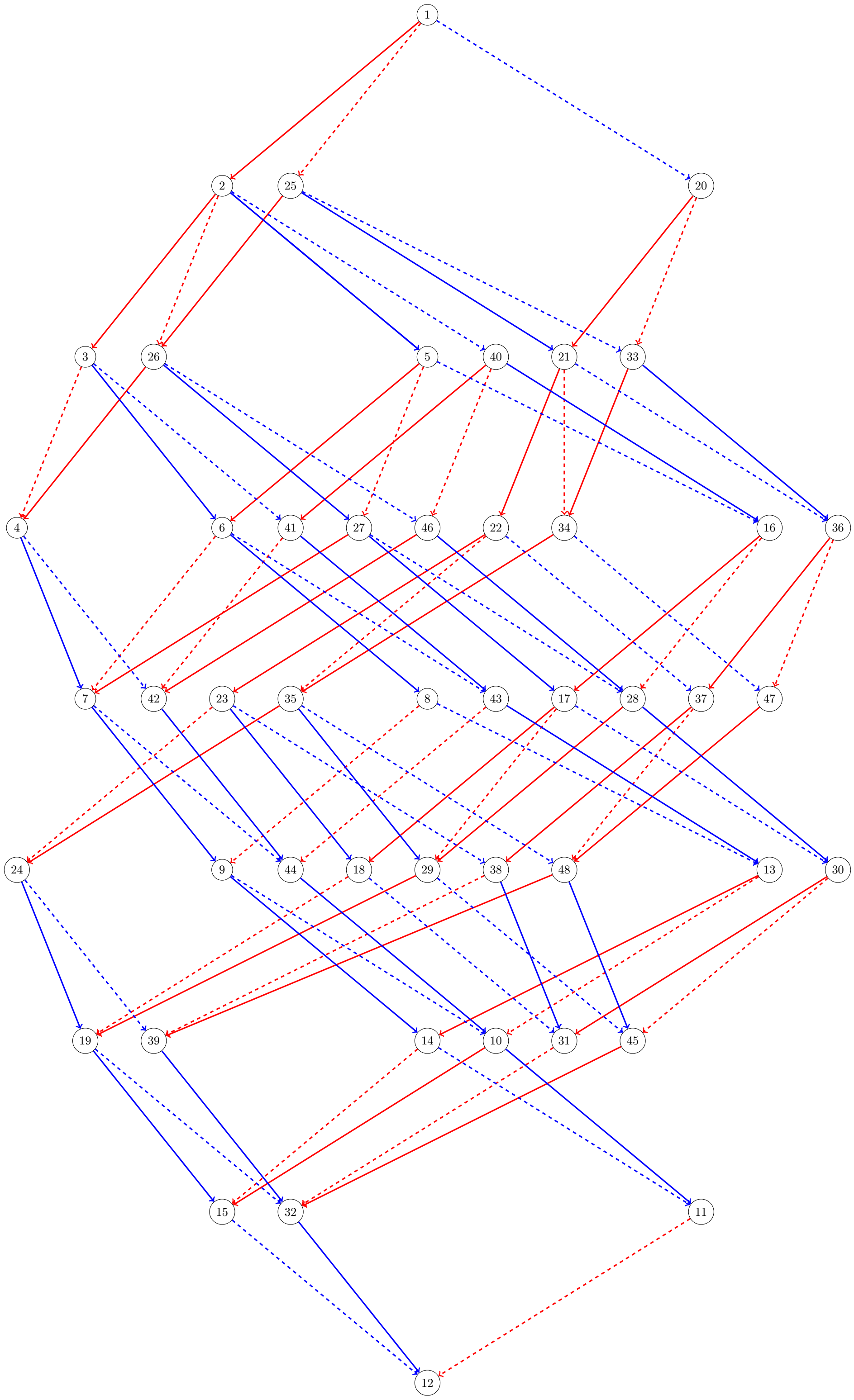


Figure A.1: Shifted tableau crystal graph $\mathsf{ShST}(\nu, 3)$, with $\nu = (5, 2, 1)$. The operators F_1, F_1' are in red and the F_2, F_2' are in blue. Vertices with the same weight are grouped together.

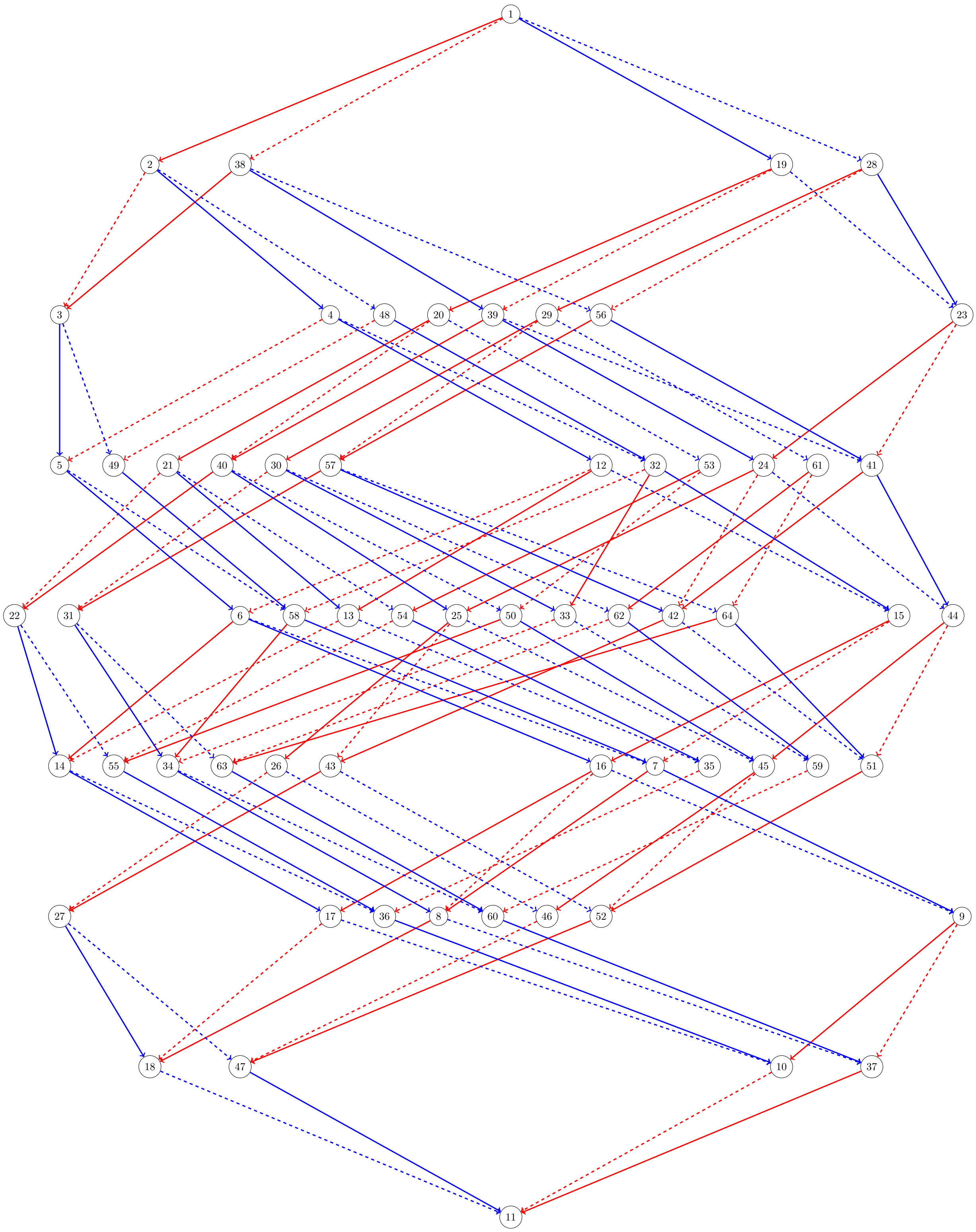


Figure A.2: Shifted tableau crystal graph $\mathsf{ShST}(\nu, 3)$, with $\nu = (5, 3, 1)$. The operators F_1, F_1' are in red and the F_2, F_2' are in blue. Vertices with the same weight are grouped together.

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