

Linear time equivalent Littlewood-Richardson coefficient symmetry maps

O. Azenhas, A. Conflitti, R. Mamede

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Littlewood-Richardson number symmetries

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- Littlewood-Richardson coefficients $c_{\mu\nu\lambda}$ are invariant under the action of $\mathbb{Z}_2 \oplus S_3$ as follows: the non-identity element of \mathbb{Z}_2 transposes simultaneously μ , ν and λ , and S_3 permutes μ , ν and λ

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- Purbhoo has defined the operation *migration* on mosaics a sort of *jeu de taquin* moves on puzzles.

Linear time reductions

- Let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map. δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit-size of A .

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 - ▶ A map β is an α -based ps-circuit \square if there is a parallel sequential algorithm which uses only a finite number of linear cost maps and a finite number of application of map α .

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 - ▶ A map β is *linearly reducible* to α , write $\beta \hookrightarrow \alpha$, if there exist a finite α -based ps-circuit \sqsupseteq which computes β . We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β , and β is linearly reducible to α .

Linear reduction of LR-symmetry maps

- **Pak-Vallejo Theorem** The following maps are linearly equivalent:
 - (1) RSK correspondence.
 - (2) Jeu de taquin map.
 - (3) Littlewood–Robinson map.
 - (4) Tableau switching map s .
 - (5) Schützenberger involution E for normal shapes.
 - (6) Reversal e .
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Pak-Vallejo's question

- The conjugation symmetry on LR tableaux is any bijection

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- ϱ^{WHS} , ϱ^{BSS} and ϱ^{AZ} are identical.

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LR-coefficient conjugation symmetry map is linearly reducible to the Schützenberger involution

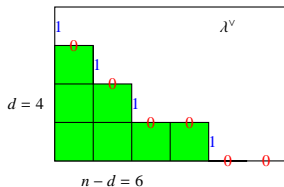
Partitions

- Fix positive integers $0 < d < n$ and consider a $d \times (n - d)$ rectangle.

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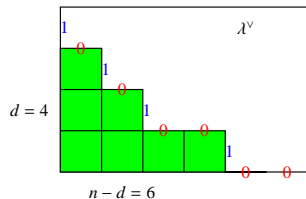
- Fix positive integers $0 < d < n$ and consider a $d \times (n - d)$ rectangle.
- $d = 4$ $n = 10$



$$\lambda = (4, 2, 1, 0) \leftrightarrow 0010010101$$

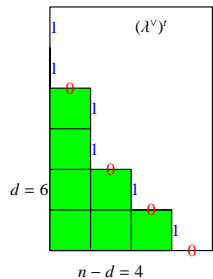
$$\lambda^v = (6, 5, 4, 2) \leftrightarrow 1010100100$$

Conjugate partitions



$$\lambda = (4, 2, 1, 0) \leftrightarrow 0010010101$$

$$\lambda^v = (6, 5, 4, 2) \leftrightarrow 1010100100$$



$$\lambda^t = (3, 2, 1, 1, 0, 0) \quad 0101011011$$

$$(\lambda^v)^t = (4, 4, 3, 3, 2, 1) \quad 1101101010$$

Littlewood-Richardson rule

- $c_{\mu \nu}^{\lambda}$ is the number of semistandard Young tableaux with shape λ^{\vee} / μ and content ν , with the following property:
 - ▶ If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top, at any stage, the number of i 's encountered is at least as large as the number of $(i + 1)$'s encountered, $\#1\text{'s} \geq \#2\text{'s} \dots$

2	3	3	λ		
μ	1	2			
		1	1	1	1

$$\nu = (5, 3, 2)$$

Benkart-Sottile-Stroomer bijection ϱ^{BSS}



$$\begin{array}{ccc} \varrho^{BSS} : LR(\mu, \nu, \lambda) & \longrightarrow & LR(\mu^t, \nu^t, \lambda^t) \\ T & \mapsto & \varrho(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} \end{array}$$

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- ▶ Facts: [Haiman] Consider two equivalence relations on a pair of tableaux. Two tableaux are Knuth equivalent if one can be obtained from the other by a sequence of (reverse) *jeu de taquin slides*. They are dual Knuth equivalent if such a (any) sequence results in tableaux of the same shape.

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- ▶ Tableaux of the same (anti) normal shape are dual equivalent. A pair of tableaux that are both Knuth and dual Knuth equivalent must be equal. If \mathcal{D} is a dual Knuth equivalence class and \mathcal{K} is a Knuth equivalence class, both corresponding to the same straight shape. Then, there is a *unique* tableau in $\mathcal{D} \cap \mathcal{K}$.

Q^{BSS} bijection

- $LR(\mu \nu \lambda) \mapsto LR(\mu^t \lambda^t \nu^t)$

$$T = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 3 & \\ \hline & 2 & \\ \hline & & 1 \\ \hline \end{array} \longrightarrow \hat{T} = \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 1 & 4 & \\ \hline & 3 & \\ \hline & & 2 \\ \hline \end{array} \longrightarrow \hat{T}^t = \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 3 & 4 & \\ \hline & & 1 & 5 \\ \hline \end{array} \longrightarrow$$

$$\hat{T}^t \cup Y(\lambda^t)^a = \begin{array}{|c|c|c|c|} \hline 2 & \bar{1} & \bar{1} & \bar{2} \\ \hline & 3 & 4 & \bar{1} \\ \hline & & 1 & 5 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline & \bar{1} & \bar{2} & \bar{1} \\ \hline & & \bar{1} & \bar{1} \\ \hline \end{array} = Z \cup \hat{Y}(\nu^t)^a \longrightarrow$$

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$$Z \cup Y(\nu^t)^a = \begin{array}{|c|c|c|c|} \hline 1 & \bar{1} & \bar{1} & 2 \\ \hline & \bar{1} & \bar{2} & \bar{1} \\ \hline & & \bar{1} & \bar{1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 1 & \bar{1} & \bar{1} & \bar{2} \\ \hline & 1 & 2 & \bar{1} \\ \hline & & 1 & 1 \\ \hline \end{array} = \varrho^{BSS}(T) \cup Y(\lambda^t)$$

Bijection ϱ^{AZ}

- $LR(\mu \nu \lambda) \mapsto LR(\lambda \nu \mu)$

$$\begin{array}{ccccc} LR(\mu, \nu, \lambda) & \xrightarrow{e} & LR(\mu, \nu^*, \lambda) & \xrightarrow{\bullet} & LR(\lambda, \nu, \mu) \\ \underset{T}{\longrightarrow} & & \underset{T^e}{\longrightarrow} & & \underset{T^{e\bullet}}{\longrightarrow} \end{array}$$

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$$\begin{array}{ccc} LR(\mu, \nu, \lambda) & \xrightarrow{e} & LR(\mu, \nu^*, \lambda) & \xrightarrow{\bullet} & LR(\lambda, \nu, \mu) \\ T & \longrightarrow & T^e & \longrightarrow & T^{e\bullet} \end{array}$$

- $LR(\mu \nu \lambda) \rightarrow LR(\lambda^t \nu^t \mu^t)$

$$\begin{array}{ccc} LR(\mu, \nu, \lambda) & \xrightarrow{\blacklozenge} & LR(\lambda^t, \nu^t, \mu^t) \\ T & \longrightarrow & T^{\blacklozenge} \end{array}$$

Bijection ϱ^{AZ}

- $LR(\mu \nu \lambda) \mapsto LR(\lambda \nu \mu)$

$$LR(\mu, \nu, \lambda) \xrightarrow[T \longrightarrow]{e} LR(\mu, \nu^*, \lambda) \xrightarrow[T^e \longrightarrow]{\bullet} LR(\lambda, \nu, \mu)$$

- $LR(\mu \nu \lambda) \rightarrow LR(\lambda^t \nu^t \mu^t)$

$$LR(\mu, \nu, \lambda) \xrightarrow[T \longrightarrow]{\blacklozenge} LR(\lambda^t, \nu^t, \mu^t)$$

-

$$\varrho^{AZ} : LR(\mu, \nu, \lambda) \xrightarrow[T \longrightarrow]{e} LR(\mu, \nu^*, \lambda) \xrightarrow[T^e \longrightarrow]{\bullet} LR(\lambda, \nu, \mu) \xrightarrow[T^{e\bullet} \longrightarrow]{\blacklozenge} LR(\mu^t, \nu^t, \lambda^t)$$

Bijection \blacklozenge

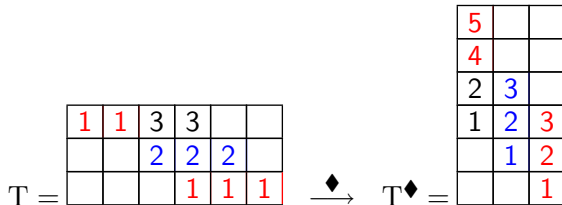
- $LR(\mu, \nu, \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t, \nu^t, \mu^t);$

Bijection \blacklozenge

- $LR(\mu, \nu, \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t, \nu^t, \mu^t);$
- $c_{\mu \nu \lambda} = c_{\lambda^t \nu^t \mu^t}$

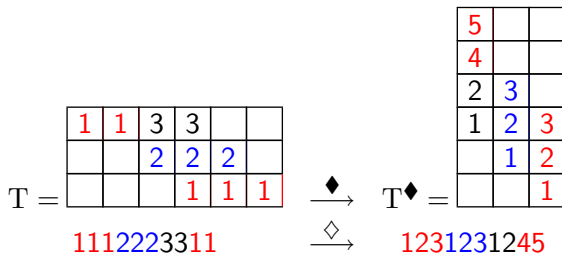
Bijection \blacklozenge

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-



Bijection \blacklozenge

- $LR(\mu, \nu, \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t, \nu^t, \mu^t)$;
- $c_{\mu \nu \lambda} = c_{\lambda^t \nu^t \mu^t}$
-



Complexity of bijection ♦

Algorithm (Bijection ♦.)

Input: LR tableau T of skew shape λ/μ , with $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$,
 $\mu = (\mu_1 \geq \dots \geq \mu_n)$, and filling $\nu = (\nu_1 \geq \dots \geq \nu_n)$, having $A = (a_{i,j}) \in M_{n \times n}(\mathbb{N})$ ($a_{i,j} = 0$
if $j > i$) as (lower triangular) recording matrix.

Write \tilde{A} , a copy of the matrix A .

For $j := n$ down to 2 do

 For $i := 1$ to n do

 Begin

 If $i = j$ then $\tilde{a}_{i,i} := \tilde{a}_{i,i} + \lambda_1 - \lambda_i$

 else

 If $j > i$ then $\tilde{a}_{i,j} = 0$ else $\tilde{a}_{i,j} := \tilde{a}_{i,j} + \tilde{a}_{i,j+1}$.

 End

So far the computational cost is $O(n^2) = O(\langle A \rangle)$.

Remark: For all $1 \leq i \leq n$ and $0 \leq j \leq n - i + 1$, we have

$$\tilde{a}_{i+j+1,i} - \tilde{a}_{i+j,i} \geq a_{i+j+1,i}.$$

Complexity of bijection ♦ continued

Algorithm (Bijection ♦.)

Set a matrix $B = (b_{i,j}) \in M_{\lambda_1 \times \lambda_1}(\mathbb{N})$ such that $b_{i,j} = 0$ for all i, j .

For $i := 1$ to n do

 Begin

 Set $c := 0$.

 For $j := 0$ to n do

 Begin

$r := \tilde{a}_{i+j,i} - a_{i+j,i}$.

 For $t := 1$ to $a_{i+j,i}$ do $b_{r+t,c+t} := b_{r+t,c+t} + 1$.

$c := c + a_{i+j,i}$.

 End

 End

This part has total computational cost at most equal to

$$O\left(\sum_{1 \leq i, j \leq n} a_{i,j}\right) = O(|\lambda \setminus \mu|) = O(|\lambda| - |\mu|) = O(\langle T \rangle).$$

Output: B recording matrix of the output tableau.

Relative Complexity of map $\varrho = \varrho^{AZ} = \varrho^{BSS} = \varrho^{WHS}$

Theorem The conjugation symmetry maps ϱ^{BSS} , ϱ^{WHS} and ϱ^{AZ} are identical, and linear equivalent to the Schützenberger involution E ,

$$\begin{array}{ccccc}
 T & \xleftrightarrow{e \bullet} & Te \bullet & \xleftrightarrow{\blacklozenge} & Te \bullet \blacklozenge \\
 \tau \updownarrow & & \tau \updownarrow & & \\
 P & \xleftrightarrow[\text{evacuation}]{E} & PE & &
 \end{array}$$

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 \tau \updownarrow & & \tau \updownarrow & & \\
 P & \xleftrightarrow[\text{evacuation}]{E} & PE & &
 \end{array}$$

Word of $T^{e \bullet \blacklozenge} = (\sigma_0 w)^{* \blacklozenge}$.

$$\sigma_0 = s_1 s_2 s_1$$

$$\begin{aligned}
 w = 11(1(12)2)(1332) &\longrightarrow 22(1(12)2)(1332) \longrightarrow 2211(2(213)3)2 \longrightarrow 3311(2(213)3)3 \\
 &\longrightarrow 33(1(12)2)1333 \longrightarrow \sigma_0 w = 3311222333
 \end{aligned}$$

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 &\xrightarrow{*} 1112223311
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 w = 11(1(12)2)(1332) &\longrightarrow 22(1(12)2)(1332) \longrightarrow 2211(2(213)3)2 \longrightarrow 3311(2(213)3)3 \\
 &\longrightarrow 33(1(12)2)1333 \longrightarrow \sigma_0 w = 3311222333 \\
 &\xrightarrow{*} 1112223311 \xrightarrow{\blacklozenge} 1231231245.
 \end{aligned}$$

The $\mathbb{Z}_2 \oplus \underline{S}_3$ -symmetries are linearly equivalent modulus the fundamental symmetry

$$c_{\mu \nu \lambda} = c_{\lambda \mu \nu} = c_{\nu \lambda \mu}$$

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 c_{\nu \mu \lambda} &= c_{\mu \lambda \nu} = c_{\lambda \nu \mu}
 \end{aligned}$$

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 & \quad c_{\mu \nu \lambda} &= c_{\nu \mu \lambda} \\
 c_{\nu \mu \lambda} &= c_{\mu \lambda \nu} = c_{\lambda \nu \mu} \\
 & &= c_{\mu^t \nu^t \lambda^t}
 \end{aligned}$$

Contents

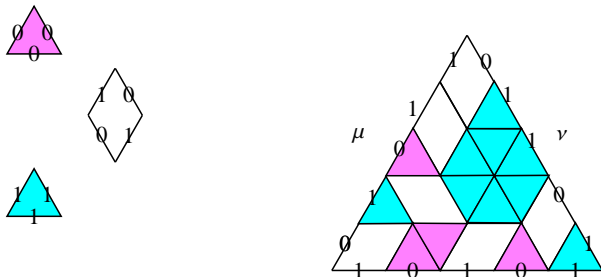
- 1 Reduction of LR-symmetry maps: An outline
- 2 LR-coefficient conjugation symmetry map is linearly reducible to the Schützenberger involution/fundamental symmetry
- 3 LR-tableaux, Knutson-Tao-Woodward puzzles, and Purbhoo mosaics: conjugation symmetry maps coincide

Puzzle rule

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length.
 - ▶ Puzzle pieces may be rotated in any orientation *but not reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.

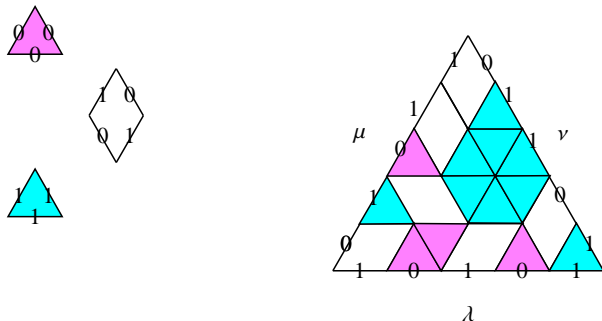
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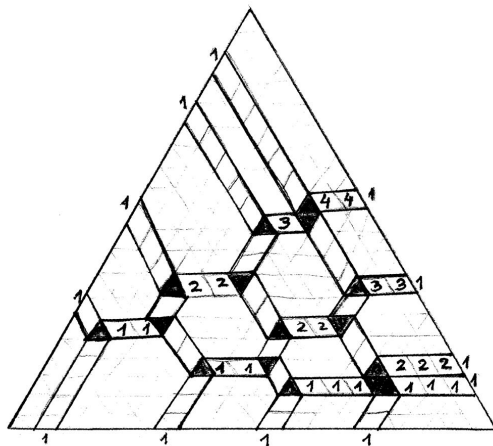


Puzzle rule

- (Knutson-Tao-Woodward) $c_{\mu \nu \lambda}$ is the number of puzzles with μ , ν and λ appearing clockwise as 01-strings along the boundary.



A bijection between Puzzles and LR tableaux: Tao's bijection

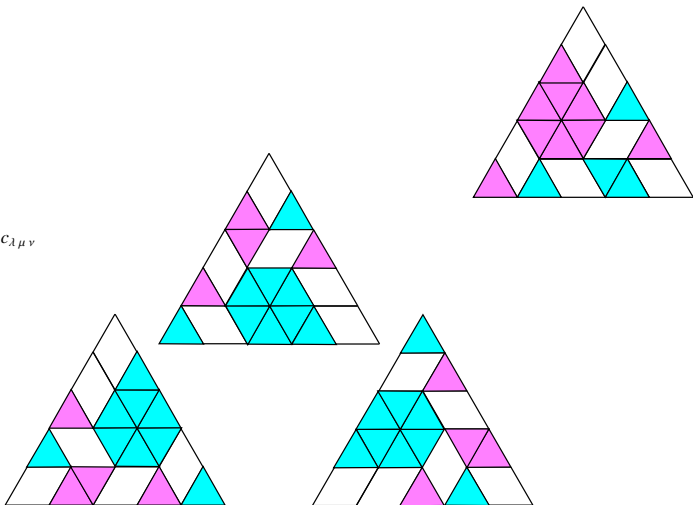


1	1	2	2	3	4	4									
				1	1	2	2	3	3						
							1	1	1	2	2	2			
											1	1	1		

Rotation and reflection

- $c_{\mu \nu \lambda} = c_{\lambda \mu \nu} = c_{\nu \lambda \mu}$
- $c_{\mu \nu \lambda} = c_{\nu^t \mu^t \lambda^t} = c_{\lambda^t \nu^t \mu^t} = c_{\mu^t \lambda^t \nu^t}$

$$c_{\mu \nu \lambda} = c_{\nu \lambda \mu} = c_{\lambda \mu \nu}$$



- ϱ^{BSS} = rotation+reflection+fundamental symmetry
- $C_{\mu \nu \lambda} = C_{\mu^t \lambda^t \nu^t}$ $C_{\mu^t \lambda^t \nu^t} = C_{\mu^t \nu^t \lambda^t}$

Q^{BSS} bijection

- $LR(\mu \nu \lambda) \mapsto LR(\mu^t \lambda^t \nu^t)$

$$T = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 3 & \\ \hline & 2 & \\ \hline & & 1 \\ \hline \end{array} \longrightarrow \hat{T} = \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 1 & 4 & \\ \hline & 3 & \\ \hline & & 2 \\ \hline \end{array} \longrightarrow \hat{T}^t = \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 3 & 4 & \\ \hline & & 1 & 5 \\ \hline \end{array} \longrightarrow$$

$$\hat{T}^t \cup Y(\lambda^t)^a = \begin{array}{|c|c|c|c|} \hline 2 & \bar{1} & \bar{1} & \bar{2} \\ \hline & 3 & 4 & \bar{1} \\ \hline & & 1 & 5 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline & \bar{1} & \bar{2} & \bar{1} \\ \hline & & \bar{1} & \bar{1} \\ \hline \end{array} = Z \cup \hat{Y}(\nu^t)^a \longrightarrow$$

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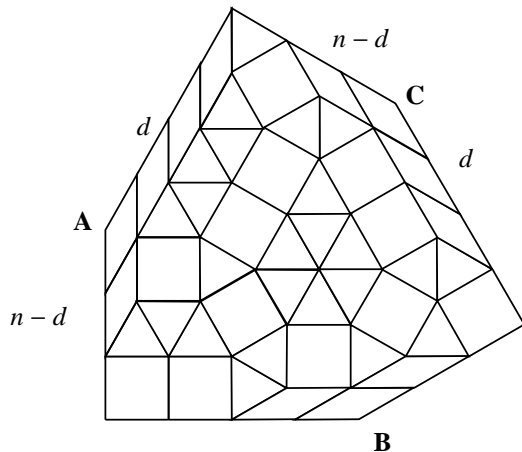
$$\hat{T}^t \cup Y(\lambda^t)^a = \begin{array}{|c|c|c|c|} \hline 2 & \bar{1} & \bar{1} & \bar{2} \\ \hline & 3 & 4 & \bar{1} \\ \hline & & 1 & 5 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline & \bar{1} & \bar{2} & \bar{1} \\ \hline & & \bar{1} & \bar{1} \\ \hline \end{array} = Z \cup \hat{Y}(\nu^t)^a \longrightarrow$$

- $LR(\mu^t \lambda^t \nu^t) \mapsto LR(\mu^t \nu^t \lambda^t)$

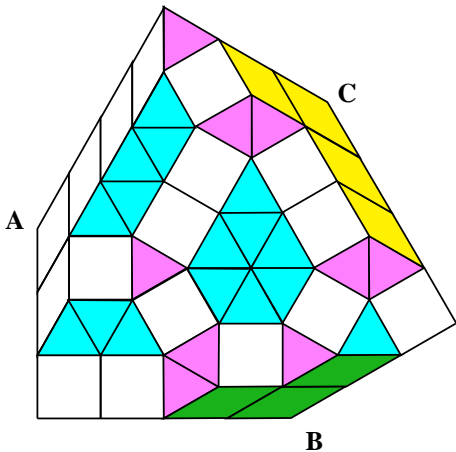
$$Z \cup Y(\nu^t)^a = \begin{array}{|c|c|c|c|} \hline 1 & \bar{1} & \bar{1} & \bar{2} \\ \hline & \bar{1} & \bar{2} & \bar{1} \\ \hline & & \bar{1} & \bar{1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 1 & \bar{1} & \bar{1} & \bar{2} \\ \hline & 1 & 2 & \bar{1} \\ \hline & & 1 & 1 \\ \hline \end{array} = \varrho^{BSS}(T) \cup Y(\lambda^t)$$

Purbhoo mosaics

A mosaic is a tiling of a hexagon by the following three shapes such that all rhombi are packed into the three nests A,B, and C.



Mosaics are in bijection with puzzles



Migration

- Migration is an operation that takes a flock to a new nest. The rhombi must move in the **standard order**. (The standard order in a tableau is the numerical ordering of the entries with priority by the rule left=smaller, right=larger, in case of equality.)

Migration

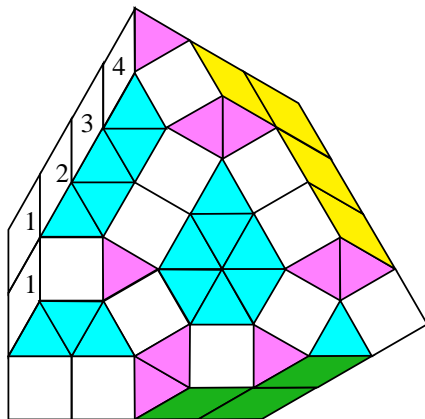
- Migration is an operation that takes a flock to a new nest. The rhombi must move in the **standard order**. (The standard order in a tableau is the numerical ordering of the entries with priority by the rule left=smaller, right=larger, in case of equality.)
- Choose the target nest. Rhombi move in the chosen direction of migration, inside a smallest hexagon in which \diamond is contained:



The move is such that the rhombus is either in its initial orientation, or its final orientation.

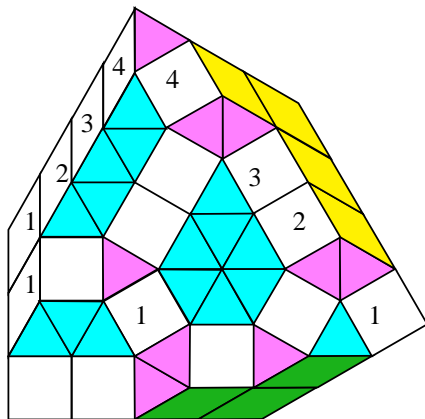
Migration(\equiv j.t.)/ ϱ^{BSS}

4	●	●
1	3	●
●	2	●
●	●	1



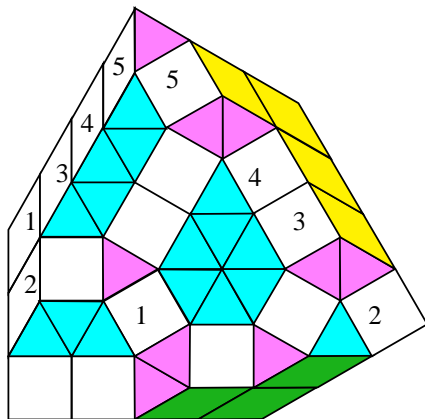
Migration(\equiv j.t.)/ ϱ^{BSS}

4	●	●
1	3	●
●	2	●
●	●	1



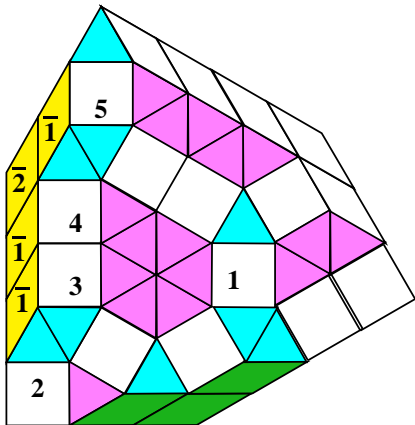
Migration(\equiv j.t.)/ ϱ^{BSS}

5	●	●
1	4	●
●	3	●
●	●	2



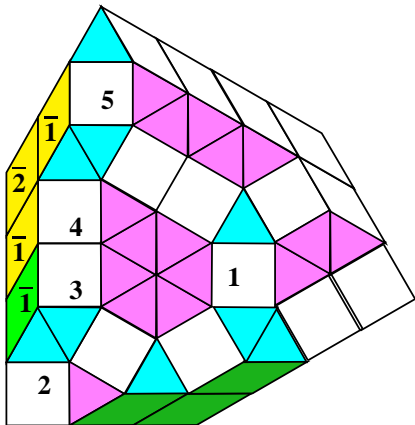
Migration(\equiv j.t.)/ ϱ^{BSS}

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
•	3	4	$\bar{1}$
•	•	1	5



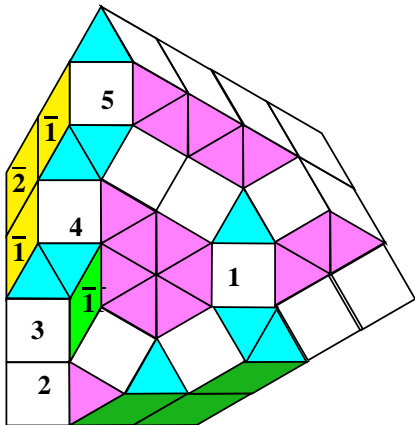
Migration(\equiv j.t.)/ ϱ^{BSS}

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
•	3	4	$\bar{1}$
•	•	1	5



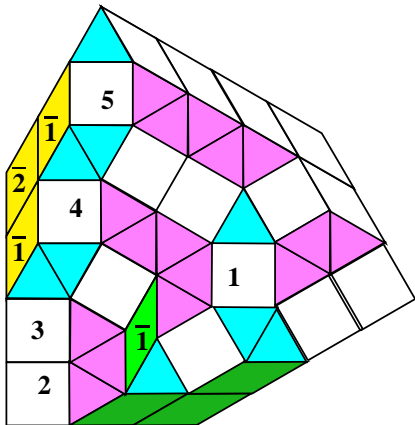
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



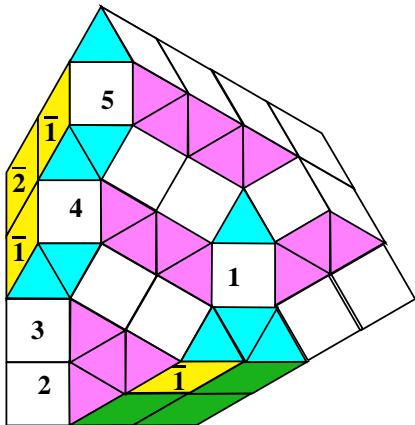
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



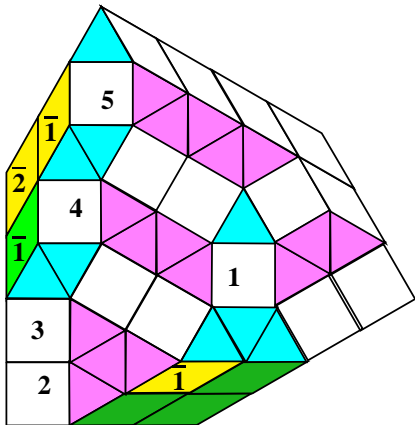
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



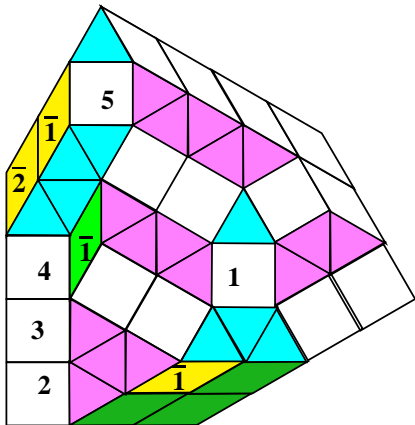
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



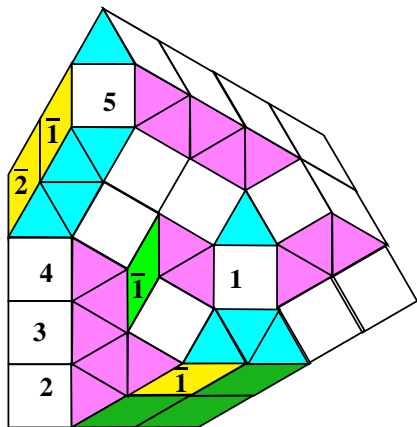
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2	3	4	$\bar{2}$
•	1	1	1
•	•	1	5



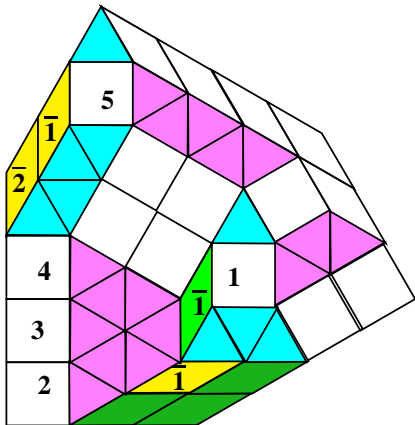
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	1	1	1
•	•	1	5



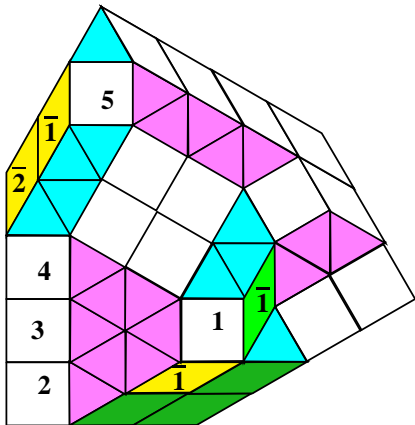
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	2
•	1	1	1
•	•	1	5



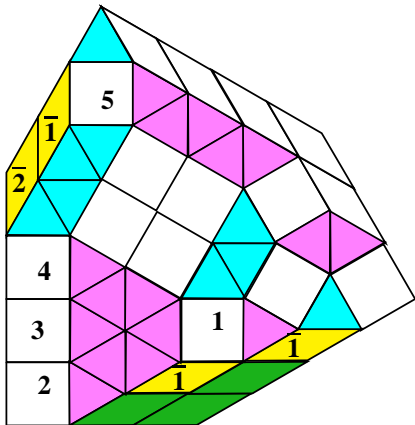
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	1	1	1
•	•	1	5



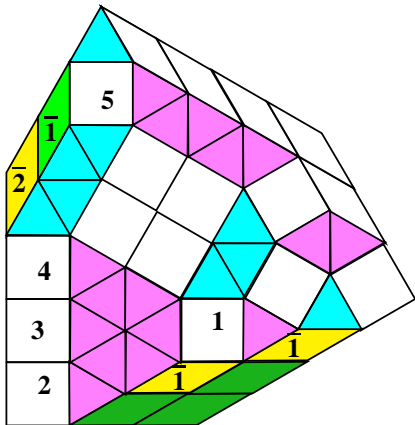
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	2
•	1	1	1
•	•	1	5



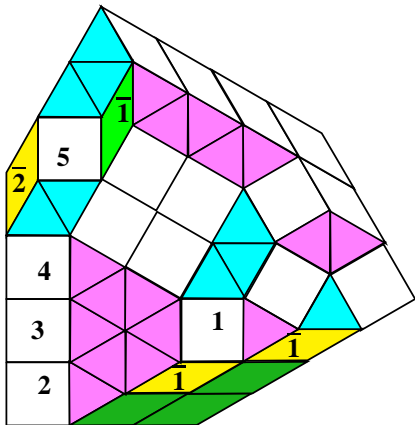
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	2
•	1	1	1
•	•	1	5



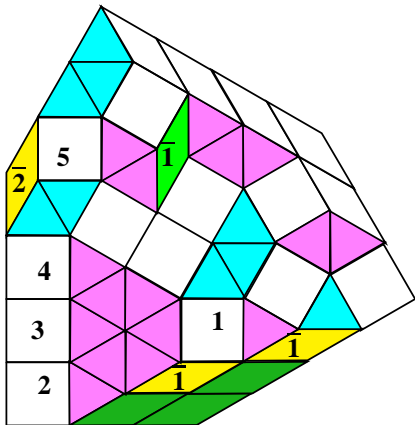
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	2
•	1	1	5
•	•	1	1



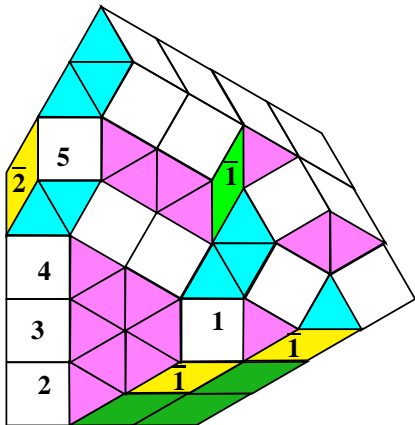
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	5
•	•	$\bar{1}$	$\bar{1}$



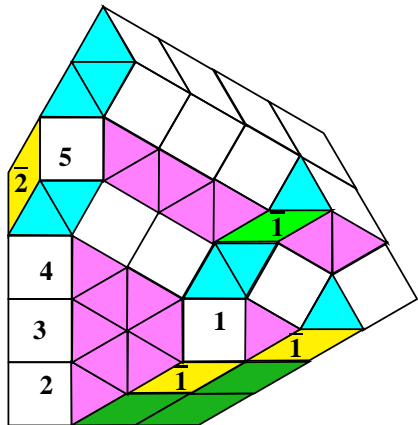
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	5
•	•	$\bar{1}$	$\bar{1}$



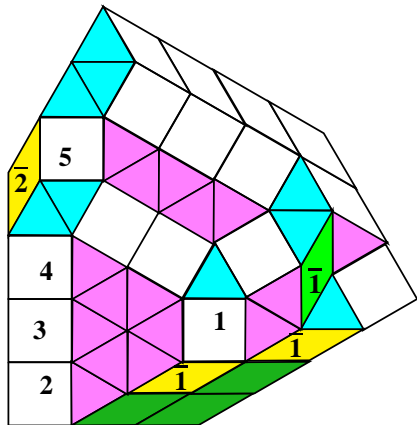
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	2
•	1	1	5
•	•	1	1



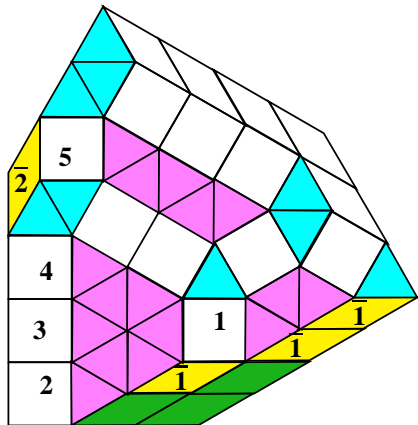
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	5
•	•	$\bar{1}$	$\bar{1}$



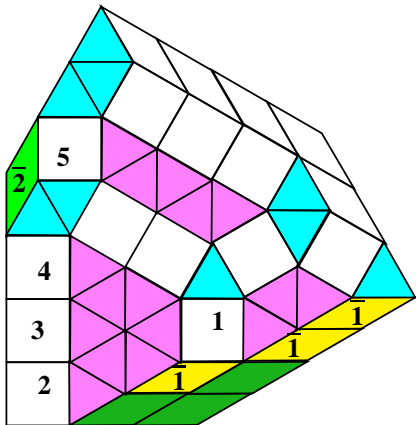
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	5
•	•	$\bar{1}$	$\bar{1}$



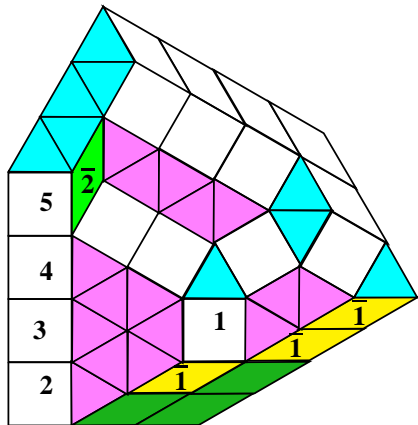
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	5
•	•	$\bar{1}$	$\bar{1}$



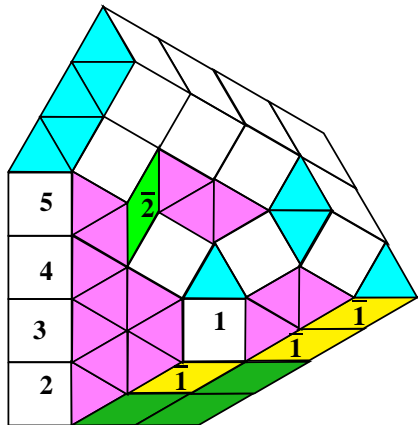
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	1	2
•	•	1	1



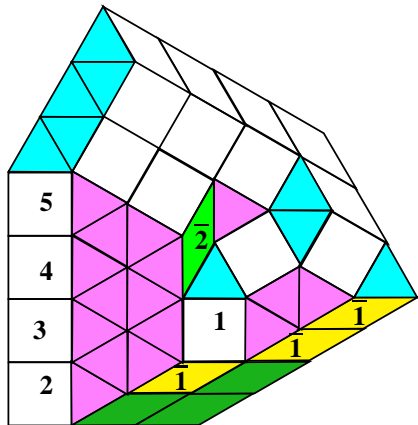
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	1	2
•	•	1	1



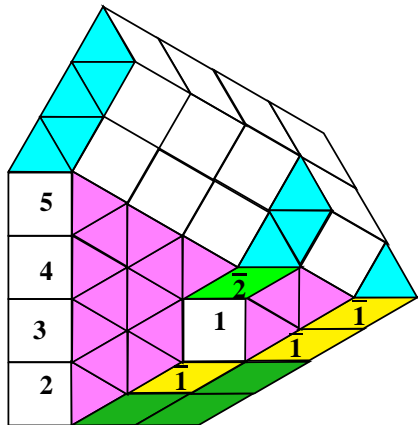
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	1	2
•	•	1	1



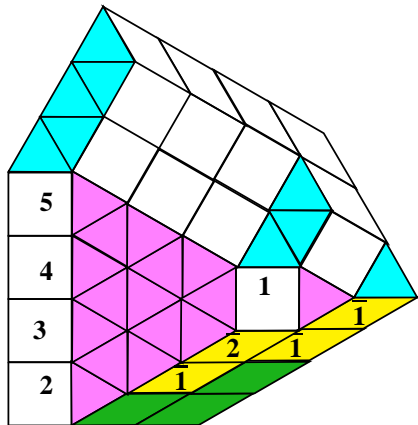
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	1	2
•	•	1	1



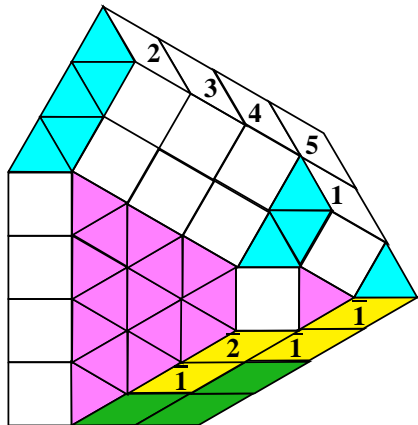
Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	2	1
•	•	1	1



Migration(\equiv j.t.)/ ϱ^{BSS}

2	3	4	5
•	1	2	1
•	•	1	1



Migration(\equiv j.t.)/ ϱ^{BSS}

1	1	1	2
•	1	2	1
•	•	1	1

