A linear time index-two subgroup of Littlewood-Richardson coefficient $\mathbb{Z}_2 \times S_3$-symmetries

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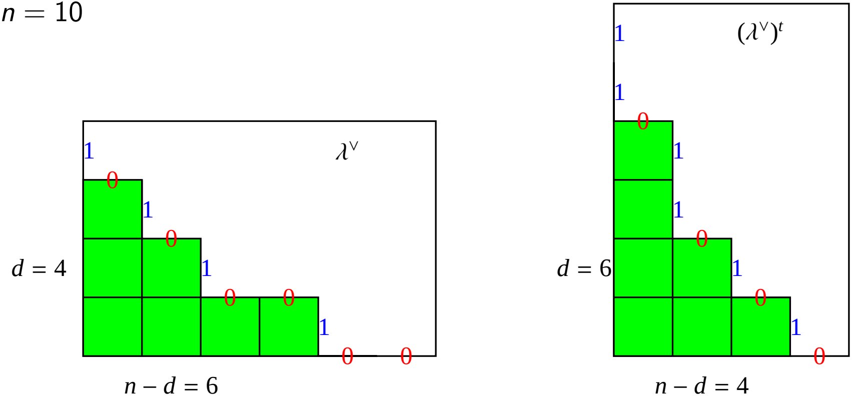
- There exist $d\times d$ non singular matrices A, B and C, over a *pid*, with Smith invariants μ, ν and λ respectively, such that $AB = C$ iff $c_{\mu\,\nu}^{\lambda} > 0$.

Partitions and 0-1 strings

Fix $0 < d < n$. Partitions which fit a $d \times (n-d)$ rectangle are in bijection with 0-1-strings of $n-d$ 0's and d 1's.

$n = 10$



$\lambda = (4,2,1,0) \leftrightarrow 0010010101$ $\qquad$ $\lambda^t = (3,2,1,1,0,0)$ $\quad 0101011011$

$\lambda^\vee = (6,5,4,2) \leftrightarrow 1010100100$ $\qquad$ $(\lambda^\vee)^t = (4,4,3,3,2,1)$ $\quad 1101101010$

Littlewood-Richardson rules

- $c_{\mu\,\nu}^{\lambda^\vee} =: c_{\mu\,\nu\,\lambda}$.
- Each Littlewood-Richardson coefficient $c_{\mu\,\nu\,\lambda}$ is a non-negative integer that may be evaluated by counting combinatorial objects with boundary data (μ, ν, λ):

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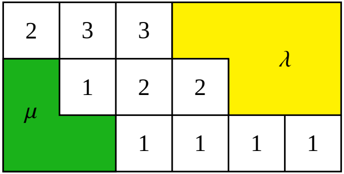
  - Knutson-Tao-Woodward puzzles
  - Purbhoo mosaics

Littlewood-Richardson tableaux

- $c_{\mu\,\nu\,\lambda}$ is the number of semistandard Young tableaux with shape $\lambda^{\vee}/\mu$ and content ν, with the following property:
  - If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top, at any stage, the number of i's encountered is at least as large as the number of $(i+1)$'s encountered, $\#1's \geq \#2's \ldots$.

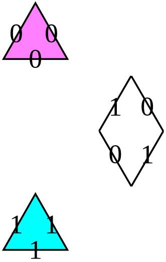
$$c_{210,532,320} = c^{643}_{210,532} = c_{000010101\ 010010100\ 000101001}$$

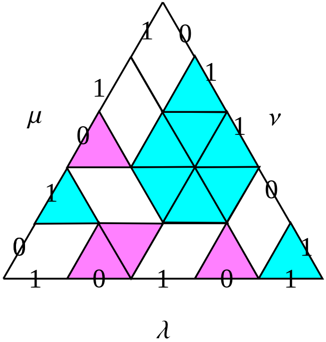


$$\nu = (5, 3, 2)$$

Knutson-Tao-Woodward puzzle rule

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length such that wherever two pieces share an edge, the numbers (colours) on the edge must agree.
- Puzzle pieces may be rotated in any orientation *but not reflected*.
- (Knutson-Tao-Woodward) $c_{\mu\,\nu\,\lambda}$ is the number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary.





2. Littlewood-Richardson coefficient $\mathbb{Z}_2 \times S_3$-symmetries

- (Benkart-Sottile-Stroomer, 96) Littlewood-Richardson coefficients $c_{\mu\,\nu\,\lambda}$ are invariant under the action of the dihedral group $\mathbb{Z}_2 \times S_3$ as follows: the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and λ, and S_3 permutes μ, ν and λ

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$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu} \qquad \begin{aligned} c_{\mu\,\nu\,\lambda} &= c_{\nu\,\mu\,\lambda} \\ c_{\mu\,\nu\,\lambda} &= c_{\mu\,\lambda\,\nu} \\ c_{\mu\,\nu\,\lambda} &= c_{\lambda\,\nu\,\mu} \end{aligned}$$

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Littlewood-Richardson coefficient $\mathbb{Z}_2 \times S_3$-symmetries

- Six of the twelve $\mathbb{Z}_2 \times S_3$-symmetries, in particular, three of the six S_3-symmetries, can be *easily exhibited* in the Littlewood-Richardson rules

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu} \qquad \begin{aligned} c_{\mu\,\nu\,\lambda} &= c_{\nu^t\,\mu^t\,\lambda^t} \\ c_{\mu\,\nu\,\lambda} &= c_{\lambda^t\,\nu^t\,\mu^t} \\ c_{\mu\,\nu\,\lambda} &= c_{\mu^t\,\lambda^t\,\nu^t} \end{aligned}$$

Either for the conjugation symmetry or for the commutativity no simple means are known to exhibit them in the Littlewood-Richardson rules.

$$\begin{aligned} c_{\mu\,\nu\,\lambda} &= c_{\nu\,\mu\,\lambda} \\ c_{\mu\,\nu\,\lambda} &= c_{\mu\,\lambda\,\nu} \\ c_{\mu\,\nu\,\lambda} &= c_{\lambda\,\nu\,\mu} \end{aligned} \qquad \begin{aligned} c_{\mu\,\nu\,\lambda} &= c_{\mu^t\,\nu^t\,\lambda^t} \\ c_{\mu\,\nu\,\lambda} &= c_{\lambda^t\,\mu^t\,\nu^t} \\ c_{\mu\,\nu\,\lambda} &= c_{\nu^t\,\lambda^t\,\mu^t} \end{aligned}$$

Linear time reductions

- Let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map. δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit–size of A.

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  - A tableau A is encoded through its recording matrix $(c_{i,j})$, where $c_{i,j}$ is the number of j's in the ith row of A.

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  - A tableau A is encoded through its recording matrix $(c_{i,j})$, where $c_{i,j}$ is the number of j's in the ith row of A.
- A function f reduces linearly to g, if it is possible to compute f in time linear in the time it takes to compute g; f and g are linearly equivalent if f reduces linearly to g and vice versa. This defines an equivalence relation on functions.

Igor Pak, Ernesto Vallejo, Reductions of Young tableau bijections, SIAM J. Discrete Mathematics, 2009, also available at arXiv:math/0408171

3. An index 2 subgroup of $\mathbb{Z}_2 \times S_3$-symmetries easy to exhibit

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  - τ the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and λ
  - $s_1 \in S_3$ switches the first and the second partition μ and ν
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- Claim: The subgroup of symmetries $\mathbf{H} =< \tau \mathbf{s_1}, \tau \mathbf{s_2} >= \{\mathbf{1}, \tau \mathbf{s_1}, \tau \mathbf{s_2 s_1 s_2}, \tau \mathbf{s_2}, \mathbf{s_1 s_2}, \mathbf{s_2 s_1}\}$ with index two of $\mathbb{Z}_2 \times S_3$, may be exhibited by maps of linear cost.

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- $\mathbf{H}\tau = \mathbf{H s_1} = \mathbf{H s_2} = \mathbf{H s_1 s_2 s_1} = \mathbf{H}\tau\mathbf{s_1 s_2} = \mathbf{H}\tau\mathbf{s_2 s_1}$

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- $\mathbf{H}\tau = \mathbf{H s_1} = \mathbf{H s_2} = \mathbf{H s_1 s_2 s_1} = \mathbf{H}\tau\mathbf{s_1 s_2} = \mathbf{H}\tau\mathbf{s_2 s_1}$
- Conjugation and commutative symmetry maps are linearly reducible to each other

♦, ♠ and ♣ involutions of linear cost

- LR-tableaux
  - ♦ $\leftrightarrow \tau s_1 s_2 s_1 = \tau s_2 s_1 s_2$, the involution showing the symmetry $c_{\mu\ \nu\ \lambda} = c_{\lambda^t\ \nu^t\ \mu^t}$

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  - ♠ $\leftrightarrow \tau s_1$, the involution showing the symmetry $c_{\mu\ \nu\ \lambda} = c_{\nu^t\ \mu^t\ \lambda^t}$
  - ♣ $\leftrightarrow \tau s_2$, the involution showing the symmetry $c_{\mu^t\ \lambda^t\ \nu^t}$

♦ involution

- $LR(\mu,\ \nu,\ \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t,\ \nu^t,\ \mu^t)$
- $c_{\mu\ \nu\ \lambda} = c_{\lambda^t\ \nu^t\ \mu^t}$

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T =

1	1	3	3		
		2	2	2	
			1	1	1

$\xrightarrow{\blacklozenge}$ $T^{\blacklozenge}$ =

5		
4		
2	3	
1	2	3
	1	2
		1

1112223311 $\xrightarrow{\lozenge}$ 1231231245

♦ involution

- $LR(\mu,\ \nu,\ \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t,\ \nu^t,\ \mu^t)$
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4		
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	1	2
		1

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♠ Involution

- $LR(\mu, \nu, \lambda) \xrightarrow{\spadesuit} LR(\nu^t, \mu^t, \lambda^t)$
- $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t}$

T =

1	3					
	2	2	3			
		1	2	2		
				1	1	1

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$\mathrm{T} =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3					
a	2	2	3			
a	b	1	2	2		
a	b	c	d	1	1	1

$\longrightarrow$

Involution

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T =

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3					
a	2	2	3			
a	b	1	2	2		
a	b	c	d	1	1	1

$\longrightarrow$

a	b					
a	3	c	3			
a	2	2	2	2		
1	b	1	d	1	1	1

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	2	2	3			
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1	3					
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a	b	c	d	1	1	1

$\longrightarrow$

a	b					
a	3	c	3			
a	2	2	2	2		
1	b	1	d	1	1	1

$\longrightarrow$

a	b					
3	3	a	c			
2	2	2	2	a		
1	1	1	1	1	b	d

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T =

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3					
a	2	2	3			
a	b	1	2	2		
a	b	c	d	1	1	1

$\longrightarrow$

a	b					
a	3	c	3			
a	2	2	2	2		
1	b	1	d	1	1	1

$\longrightarrow$

a	b					
3	3	a	c			
2	2	2	2	a		
1	1	1	1	1	b	d

$\longrightarrow$

d			
b			
	a		
		c	
		a	
			b
			a

$= T^{\spadesuit}$

♠ is a shortcut

- $$T \xrightarrow[\text{standardization}]{} \widehat{T} \xrightarrow[\text{transposition}]{t} \widehat{T}^t \xrightarrow[\text{tableau}-\text{switching}]{} T^{\spadesuit}$$

1	3					
a	2	2	3			
a	b	1	2	2		
a	b	c	d	1	1	1

$\longrightarrow$

1	10					
a	6	7	11			
a	b	2	8	9		
a	b	c	d	3	4	5

$\longrightarrow$

5			
4			
3	9		
d	8	11	
c	2	7	
b	b	6	10
a	a	a	1

$\longrightarrow$

5			
4			
3	9		
d	8	11	
2	c	7	
b	b	6	10
1	a	a	a

$\longrightarrow$

5			
4			
3	9		
d	8	11	
2	7	c	
b	6	10	b
1	a	a	a

$\longrightarrow$

d			
b			
5	a		
4	9	c	
3	8	a	
2	7	11	b
1	6	10	a

$\longrightarrow$

d			
b			
	a		
		c	
		a	
			b
			a

$= \mathrm{T}^{\spadesuit}$

♣ involution

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- $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\lambda^t\,\nu^t}$

$\mathrm{T} =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longrightarrow$

1	3	a	b	c	d	e
	2	2	3	a	b	c
		1	2	2	a	b
				1	1	1

$\longrightarrow$

a	b	1	c	d	3	e
	a	b	2	2	c	3
		a	1	b	2	2
				1	1	1

$\longrightarrow$

a	b	1	2	2	3	3
	a	b	1	1	2	2
		a	c	d	1	1
				b	c	e

$\longrightarrow$

e			
c			
b	d		
	c		
	a	b	
		a	b
			a

$= \mathrm{T}^{\clubsuit}$

♣ involution

- $LR(\mu, \nu, \lambda) \xrightarrow{\clubsuit} LR(\mu^t, \lambda^t, \nu^t)$
- $c_{\mu\ \nu\ \lambda} = c_{\mu^t\ \lambda^t\ \nu^t}$

T =

1	3					
	2	2	3			
		1	2	2		
				1	1	1

→

1	3	a	b	c	d	e
	2	2	3	a	b	c
		1	2	2	a	b
				1	1	1

→

a	b	1	c	d	3	e
	a	b	2	2	c	3
		a	1	b	2	2
				1	1	1

→

a	b	1	2	2	3	3
	a	b	1	1	2	2
		a	c	d	1	1
				b	c	e

→

e			
c			
b	d		
	c		
	a	b	
		a	b
			a

= T♣

- ♣ is a shortcut of

$$T \xrightarrow[\text{standardization}]{} \widehat{T} \xrightarrow[\text{transposition}]{t} \widehat{T}^t \xrightarrow[\text{tableau-switching}]{} T^{\clubsuit}$$

♣♦, ♦♣ bijections of linear cost

- $LR(\mu, \nu, \lambda) \xrightarrow{\clubsuit\blacklozenge} LR(\lambda, \mu, \nu)$
- $c_{\mu\nu\lambda} = c_{\lambda\mu\nu}$
- ♣♦

$$T \underset{180^o\, rotation}{\xrightarrow{\bullet}} T^{\bullet} \underset{tableau-switching}{\longrightarrow} T^{\clubsuit\blacklozenge}$$

$\clubsuit(\spadesuit), \diamondsuit$ generate a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

- LR-tableaux

Claim:

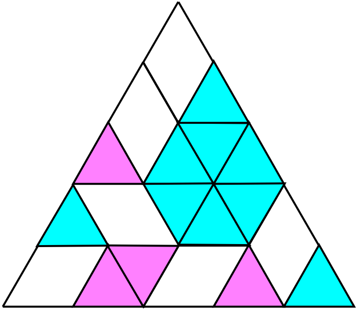
$$\{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge = \spadesuit\} \simeq S_3$$

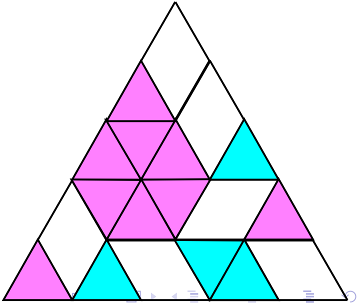
form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$.

Puzzle mirror reflections with 0's and 1's swapped

- $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t}$ ♠
- $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$ ♦
- $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\lambda^t\,\nu^t}$ ♣ = ♠♦♠ = ♦♠♦

♠

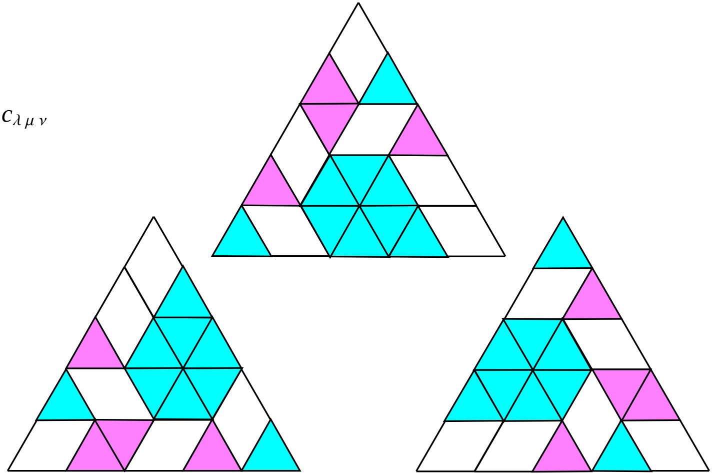




Puzzle $2\pi/3$–rotations

- $c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu}$ ♣♦
- $c_{\mu\,\nu\,\lambda} = c_{\nu\,\lambda\,\mu}$ ♦♣

$c_{\mu\,\nu\,\lambda} = c_{\nu\,\lambda\,\mu} = c_{\lambda\,\mu\,\nu}$



Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* $\clubsuit, \blacklozenge$ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 > \simeq S_3$$

$$< \spadesuit, \blacklozenge > = \{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit\} \simeq S_3$$

Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* $\clubsuit, \blacklozenge$ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 > \simeq S_3$$

$$< \spadesuit, \blacklozenge >= \{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit\} \simeq S_3$$

$$< \text{puzzle } 2\pi/3 \text{ rotations } >$$

$$\{\mathbf{1}, \clubsuit\blacklozenge, \blacklozenge\clubsuit\}$$

Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* $\clubsuit, \blacklozenge$ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections \& } 0 \leftrightarrow 1 > \simeq S_3$$

$$< \spadesuit, \blacklozenge > = \{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit\} \simeq S_3$$

$$< \text{puzzle } 2\pi/3 \text{ rotations } >$$

$$\{\mathbf{1}, \clubsuit\blacklozenge, \blacklozenge\clubsuit\}$$

- Conjugation and commutative symmetry maps are linearly reducible to each other

Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* $\clubsuit, \blacklozenge$ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 >$$
$$\|$$
$$< \spadesuit, \blacklozenge >= \{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit\} \simeq S_3$$

Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* ♣, ♦ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 >$$
$$\|$$
$$< \spadesuit, \diamondsuit >= \{\mathbf{1}, \clubsuit, \diamondsuit, \clubsuit\diamondsuit\clubsuit = \diamondsuit\clubsuit\diamondsuit, \clubsuit\diamondsuit, \diamondsuit\clubsuit\} \simeq S_3$$

$$< \text{puzzle } 2\pi/3 \text{ rotations } >$$
$$\|$$
$$\{\mathbf{1}, \clubsuit\diamondsuit, \diamondsuit\clubsuit\}$$

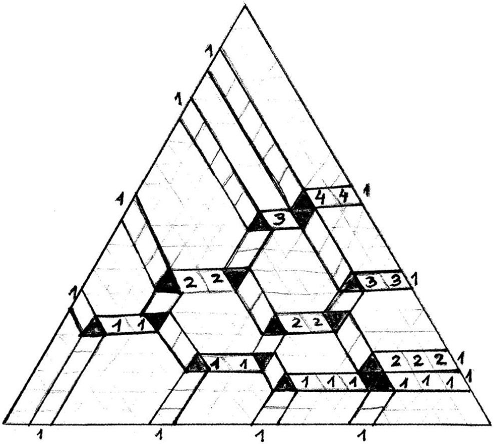
Action of an index 2 subgroup of $\mathbb{Z}_2 \times S_3$ on KTW-puzzles/LR-tableaux

- The group generated by the puzzle *mirror reflections with the 0's and 1's swapped* /*LR-tableau simple involutions* $\clubsuit, \blacklozenge$ form a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$

$$< \text{puzzle mirror reflections } \& \ 0 \leftrightarrow 1 >$$
$$\|$$
$$< \spadesuit, \blacklozenge >= \{\mathbf{1}, \clubsuit, \blacklozenge, \clubsuit\blacklozenge\clubsuit = \blacklozenge\clubsuit\blacklozenge, \clubsuit\blacklozenge, \blacklozenge\clubsuit\} \simeq S_3$$

$$< \text{puzzle } 2\pi/3 \text{ rotations } >$$
$$\|$$
$$\{\mathbf{1}, \clubsuit\blacklozenge, \blacklozenge\clubsuit\}$$

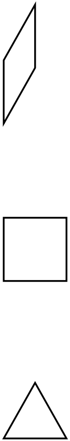
Puzzles and LR tableaux are in bijection: Tao's bijection

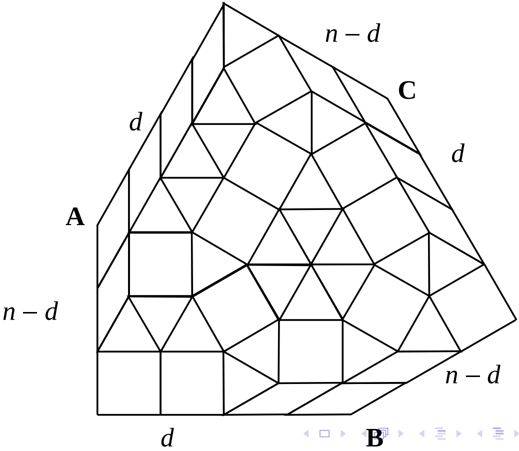


	1	1	2	2	3	4	4								
					1	1	2	2	3	3					
								1	1	1	2	2	2		
											1	1	1		

Purbhoo mosaics

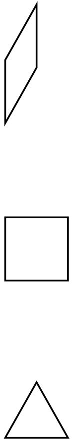
A mosaic is a tiling of an hexagon, with angles and side lengths as below, by the following three shapes of unitary triangles, unitary squares, and unitary rhombi with angles 30^o and 150^o such that all rhombi are packed into the three 150 nests A,B, and C.

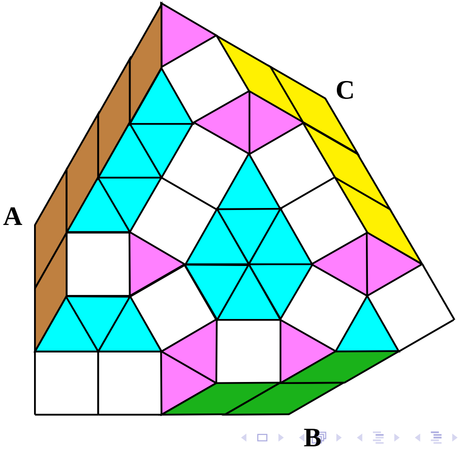




Mosaics are in bijection with puzzles

A mosaic is a tiling of an hexagon, with angles and side lengths as below, with unitary triangles, unitary squares, and unitary rhombi with angles 30^o and 150^o all packed into the three 150^o nests.



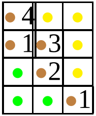


Migration/*jeu de taquin*

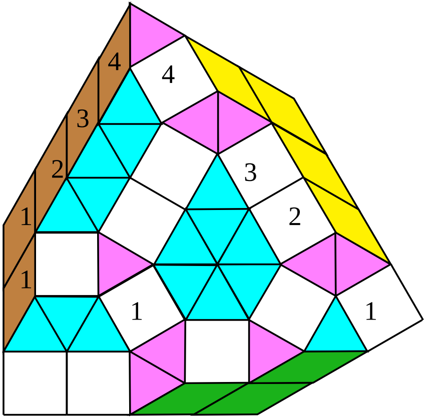
- Migration is an operation that take the rhombi from one nest to a new one The rhombi must move in the standard order.(The standard order in a tableau is the numerical ordering of the entries with priority by the rule left=smaller, right=larger, in case of equality.)
- Choose the target nest. Rhombi move in the chosen direction of migration, inside a smallest hexagon in which $\diamondsuit$ is contained:

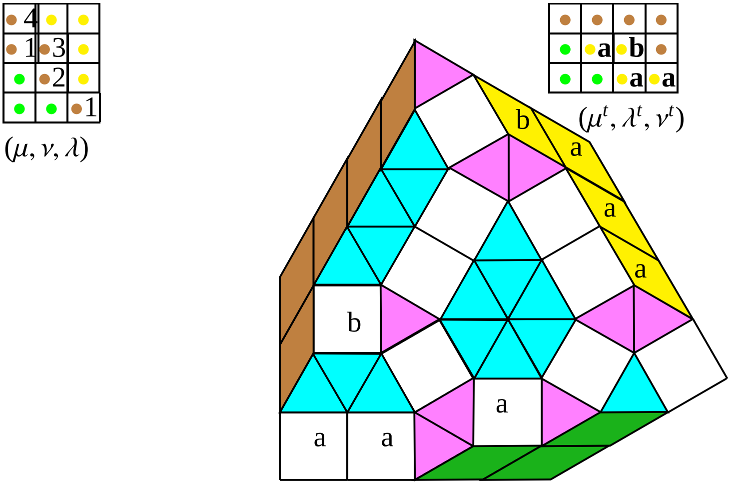
The move is such that the rhombus is either in its initial orientation, or its final orientation.

Purbhoo mosaics are in bijection with puzzles and LR tableaux



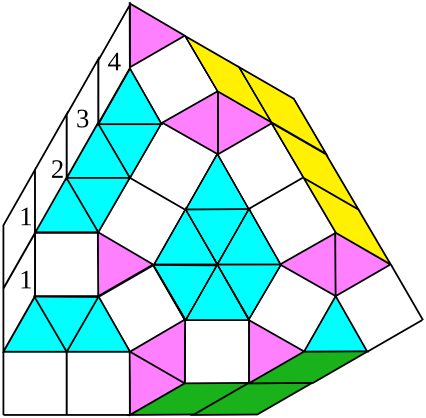
(μ, ν, λ)




4
1
3
2
1
(μ, ν, λ)
a
b
a
a
$(\mu^t, \lambda^t, \nu^t)$
b
a
a
a
b
a
a
a

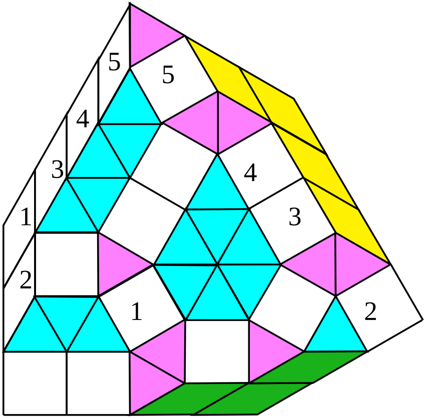
Migration ($\equiv$ j.t.)

4	●	●
1	3	●
●	2	●
●	●	1



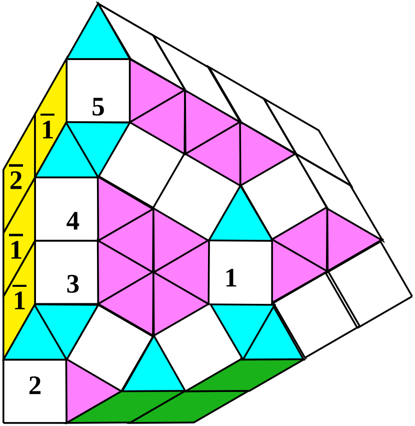
Migration(≡ j.t.)

5	●	●
1	4	●
●	3	●
●	●	2



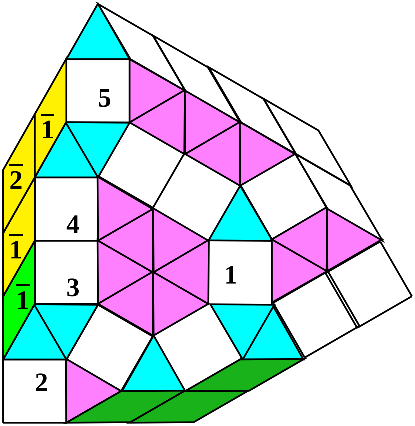
Migration($\equiv$ j.t.)

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
●	3	4	$\bar{1}$
●	●	1	5



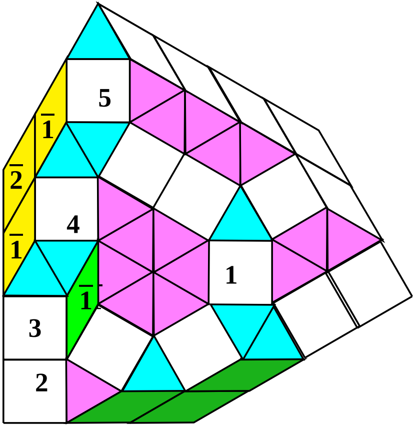
Migration($\equiv$ j.t.)

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
•	3	4	$\bar{1}$
•	•	1	5



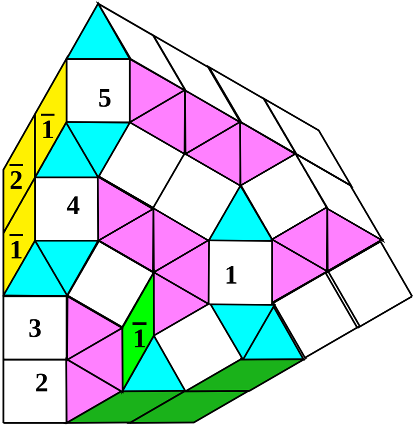
Migration($\equiv$ j.t.)

2	3	$\bar{1}$	$\bar{2}$
●	$\bar{1}$	4	$\bar{1}$
●	●	1	5



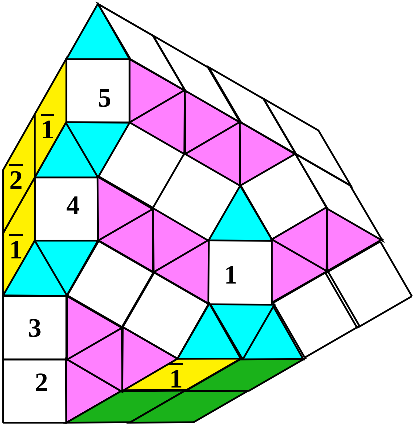
Migration($\equiv$ j.t.)

2	3	$\bar{1}$	$\bar{2}$
●	$\bar{1}$	4	$\bar{1}$
●	●	1	5



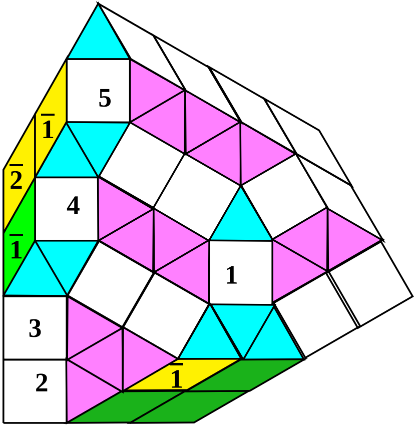
Migration($\equiv$ j.t.)

2	3	$\bar{1}$	$\bar{2}$
●	$\bar{1}$	4	$\bar{1}$
●	●	1	5



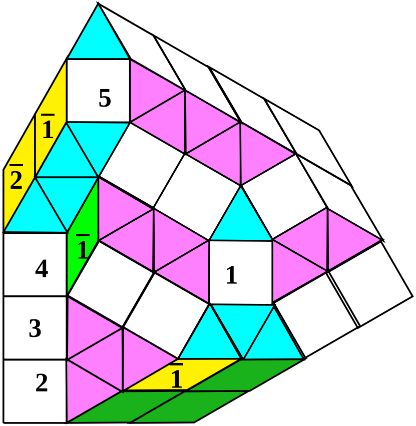
Migration($\equiv$ j.t.)

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



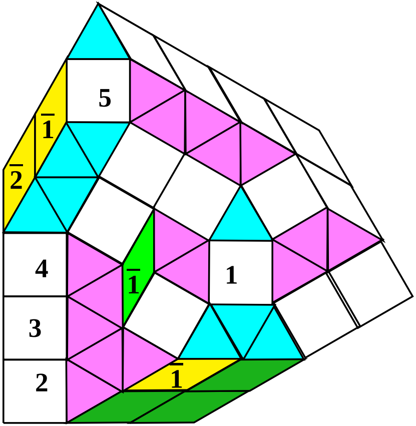
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	$\bar{1}$
•	•	1	5



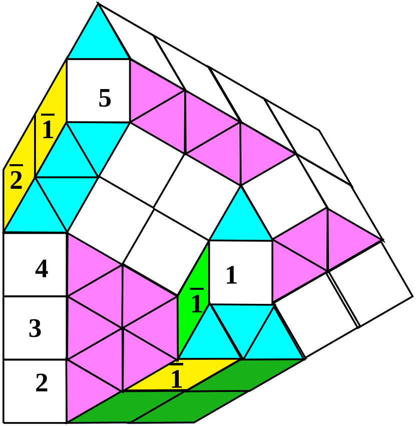
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	$\bar{1}$	$\bar{1}$
●	●	1	5



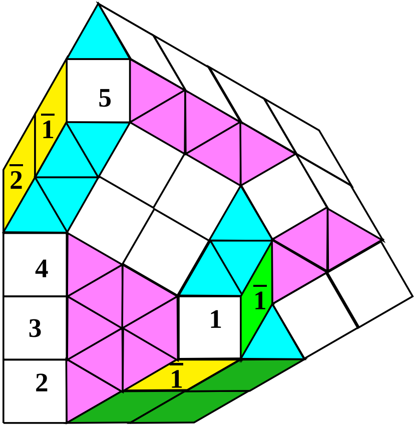
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	$\bar{1}$	$\bar{1}$
●	●	1	5



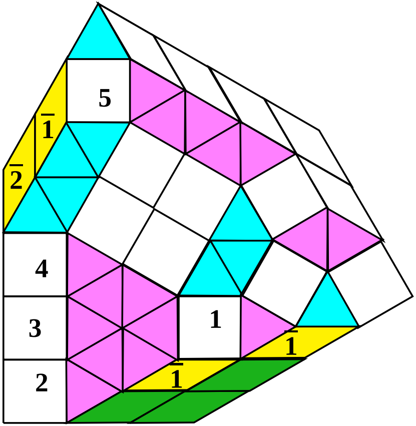
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	$\bar{1}$	$\bar{1}$
●	●	1	5



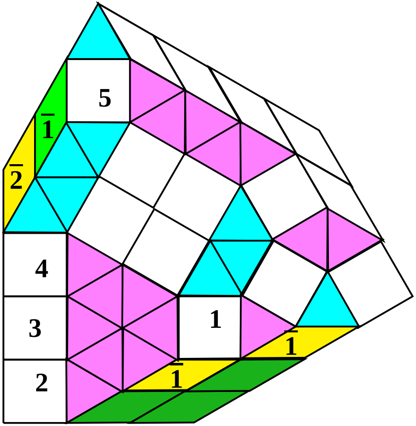
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	1	$\bar{1}$
•	•	$\bar{1}$	5



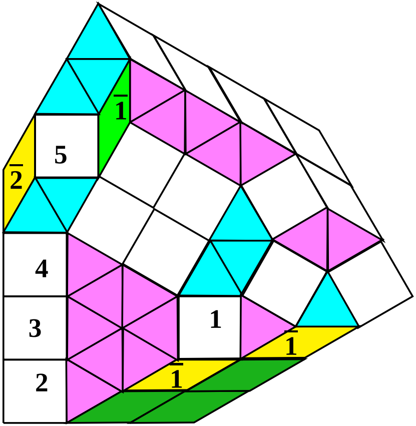
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	1	$\bar{1}$
•	•	$\bar{1}$	5



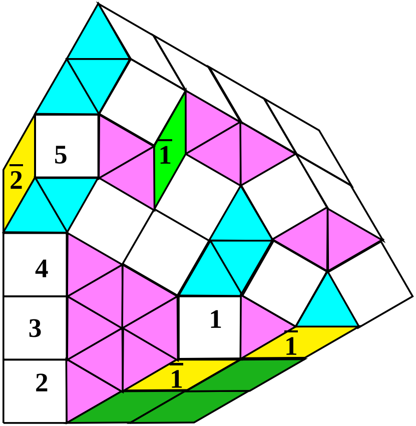
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



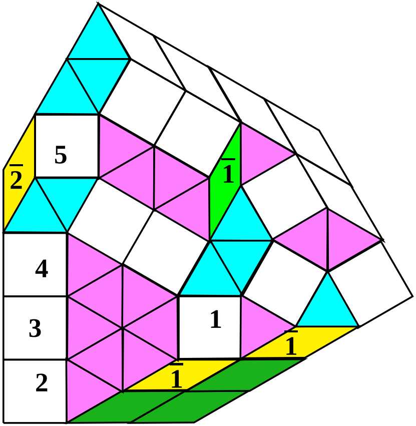
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



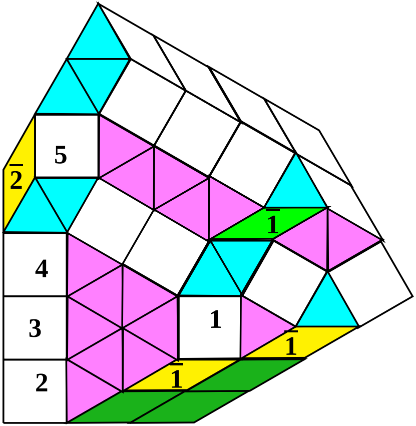
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



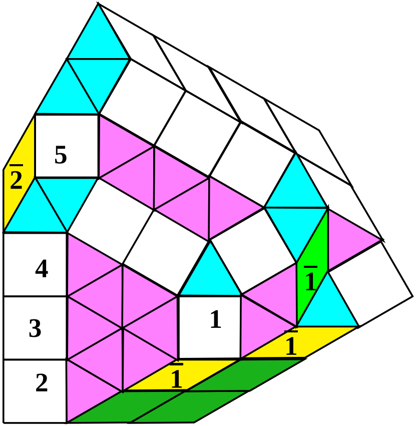
Migration(≡ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



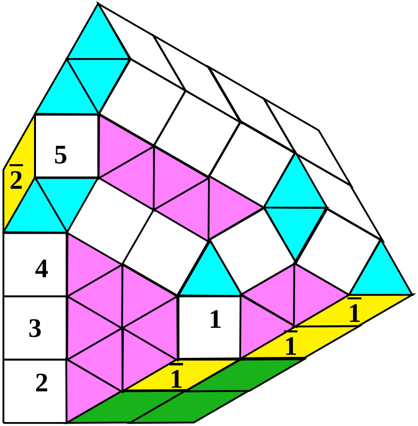
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



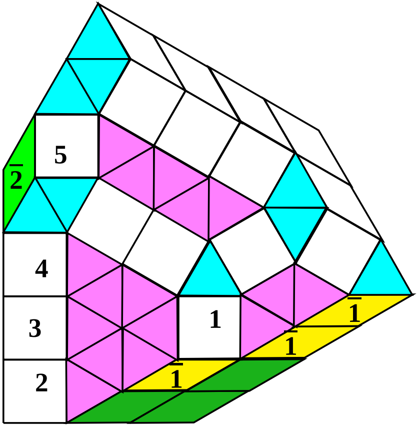
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



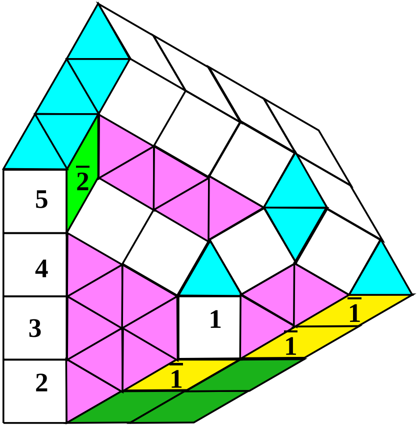
Migration($\equiv$ j.t.)

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



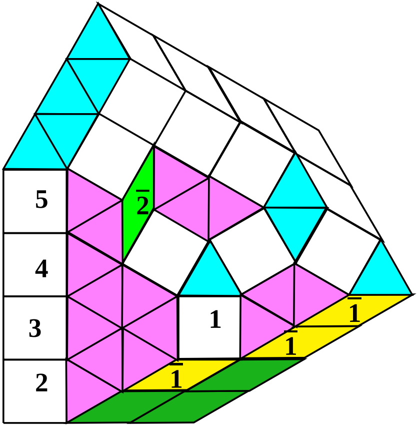
Migration($\equiv$ j.t.)

2	3	4	5
•	$\bar{1}$	1	$\bar{2}$
•	•	$\bar{1}$	$\bar{1}$



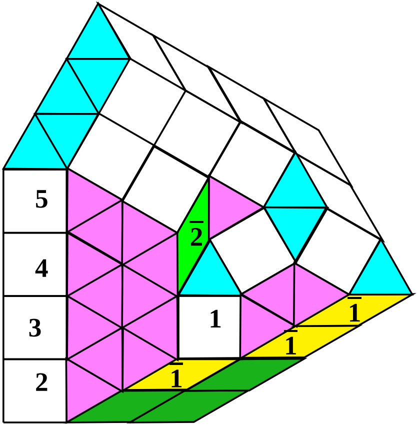
Migration($\equiv$ j.t.)

2	3	4	5
•	$\bar{1}$	1	$\bar{2}$
•	•	$\bar{1}$	$\bar{1}$



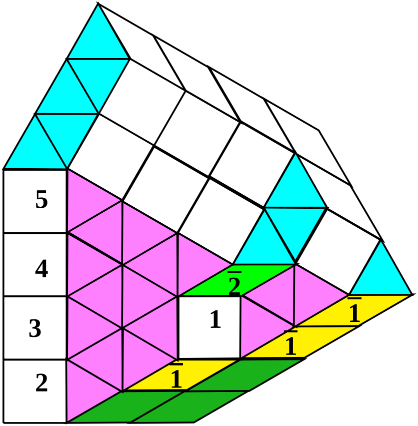
Migration($\equiv$ j.t.)

2	3	4	5
●	$\bar{\mathbf{1}}$	1	$\bar{\mathbf{2}}$
●	●	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$



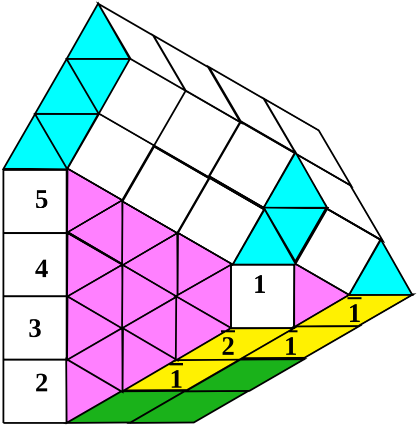
Migration($\equiv$ j.t.)

2	3	4	5
●	$\bar{1}$	1	$\bar{2}$
●	●	$\bar{1}$	$\bar{1}$



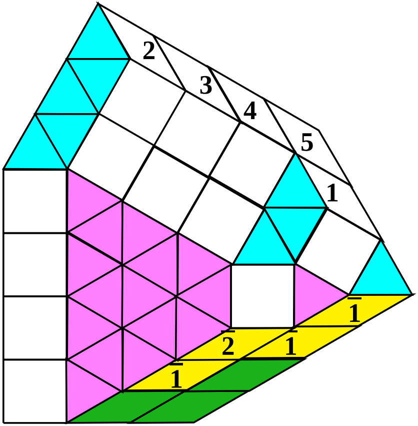
Migration($\equiv$ j.t.)

2	3	4	5
•	$\bar{1}$	$\bar{2}$	1
•	•	$\bar{1}$	$\bar{1}$



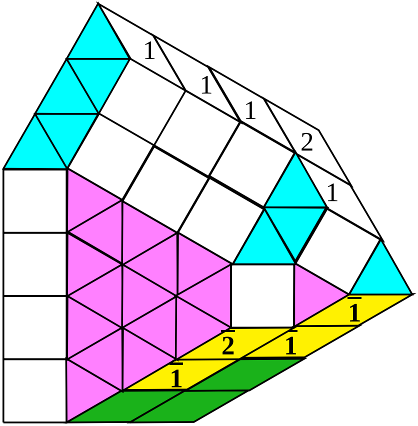
Migration($\equiv$ j.t.)

2	3	4	5
●	$\bar{1}$	$\bar{2}$	1
●	●	$\bar{1}$	$\bar{1}$



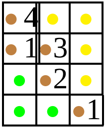
Migration($\equiv$ j.t.)

1	1	1	2
•	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
•	•	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$

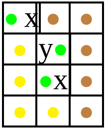


Mosaic 120^o clockwise rotation

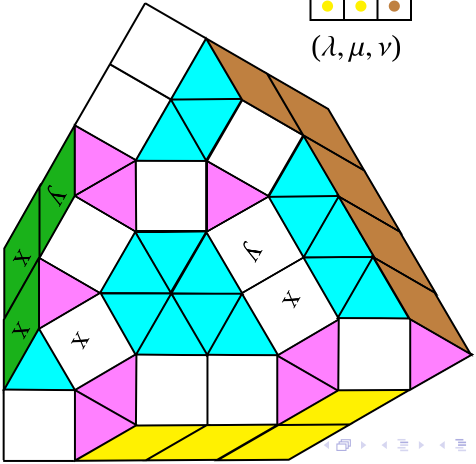
♣♦

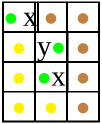


(μ, ν, λ)

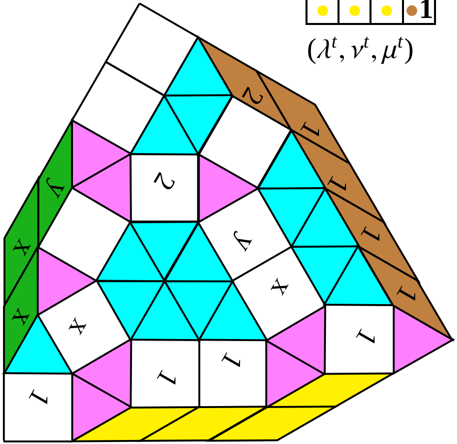


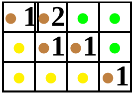
(λ, μ, ν)





(λ, μ, ν)





$(\lambda^t, \nu^t, \mu^t)$

Linear reductions and the Schützenberger involution

- Pak-Vallejo Theorem(SIAM Dis. Math. 09) The following maps are linearly equivalent:
  (1) RSK correspondence.
  (2) *Jeu de taquin* map.
  (3) Littlewood–Robinson map.
  (4) Tableau–switching map.
  (5) Schützenberger involution E for normal shapes.
  (6) Reversal e.
  (7) (Fundamental) commutative symmetry map $\rho_1 : LR(\mu, \nu, \lambda) \to LR(\nu, \mu, \lambda)$.

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Theorem(A., C., M, DMTCS Proceedings, 09)

- The LR-conjugation symmetry maps are identical.

$$\varrho = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} = \spadesuit\, \rho_1 = \blacklozenge\, \rho = \clubsuit \rho_2.$$

- The LR-commutative and transposition symmetry maps are linearly equivalent to the Schützenberger involution E,
  $\rho = e\bullet$

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau \updownarrow & & \tau \updownarrow & & \\ P & \underset{E}{\overset{\text{evacuation}}{\longleftrightarrow}} & P^E. & & \end{array}$$

- $\rho_1 = \spadesuit\blacklozenge\, e\,\bullet$

Action of $\mathbb{Z}_2 \times S_3$ on LR-tableaux/KTW-puzzles

- $$\mathbb{Z}_2 \times S_3 =< \clubsuit, \blacklozenge, \rho : \rho^2 = \clubsuit^2 = \blacklozenge^2 = (\clubsuit\blacklozenge)^3 = (\clubsuit\rho)^2 = (\blacklozenge\rho)^2 = 1 >$$
- $\rho = e\bullet$

Remarks/Further links

- Why the involutions exhibiting a specific LR-symmetry always coincide?

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- Why is it *difficult* to exhibit the commutative symmetry in either Littlewood-Richardson rule?