HETEROCLINIC DYNAMICS

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1. Course description

We aim to address the questions What? Why? and How? as they apply to heteroclinic cycles and networks. More specifically:

(1) What is a heteroclinic network? Where do they come from and what are they useful for?
(2) Why are heteroclinic networks interesting in dynamics?
(3) How do we construct heteroclinic networks and how do we prove dynamical results about them?

The lectures will be responsive to the needs and background of the audience. In general terms, I hope to spend some time discussing where heteroclinic networks come from (and where to look for them) with particular reference to Lotka-Volterra systems, symmetric dynamics and network dynamics. I also want to look at some possible applications (including to neuroscience and winnerless competition) as well as ways in which heteroclinic networks and cycles can be viewed as templates for complex dynamics (for example, switching). Overall, I am interested in communicating the basic ideas and interest rather than in developing highly technical analysis (there is plenty of that to be found). I intend to provide a detailed set of notes.

2. Summary of past work

Here is a summary – with references – of some work and directions in the area over the past twenty or so years. This is neither required reading for the course nor a description of everything I hope to cover! Nevertheless, there may be references here that could be of interest to potential members of the course.

Heteroclinic and homoclinic cycles can occur in low codimension bifurcations of vector fields and have been intensively studied by many authors (see, for example, the volumes by L Shilnikov *et al.* [49, 50]). Of particular interest are the mechanisms whereby homoclinic bifurcations can lead to complex dynamics and chaos. In a related direction that dates back to Duffing [16], and follows on earlier work by Wang & Ott [51], Mohapatra & Ott have shown how periodic forcing of homoclinic loops and heteroclinic cycles can lead to the formation of non-uniformly hyperbolic attractors [41] – these works address the difficult problem of finding explicit examples of non-uniform hyperbolicity and build on earlier work of L-S Young *et al.* on shear induced chaos and rank one attractors [37, 52].

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On account of the Kupka-Smale theorem, *robust* heteroclinic cycles and networks only occur when vector fields possess additional structure which is invariably associated with the presence of invariant subspaces. They are a well-known phenomenon in models of population dynamics, ecology and game theory based on the Lotka-Volterra equations (for example, [38, 27, 28, 29, 30]). Typically, these systems are defined on a simplex or the positive orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n\}$ and have the 'extinction' hyperplanes $x_i = 0$ as invariant subspaces. The first example of a heteroclinic cycle that the author is aware of in the literature appears in the 1975 paper by May & Leonard [38][1] – and has been used to model the 'rock-paper-scissors' game and winnerless competition (for example, [38, 14, 2, 47]).

A large and well-studied class of dynamical systems with invariant subspaces are differential equations which are *equivariant* with respect to a compact Lie group of symmetries (for example, [48, 39, 35, 22, 36, 34, 10, 23, 15, 24]). Robust heteroclinic cycles and networks occur because generic intersections of stable and unstable manifolds of equilibria in equivariant dynamics need not be transverse [17, 18, 19]. This breakdown of transversality is closely associated with a rich invariant subspace structure. Specifically, if a finite or compact Lie group G acts smoothly on the phase space M, and H is any non-empty subset of G, then the submanifold $M^H = \{x \in M \mid hx = x, \forall h \in H\}$ is invariant by the flow of every C^1 G-equivariant vector field on M.

From the mathematical point of view, robust heteroclinic networks often lead to interesting complex dynamics. For example, the phenomenon of random *switching* between nodes [31, 5, 8, 32]. Evidence of heteroclinic switching has even been observed *in vivo* in Abeles *et al.* [1]. From the point of view of applications, there has been recent interest in robust heteroclinic cycles that appear in neural microcircuits where they give nonlinear models with 'winnerless competition' – there is a local competition between different states but not necessarily a global winner [46]. These models seem useful for explaining sequence generation and spatio-temporal encoding and have been found in rate-based [3] and other models [44]. They can also be found in phase oscillator models derived from Hodgkin-Huxley models [25] or more general phase oscillator models [9]. Heteroclinic networks can be used to perform finite-state computations in phase oscillator systems [6, 8] (see also [42, 43] for pulse coupled systems). Analogous behaviour is also found in hybrid models of neural systems such as the networks of unstable attractors in systems of delay-pulse coupled oscillators [40] as well as in coupled chemical reaction systems [33].

Constructions of heteroclinic networks with specific properties may be found in [23, 11, 12, 20, 21].

From the point of view of much contemporary research on networks, in particular neural dynamics, it is interesting to consider networks of coupled nonlinear oscillators and look for patterns of synchronization. The simplest models of this type are Kuramoto style coupled phase oscillators where node dynamics is defined on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. One approach, due to Ashwin & Swift [13], is

[1]Interest has continued since that time notwithstanding the authors remark that "Biologically, the behaviour illustrated in Figs. 4 and 5 is nonsense".

to look at all-to-all coupled systems of n-cells with S_n symmetry where there is often synchronization into clusters of cells combined with heteroclinic phenomena (see, for example, [26, 45, 8]). We will consider networks of interacting dynamical systems with no symmetries (local or global) but where the network architecture can lead to invariant subspaces comprised of groups of synchronized cells and interesting heteroclinic networks (see [4, 20, 21]).

References

[1] M Abeles, H Bergman, I Gat, I Meilijson, E Seidmann & M Tishby, 'Cortical activity flips among quasi-stationary states', *PNAS* **92** (1995), 8616–8620.

[2] V S Afraimovich, M I Rabinovich & P Varona. 'Heteroclinic contours in neural ensembles and the winnerless competition principle', *Inter. J. Bifur. Chaos* **14** (2004), 1195–1208.

[3] V S Afraimovich, V P Zhigulin & M I Rabinovich. 'On the origin of reproducible sequential activity in neural circuits', *Chaos* **14**(4) (2004), 1123–1129.

[4] M A D Aguiar, P Ashwin, A Dias & M Field. 'Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation', *Journal of Nonlinear Science*, **21**(2) (2011), 271–323.

[5] M A D Aguiar, S B S D Castro & I S Labouriau. 'Dynamics near a heteroclinic network', *Nonlinearity* **18**(1) (2005), 391–414.

[6] P Ashwin & J Borresen, 'Discrete computation using a perturbed heteroclinic network', *Phys. Rev. E* **70** (2004), 026203.

[7] P Ashwin, O Burylko & Y Maistrenko. 'Bifurcation to heteroclinic cycles and sensitivity in three and four coupled phase oscillators', *Phys. D* **237**(4) (2008), 454–466.

[8] P Ashwin, G Orosz & J Borresen. 'Heteroclinic Switching in Coupled Oscillator Networks: Dynamics on Odd Graphs', *Nonlinear Dynamics and Chaos: Advances and Perspectives, Understanding Complex Systems*, M Thiel, J Kurths, M C Romano, A Moura & G Károlyi eds., Springer, (2010), 31–50.

[9] P Ashwin, G Orosz, J Wordsworth & S Townley, 'Reliable switching between cluster states for globally coupled phase oscillators', *SIAM Journal on Applied Dynamical Systems* **6**, (2007), 728–758.

[10] P Ashwin & M Field. 'Heteroclinic networks in coupled cell systems', *Arch. Rational Mech. Anal.* **148** (1999), 107–143.

[11] P Ashwin & C M Postlethwaite. 'On designing heteroclinic networks from graphs', *Physica D* **265** (2013), 26–39.

[12] P Ashwin and C Postlethwaite. 'Designing Heteroclinic and Excitable Networks in Phase Space Using Two Populations of Coupled Cells', *Jn. Nonlinear Science* **26**(2) (2016), 345–364,

[13] P Ashwin & J W Swift. 'The dynamics of n weakly coupled identical oscillators', *J. Nonlinear Science* **2** (1992), 69–108.

[14] C-W Chi, S-B Hsu & L-I Wu. 'On the asymmetric May-Leonard model of three competing species', *SIAM J. Appl. Math.* **58** (1998), 211–226.

[15] A P S Dias, B Dionne & I Stewart. 'Heteroclinic Cycles and Wreath Product Symmetries', *Dynamics and Stability of Systems* **15** (2000), 353–385. *Nonlinearity* **18** (2005), 1003–1020.

[16] G Duffing. *Erzwungene schwingungen bei ver anderlicher eigenfrequenz und ihre technische Bedeutung* (Verlag Friedr, Vieweg & Sohn, Braunschweig, 1918).

[17] M J Field. 'Equivariant dynamical systems', *Bull. Amer. Math. Soc.* **76** (1970), 1314–1318.

[18] M J Field. 'Transversality in G-manifolds', *Trans. Amer. Math. Soc.* **231** (1977), 429–450.

[19] M J Field. 'Equivariant dynamical systems', *Trans. Amer. Math. Soc.* **259**(1) (1980), 185–205.

[20] M J Field. 'Heteroclinic Networks in Homogeneous and Heterogeneous Identical Cell Systems', *J. Nonlinear Science* **25**(3) (2015), 779–813.

[21] M J Field. 'Patterns of desynchronization and resynchronization in heteroclinic networks', preprint 2016.

[22] M J Field & R W Richardson. 'Symmetry breaking and branching patterns in equivariant bifurcation theory II', *Arch. Rational Mech. and Anal.* **120** (1992), 147–190.

[23] M J Field. *Dynamics, Bifurcation and Symmetry*, Pitman Research Notes in Mathematics, **356**, 1996.

[24] M J Field. *Dynamics and Symmetry* (Imperial College Press Advanced Texts in Mathematics — Vol. 3, 2007.)

[25] D Hansel, G Mato & C Meunier. 'Phase dynamics for weakly coupled Hodgkin-Huxley neurons', *Europhys Letts* **23** (1993), 367–372.

[26] D Hansel, G Mato & C Meunier. 'Clustering and slow switching in globally coupled phase oscillators', *Phys. Rev. E* **48**(5) (1993), 3470-3477. 'Phase dynamics for weakly coupled Hodgkin-Huxley neurons', *Europhys Letts* **23** (1993), 367–372.

[27] J Hofbauer. 'Heteroclinic cycles on the simplex', *Proc. Int. Conf. Nonlinear Oscillations, Janos Bolyai Math. Soc. Budapest*, 1987.

[28] J Hofbauer. 'Heteroclinic cycles in ecological differential equations', *Tatra Mountains Math. Publ.* **4** (1994), 105–116.

[29] J Hofbauer & K Sigmund. *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, 1988.

[30] J Hofbauer & K Sigmund. *Evolutionary Games and Replicator Dynamics*, Cambridge University Press, Cambridge, 1998.

[31] A J Homburg & J Knobloch. 'Switching homoclinic networks', *Dynamical Systems* **25**(3) (2010), 351–358.

[32] V Kirk, E Lane, C M Postlethwaite, A M Rucklidge & M Silber. 'A mechanism for switching near a heteroclinic network', *Dynamical Systems: An International Journal*, **25**(3) (2010), 323–349.

[33] I Kiss, C Rusin, H Kori & J Hudson. 'Engineering complex dynamical structures: sequential patterns and desynchronization', *Science* **316** (2007), 1886-9. *Nonlinearity* **7**(6) (1994), 1605–1622.
[34] M Krupa. 'Robust heteroclinic cycles', *Journal of Nonlinear Science* **7** (1997), 129–176.
[35] M Krupa & I Melbourne. 'Asymptotic stability of heteroclinic cycles in systems with symmetry', *Ergod. Th. & Dynam. Sys.* **15** (1995), 121–147.
[36] M Krupa & I Melbourne. 'Asymptotic stability of heteroclinic cycles in systems with symmetry, II', *Proc. Roy. Soc. Edinburgh A* **134A** (2004), 1177-1197.
[37] K K Lin & L-S Young. 'Shear-induced chaos', *Nonlinearity* **21** (2008), 899–922.
[38] R M May & W J Leonard. 'Nonlinear aspects of competition between three species', *SIAM J. Appl. Math.*, **29** (1975), 243–253.
[39] I Melbourne, P Chossat & M Golubitsky. 'Heteroclinic cycles involving periodic solutions in mode interactions with $O(2)$ symmetry', *Proc. Roy. Soc. Edinburgh* **113A** (1989), 315–345.
[40] R-M Memmesheimer & M Timme. 'Designing the Dynamics of Spiking Neural Networks', *Phys. Rev. Letts* **97** (2006), 188101.
[41] A Mohapatra & W Ott. 'Homoclinic loops, heteroclinic cycles and rank one dynamics', *SIAM Jnr. Appl. Dyn. Sys*, to appear.
[42] F S Neves & M Timme. 'Controlled perturbation-induced switching in pulse-coupled oscillator networks', *J. Phys. A: Math. Theor.* **42** (2009), 345103.
[43] F S Neves & M Timme. 'Computation by switching in complex networks', *Phy. Rev. Lett.* **109** (2012), 751–811.
[44] T Nowotny & M Rabinovich. 'Dynamical Origin of Independent Spiking and Bursting Activity in Neural Microcircuits', *Phys. Rev. Letts* **98** (2007), 128106.
[45] G Orosz, J Moehlis & P Ashwin. 'Designing the dynamics of globally coupled oscillators', *Progress of Theoretical Physics*, **122**(3) (2009), 611-630.
[46] M Rabinovich, R Huerta & G Laurent. 'Transient Dynamics for Neural Processing', *Science* **321** (2008), 48-50.
[47] M Rabinovich, A Volkovskii, P Lecanda, R Huerta, H D I Abarbanel & G Laurent. 'Dynamical encoding by networks of competing neuron groups: Winnerless competition', *Phys. Rev Lett* **8706**(8102) (2001), 149–141.
[48] G L dos Reis. 'Structural stability of equivariant vector fields on two manifolds', *Trans. Amer. Math. Soc.*, **283** (1984), 633–643.
[49] L P Shilnikov, A L Shilnikov, D V Turaev & L O Chua. *Methods of Qualitative Theory in i Nonlinear Dynamics, Part 1* (World Scientific, Singapore, 1998).
[50] A L Shilnikov, D V Turaev, L O Chua & L P Shilnikov. *Methods of Qualitative Theory in Nonlinear Dynamics, Part 2* (World Scientific, Singapore, 2002).
[51] Q Wang & W Ott. 'Dissipative homoclinic loops of two-dimensional maps and strange attractors with one direction of instability', *Comm. Pure Appl. Math.*, **64** (2011), 1439–1496.
[52] Q Wang & L-S Young. 'Strange attractors in periodically-kicked limit cycles and Hopf bifurcations', *Comm. Math. Phys.* **240** (2003), 509–529.

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