Separation in Point-Free Topology

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To our wives Jitka and Isabel and to our teacher and friend Bernhard Banaschewski

Preface

Point-free topology is a general discipline of geometry exploiting the algebraic properties of natural pieces of spaces. It originated in individual results in the late 1930s and early 1940s, developed to the current concept system in the 1960s and 1970s, and flourishes ever since. The reader can learn the basics (and certainly more than just that) in several monographs and chapters in handbooks and similar publications. There is, however, no systematic comprehensive presentation of special properties needed when treating concrete special problems. The aim of this book is to fill in this gap.

Similarly like in classical topology, when dealing with special problems the scope of the quite general concepts has to be restricted by specific conditions. In the classical theory, the most significant role in this respect is played by the so-called separation axioms of various strength. In the point-free context, the need to delimit and specify the objects suitable for particular purposes is perhaps even stronger, and it is natural to borrow and imitate the classical requirements (already in the early times of the theory, one discovered the benefits of the condition of regularity corresponding precisely to the homonymous axiom of classical topology; also, we should not forget to note the use of the so-called disjunctivity replacing the T_1-axiom in the point-free prehistory). Although the points seem to be crucial in the classical formulations of separation axioms and similar conditions, it has turned out that one can either produce exact counterparts or at least mimic them to advantage. And sometimes one discovers very useful conditions of essentially point-free nature.

This book bids a systematic study of this area. We present as much as possible of the broad range of separation type conditions, from very weak ones (in among of which the reader might be surprised by the benefits of the so far underestimated subfitness), over the pleiad of the Hausdorff type conditions, to very strong ones (the particularly strong scatteredness may be surprising to be found in this company, but in the point-free context it belongs, and plays a very interesting role). We discuss their interrelations and point out consequences.

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Introduction

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In the nineteenth century, mathematical analysis flourished. The correct formulation of newly discovered facts needed precise thinking about phenomena like convergence, continuity (and different types of continuity) or approximation. A space was not any more just the Euclidean space or a part of it. More or less explicitly, people started to think in terms of fairly general metric spaces. This was a great progress, and this is how we treat with advantage a lot of questions of analysis (and not only of analysis) since.

But it did not take long to observe that for some basic notions, notably in particular for the most fundamental concept of continuity, the concrete metric structure was unimportant: although the definitions were naturally expressed in terms of distance (metric), replacing a concrete metric by others gave the same results. What, then, is the structure that makes a set to a space? The attempts to solve this problem culminated in Hausdorff's ingenious idea published in 1914 (*Grundzüge der Mengenlehre*, Leipzig) that laid the foundation for (set-theoretic) topology, the generalized geometry as we know it today. The idea is very transparent and intuitive: for an element (point) of a set, one has to make the difference between being surrounded by the set (like a boat in the middle of a lake, surrounded by water from all sides) or just being in the set, possible on the border (like a boat landed, still in water, but already bordering the shore). It turned out, perhaps surprisingly, that for understanding continuity and related phenomena, it suffices to understand a space as a set X with the additional information whether a subset $U \subseteq X$ containing an element x surrounds it (one speaks of being a *neighbourhood* of x) or not, with a few very simple and very intuitive axioms.

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Instead of taking the idea of neighbourhood for the primitive concept, one can, equivalently, start with the concept of an *open set*. In a space defined as above, an open set is a set that is a neighbourhood of each of its points; conversely, if we start with open sets, we can define a neighbourhood V of x as a set such that there is an open U between x and V, that is, such that $x \in U \subseteq V$. This variant of the definition of a (topological) space was used very shortly after introducing the neighbourhood idea. The system of axioms needed here is even simpler than specifying the properties of neighbourhoods: it is only a simple property of a sublattice of the lattice of all subsets of the set of points. While it may be slightly less intuitive, it is technically much more expedient, and it is the definition mostly used.

In fact, it is less intuitive only from the point of view of a space as a suitably structured set of points. We can think of a space from another angle, though, and then the concept will become very natural. Points are constructs, not very realistic entities without extent. Forget about them and think of a space as about a system of places, spots of a typically nontrivial extent. They join to make bigger spots, and they "hold together" if they meet. This is all the geometry we need to start with, and this is the pivotal idea of point-free topology. The reader certainly knows that what we are speaking about is the so-called *frame* (or *locale*)—see [161, 220] or the Appendix below—the generalized space some aspects of which are the main topic of this book. Good and typical models of such generalized spaces are the lattices $\Omega(X)$ of open sets of topological spaces.[1] There are other frames, and the theory is a considerable extension of the classical one (we will speak of the point-based topological spaces as of the *classical ones*), but we will relate the explanatory examples in this Introduction mostly to the $\Omega(X)$. Just keep in mind that in the spaces one usually works with, the open sets are models of such realistic nontrivially extended easily understandable places, and that in the general theory such places have their own existence and do not have to consist of points.

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Now, let us return to the classical topological spaces. The general concept is indeed very general, for many purposes too general (in one respect, however, the Hausdorff's axiomatics was not general enough; we will come to it shortly). Therefore, one often restricts the scope of spaces in question by various extra conditions, among which a particular role is played by the so-called *separation*

[1]The spaces X of a very broad class are determined and well represented by the lattice $\Omega(X)$. Therefore, one often speaks of the lattices as of the spaces.

axioms. Already the original Hausdorff's system of axioms contained such a separation condition, requiring that

(H) *any two distinct points have disjoint neighbourhoods*

(in other words, any two distinct points are *separated* by neighbourhoods). This is what we call today the *Hausdorff axiom*.

In the first years of topology, the interest was predominantly focused to spaces that were more and more geometric (or, rather, more and more like spaces with the structure determined by a metric). Thus, one considered the *regularity* requiring that

(reg) *any point x and a closed set $A \not\ni x$ have disjoint neighbourhoods*

(a point not contained in a closed set and the closed set can be *separated* by neighbourhoods), *complete regularity* where one separates such a pair of a point and a closed set by a continuous real function f (in the sense that $f(x) = 1$ and $f[A] = \{0\}$), or the *normality* requiring that

(norm) *any two disjoint closed subsets have disjoint neighbourhoods.*

Further, there are stronger types of normality like the *complete normality* where one assumes that every subspace of the given one is normal (for the plain normality this is not generally the case, unlike for the Hausdorff property or regularity) or *perfect normality* where one assumes that any two disjoint closed subsets A, B can be separated by a continuous real function f such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$ (spaces satisfying this condition are already very similar to metrizable ones) or *full normality* where every open cover has a star refinement (in normality this holds for *finite* open covers).

Conversely, it turned out that there are applications of topology where the Hausdorff axiom is too strong. This led to the weaker assumption that

(T_1) *any point can be separated from any other by a neighbourhood*

or the even weaker (T_0) where for any two distinct points, at least one can be separated from the other.[2]

This is of course not an exhaustive list of separation axioms. Some more will be presented in Chap. I, for instance, the very important T_D, or the symmetry, and also the particularly important sobriety (which is in fact an axiom of a different nature, sometimes included into the separation area for quite good reasons), but the reader has certainly already an idea what we have in mind when speaking of conditions of this type.

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In point-free topology, the need to specify the objects suitable for particular purposes is perhaps even stronger than in the realm of classical spaces. First, of

[2] For the T_k notation, see Chap. I (Sect. 4.4); T_1 is sometimes referred to as the *Fréchet axiom*.

course, there are the questions whether that or other classical result has a point-free counterpart. The spaces in such classical facts are mostly subjected to special conditions that have to be somehow satisfied also in the extended situation. But there are also (and often) phenomena that have no classical background or that are more interesting in the point-free context. Such facts very rarely hold in absolute generality. Hence, what are the (hopefully natural) conditions under which they do hold? All this calls for studying the point-free separation.

This need was obvious from the very beginning of the theory. In particular, already in the founding years of point-free topology, the role of regularity and complete regularity was recognized,[3] and very useful results on (completely) regular frames were obtained (to name just one, the choice-free variant of the Stone–Čech compactification presented in [43, 44]). Soon there also appeared an article [81] showing that one can formulate separation axioms or suitable replacements in the point-free setting. In [152], Isbell introduced and discussed important separation conditions specific for the point-free spaces.

In the recent decades, separation was studied with a growing intensity. It turned out, a.o., that some conditions (and some of them neglected for years, e.g., subfitness) play a much more important role than had been expected. Therefore, we think that the subject deserves a comprehensive treatment. It will be the main topic of this book.

Thus, what topics we wish to discuss? First of all, we will have to analyse the conditions that extend or mimic the standard ones from classical theory. Their adaptability to the point-free situation varies. Some of them can be directly or almost directly translated (such as normality and also regularity). Then there are translations based on a necessary and sufficient condition, not obvious, but better suited for the point-free context. And there are also such that are translated only seemingly: they can lead to very good analogues or sometimes to something quite different. We will also meet cases where the translation is precise but not very useful, because at a closer scrutiny they reduce to speaking on classical spaces only.

Secondly, but about this we will speak later, there are separation conditions specific for the point-free context, not making much sense for classical spaces but very useful in the general situation.

We will be interested, even in the case of spaces, only in the facts that can have a point-free interpretation. Therefore, ***we will mostly ignore the axiom T_0***. Two points that cannot be distinguished (that is, separated) by an open set have to be treated, in our perspective, as one. Therefore, for trivial reasons, there will be made no attempt to find an analogue of this axiom for frames, and, however, when speaking of classical spaces, the axiom T_0 will be automatically assumed (perhaps now and then recalled, only not to be forgotten).

Before discussing concrete examples: let us agree to call an extension of a concept *conservative* if, when applied to the frame $\Omega(X)$, it coincides with the

[3] We speak of the 1960s and later; a very important separation condition was discovered and used in 1938 (by Wallman [282]) and then neglected for decades, see paragraphs 7 and 8 below.

homonymous property of X. It can sometimes happen that a definition is not conservative, but so useful that it deserves keeping the name; however, there are also cases of conservative extensions that are practically useless.

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From the conditions above, only one, namely normality, has an absolutely straightforward translation into the language of open sets. Replacing the closed sets A, B by their open complements C, D, we can reformulate the axiom (norm) for a space X by requiring that

for any two open sets C, D such that $C \cup D = X$, there are disjoint opens U, V such that $C \cup U = X = D \cup V$.

But also the regularity condition is very simple. For open sets, define the *rather below* relation

$$V \prec U \quad \equiv_{\mathrm{df}} \quad \overline{V} \subseteq U$$

(in the language of the lattice of open sets, $\overline{V} \subseteq U$ can be expressed, using the pseudocomplement $V^* = \mathrm{int}(X \smallsetminus V)$ of V, as $V^* \cup U = X$). Then it is easy to see that a space is regular if and only if every open U is equal to the union $\bigcup\{V \mid V \prec U\}$.

This formula is an example of an *almost direct* translation mentioned above. Note that it comes immediately from a (very straightforward) characterization of regularity standardly used in classical topology, namely that regular spaces are those in which every neighbourhood of a point contains a closed one. (Complete regularity can be expressed similarly, with a variant of the relation $\prec$.[4] It needs however some more reasoning (see Chap. VI, Sects. 1 and 2) which brings a useful insight into the classical spatial condition as well.)

To give an example of a not quite straightforward characterization yielding a conservative extension, here is the Johnstone–Paseka–Šmarda–Sun Shu–Hao formula for the Hausdorff property (why so many authors: the formula merged from very differently motivated approaches by Johnstone and Sun Shu-Hao [171] and by Paseka and Šmarda [216]):

for any two open sets A, B such that $X \neq A \nsubseteq B$, there are disjoint opens U, V such that $U \nsubseteq A$, $V \nsubseteq B$.

[4] Namely, the *completely below* relation $\prec\!\!\prec$ introduced by Banaschewski in his 1953 doctoral dissertation [19], replacing a notion Alexandroff had introduced using real-valued continuous functions [2].

(It is very easy to prove that this is equivalent to the separation of points as in (H) above, one has only to reason just a bit. Note that incomparability of A and B would not do the job.)

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The Hausdorff axiom, a point-free formulation of which we have presented in the previous paragraph, is fairly instructive. There are several approaches, two of them merging in the conservative formula we have recalled above.

Another one (Isbell [152]) was based on the following elegant (and obvious) classical characterization: a (T_0) topological space is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in the product $X \times X$. Now this is nice: we know what the products $L \oplus L$ in the category of locales are, a diagonal subobject is well defined and well understood, and also closedness of subobjects is a straightforward concept (precisely corresponding to classical closedness of subspaces). Thus, we can say that a locale L is *Hausdorff* if the diagonal in the square $L \oplus L$ is a closed sublocale (generalized subspace). But beware: we have forgotten that the product in the category of locales does not quite correspond to the classical one: the category is bigger and hence the universality of the product is checked by many more objects than in the category of topological spaces, resulting in the fact that the product of locales $\Omega(X) \oplus \Omega(X)$ does not necessarily correspond to the product of spaces $X \times X$. This is an example of a definition that is translated only seemingly. And indeed thus defined Hausdorff property (call it (sH)) is not conservative, that is, a space can be classically Hausdorff while $\Omega(X)$ is not (sH). But it would be wrong to dismiss it. The situation is strange: locales satisfying this non-conservative definition behave like Hausdorff objects should. In many respects, they behave better than those with the conservative property (just to give an example, (sH) combined with compactness yields regularity, the conservative one does not). This shows that sometimes it can be advisable to mimic a classical property by something that is just analogous and not necessarily a precise extension.

There are several interesting facets of the Hausdorff property in the point-free context. Another one is an example of a contingency that was not mentioned yet. When extending a concept we may have a choice between two necessary and sufficient conditions that are (of course) equivalent for spaces but may diverge in the broader situation. Both the extended notions are then conservative, but different (see Chaps. III, 3.4 and 4.1).

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We have seen that there are sometimes reasons for preferring a non-conservative analogue for a definition of an extended concept, even if we have a perfectly correct conservative one. The axiom T_1 will be an even better example. Consider the following necessary and sufficient condition:

A (sober[5]*) space X is T_1 if and only if all the prime elements in $\Omega(X)$ are maximal.*

Here we have a nice statement in the language of open sets. Primeness is a lattice concept, not to speak of maximality. Yet, taking it for a definition of a frame separation axiom ("a frame is said to be T_1 if each of its prime elements is maximal"), which is sometimes done, is not particularly helpful. Why? It is because the primes represent the points of the spectrum (see A.3.4) so that one actually speaks only about the spectrum ΣL, the spatial part of L. Defining a property $\widetilde{P}$ for a frame L by requiring P for ΣL is somewhat cheap, but first of all it is practically useless: it cannot say anything new in the extended point-free area.[6]

This leads to an example of a separation condition that is inherently point-free (although it does make perfect sense for spaces as well). Consider the following requirement on a (complete) lattice L:

$$\forall a, b \in L \ \ (\text{if } a \nleq b \text{ then there is a } c \in L \text{ such that } a \vee c = 1 \neq b \vee c).$$

This property is called *subfitness*. It appeared, as a suitable substitute for T_1 already in 1938, in one of the first papers using point-free reasoning (Wallman [282]), under another name, long before point-free topology started to be cultivated. (It obviously holds in T_1-spaces: If $A \nsubseteq B$ takes an $x \in A \smallsetminus B$; then, we have the open $C = X \smallsetminus \{x\}$ with $A \cup C = X \neq B \cup C$. Moreover, under a very weak condition T_D, it is equivalent with T_1.)

Subfitness is strictly weaker than T_1 in classical spaces (see II.2.2). For frames it is extremely useful. It is not only a very good replacement of T_1; we will meet it again and again as a sufficient (often, necessary and sufficient) condition for important point-free facts.

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Although subfitness makes a good sense in classical topological spaces (see Sect. 2.1 in Chap. II), we cannot view it as a translation of a classical property. It appeared in the aforementioned Wallman's article [282] as an inherent lattice theoretic concept

[5]For sobriety see I.7. We think now of the typical primes $X \smallsetminus \overline{\{x\}}$; if $\overline{\{x\}} \neq \{x\}$, we have two such open sets, both prime, comparable.

[6]Admittedly, this is a rather harsh rebuff. Thus defined T_1 has its role in analysing analogues C of stronger classical properties. The question whether such C implies T_1 is certainly legitimate (see, e.g., IV.1). But we are speaking on the property in se.

(called *disjunctivity*[7]) that could serve as a helpful *replacement* of T_1. When it reappeared after more than three decades, in a different but equivalent form, it was as a specific property of point-free spaces, already studied as objects in their own right.

Indeed, in 1972, Isbell [152] defined *subfitness* as the property that

(sfit) *every open sublocale* (*generalized subspace*) *is a join of closed ones,*

together with the *fitness* where he required that

(fit) *every closed sublocale* (*generalized subspace*) *is a meet of open ones.*

These two properties look, deceptively, as being somehow dual to each other. They are not. Actually, the latter can be characterized as hereditary subfitness (subfitness itself is not inherited by subobjects) and it is much stronger. While in spaces subfitness is weaker than T_1, fitness is almost as strong as regularity. The discrepancy may be surprising, but keep in mind that in the point-free context a space can have a lot of new sublocales, and requiring a given property for all of them is a rather strong claim.

The open-closed formulas for subfitness and fitness are good examples of conditions that look like translations from the classical theory but are indeed not so. In fact, requiring in spaces that every open subset is a union of closed ones makes a good, but different, sense (namely, it is equivalent with the so-called *symmetry*, which the subfitness is not), and by De Morgan formula the same is obtained requiring that every closed subset is an intersection of open ones. (Although open resp. closed sublocales are in perfect correspondence with open and closed subsets, the lattice structure of the system of sublocales is more complex; also, one has just a one-sided De Morgan formula.)

The formula for fitness is, perhaps surprisingly, equivalent with the formally stronger

(fit’) *every sublocale whatsoever is a meet of open ones.*

We have already stated that fitness and subfitness are, despite the appearance, far from being dual to each other. Hence, one can hardly expect that the equivalence of fitness with (fit’) should have a full analogue for subfitness. And indeed the condition that

every sublocale whatsoever is a join of closed ones

is much stronger than subfitness (actually, even much stronger than fitness). It turns out that it is equivalent with the well-known *scatteredness*, a property that is perhaps more important in point-free topology than in the classical one (in particular, *scattered spaces* are precisely those for which subspaces and sublocales coincide).

[7] Wallman had in mind the lattice of *closed* sets. Much later, when the opposite perspective of *open* sets was already firmly established, the authors of, e.g., [192, 254, 255] started to speak of *conjunctivity*.

Note that the analogous claims about spaces and subspaces as above are again the same, every subset is a union of closed ones if and only if every open subset is an intersection of open ones, and it is nothing else but T_1. Here we can see the pitfalls of extending notions by analogy: we have two necessary and sufficient conditions for T_1 and if we extend them with general subobjects instead of subspaces, we eventually obtain two very strong (distinct) conditions.

We have got acquainted with the setting. Now we can briefly outline the contents.

In Chap. I, we summarize the well-known separation axioms in classical topological spaces. We add some more: symmetry, the very important T_D, and sobriety. We start to think in terms of the lattice of open sets.

Chapter II is devoted to the basics of subfitness and fitness. It starts with T_1 and the point-free T_U (which differs from the spatial case where it coincides with symmetry). Some consequences of subfitness are presented: in particular, a somewhat surprising useful formula for the Heyting operation (and pseudocomplement) and a spatialization theorem. Isbell's formulas for subfitness and fitness are introduced, and fitness is shown to be the hereditarily modified subfitness. Specific properties of congruences under these conditions are presented.

In Chap. III, we turn to axioms of Hausdorff type. We introduce five different approaches one encounters in literature and discuss the resulting four concepts (two of the approaches merge) and show how they relate. Then, after introducing the necessary techniques, we prove some particularly nice properties of the "strong Hausdorff axiom" (the closed diagonal one) in which the others fail or most probably fail: the density theorem, the facts that under this condition compact sublocales are closed, and that strongly Hausdorff compact frames have stronger separation properties.

Chapter IV summarizes the "low separation properties". First, relations between T_1, T_U and subfitness are briefly discussed. Then, two more characterizations of the strong Hausdorff property are introduced, and the merits of the individual variants of point-free Hausdorff axioms are pointed out. After presenting some implications and (sometimes surprising) non-implications, we outline the tangle of relations in the low separation world.

In Chap. V, we discuss regularity, the first separation axiom that had been widely technically used in point-free theory and also the previously defined fitness from the perspective of its closedness to regularity. We show that regularity implies the strong Hausdorff property and all the lower separation properties. The historical role is emphasized and proofs originally used when regularity substituted weaker assumptions recalled. Some properties of fitness presenting this property as a relaxed regularity are proved, together with discussing other related assumptions.

In [220], we had an incorrect proof of reflexivity of fitness; we pay the debt now by rendering a correct one.

Chapter VI is concerned with complete regularity. First, we briefly discuss the relation “completely below” (to the analysis of which we return at the end of the chapter). Complete regularity is compared with regularity; we recall examples proving it is strictly stronger. The role of complete regularity in uniformization is recalled. The structure of cozero elements is discussed, and a Lindelöf reflection (a specifically point-free phenomenon) is presented. Finally, a choice-free variant of complete regularity is introduced (an application of which is, e.g., the choice-free Stone–Čech compactification).

In Chaps. VII and VIII, we treat the question of normality and various stronger properties (omitting paracompactness, also known as full normality, which had its own chapter in [220]). We start with plain normality and its basic properties. Next, we discuss the behaviour of finite covers and present the Wallman compactification of a subfit frame, showing that, under normality, it coincides with the Stone–Čech compactification. We conclude Chap. VII with a discussion of complete normality, the hereditary version of normality.

In Chap. VIII, we have two more variants of normality. There is the perfect normality, which turns out to be a conjunction of the classical perfectness (that is slightly different in the point-free context due to the different behaviour of sublocales and subspaces) and normality. Next, we deal with the technically important collectionwise normality, weaker than paracompactness. Then we present point-free real-valued functions and prove, in the penultimate section, the Katětov–Tong insertion theorem, using to advantage the techniques of the point-free real line. We finish with a unified view of several weaker variants of normality and a glimpse of the parallel between normality and extremal disconnectedness.

When comparing fitness and subfitness in 8 above, we mentioned that unlike fitness, where we can think of *all* sublocales as intersections of open ones, arbitrary sublocales being joins of closed ones is a property much stronger than subfitness. This and the theory of an ensuing envelope is the topic of Chap. IX. First, we recall the concept of scatteredness, both in the point-free and in the classical contexts, and show that for frames it coincides with the stronger formula from 8. For spaces, we present the Simmons (and Niefield–Rosenthal) theorems on sublocales in scattered spaces and frames. The frame of joins of closed sublocales $\mathsf{S}_\mathfrak{c}(L)$ is introduced. It is shown that for subfit frames it is a Boolean algebra, and more (for instance, that it is the maximal essential extension).

The last chapter contains some facts that did not exactly fit into the individual chapters.

For the convenience of the reader, we add a concise appendix containing some definitions and facts they may prefer to have at hand rather than having to look for them elsewhere. Besides repeating facts from our previous monograph [220], it also presents facts that are not quite so easy to look for in the literature (like, e.g., the frame coproduct $\oplus$ as a tensor product) and simpler proofs of known facts.

Contents