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Locales treated mostly in a covariant way. (English)

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The text consists of 12 chapters covering certain aspects of rudiments of pointfree topology. It is not (and it does not pretend to be) a comprehensive study of the subject. Indeed, as the authors point out, it omits a number of topics which are germane to pointfree topology, such as compactness and compactification.

The first chapter recalls Galois adjunctions, Heyting algebras, computation with the Heyting implication, Boolean algebras and sobriety. The category Loc of locales and their maps is introduced in the second chapter. The objects of Loc are frames in the traditional sense. The new idea is in the choice of the morphisms. Whereas in his pioneering paper ["Atomless parts of spaces", Math. Scand. 31, 5–32 (1972; Zbl 0246.54028)], *J. R. Isbell* simply reversed frame homomorphisms and took the resulting morphisms in $\mathrm{Frm}^{\mathrm{op}}$ as morphisms in the category he christened the category of locales, the authors define a localic map between locales L and M to be a mapping $f : L \to M$ which has a left adjoint $f^* : M \to L$ that preserves finite meets, including the top. In standard parlance, these are mappings which have left adjoints that are frame homomorphisms. The key thing to note is that localic maps are functions in the usual sense, and not merely arrows in some category. Thus, for instance, given a frame homomorphism $h : L \to M$, its right adjoint $h_* : M \to L$ is a localic map.

Points of a locale are defined to be its prime elements; which then leads to the establishment of the spectrum $\mathrm{Pt}(L)$ of L in the usual way. Localic maps send points to points, so that $\mathrm{Pt} : \mathrm{Loc} \to \mathrm{Top}$ becomes a covariant functor sending a localic map $f : L \to M$ to its restriction $\mathrm{Pt}(f) : \mathrm{Pt}(L) \to \mathrm{Pt}(M)$. In the opposite direction, there is a covariant functor $\mathrm{Lc} : \mathrm{Top} \to \mathrm{Loc}$ that sends a topological space to its frame of open sets, and a continuous function to the right adjoint of the frame homomorphism it induces. This is one of many instances demonstrating the desirability of choosing maps of locales as the authors have done.

The (by now expected) adjunction between these functors is established in Chapter 3, as are (also expected) spatiality and sobriety characterizations in terms of the attendant natural transformations

$$\mathrm{LcPt} \to \mathrm{Id} \quad \text{and} \quad \mathrm{Id} \to \mathrm{PtLc}.$$

The fourth chapter is about the basic structure of localic maps. It is shown that epimorphisms in this new category (new because the morphisms are new) are precisely the surjective localic maps, and extremal monomorphisms are exactly the injective localic maps. Furthermore, the couple of classes (*surjective localic maps, injective localic maps*) constitutes a factorization system in Loc.

Sublocales, treated in Chapters 5 and 6, are defined as follows: $S \subseteq L$ is a *sublocale* in case it is closed under meets and for every $s \in S$ and $a \in L$, $x \to s \in S$, where $\to$ denotes the Heyting implication. Of course this is not a new idea; indeed, it appears as Exercise 2.3 in [*P. T. Johnstone*, Stone spaces. Cambridge Studies in Advanced

Mathematics, 3. Cambridge: Cambridge University Press (1982; Zbl 0499.54001)] and characterizes a sublocale when defined as a fix-set of some nucleus. An elegant new characterization is that a subset of a locale is a sublocale iff its identical embedding in the locale is a localic map. Although the notion of sublocale is not new, the method of treatment brings to the fore in a very transparent way known properties of open, closed and Boolean sublocales; culminating with the delectable *Every sublocale is a union of Boolean sublocales*, and a useful spatiality criterion with a neat proof.

Since a localic map $f : L \to M$ is a function, if S is a sublocale of L, then $f[S] = \{f(x) \mid x \in S\}$ is a subset of M. Chapter 7 starts by showing that, in this case, $f[S]$ is in fact a sublocale of M, referred to as the *image of* S under f. Regarding preimages, some ingenuity was needed. Every subset of a locale which is closed under meets contains a largest sublocale. If $f : L \to M$ is a localic map and T is a sublocale of M, then $f^{-1}[T]$ is a subset of L which is closed under meets. The largest sublocale it contains is defined to be the *preimage of* T under f. This is as close to classical topology as it gets. To paraphrase the authors, "images have now become really images, and preimages are really preimages modulo slight modifications". Open and closed localic maps are defined as in classical topology. The rest of the chapter establishes properties of images, preimages, open maps and closed maps that one expects.

Chapter 8 is pretty much preparatory for its successor, in which products are defined. Incidentally, this is one instance where most calculations are done in Frm rather than Loc per se. It is established that the category Loc (remember that it is not the old category of locales) is complete and cocomplete.

Chapters 10 and 11 are about separation – the former concentrating on regularity, fitness and subfitness; and the latter on Hausdorff axioms. The last (and longest) chapter deals with entourages and quasi-uniformities on a locale; leading to an introduction of localic groups as group objects in the category of locales.

To summarize: this is a well-written text that brings a fresh view of pointfree topology. Some terseness here and there notwithstanding, a beginner in the field will find it very readable and informative. It should of course be read (by a beginner) in conjunction with other texts mentioned in its bibliography. A seasoned pointfree topologist (whether "algebraic" or "geometric" in inclination) should also find it very informative and its approach legitimately covariant vis-à-vis classical topology, and not just a sleight of hand that swiftly turns arrows around to make things look covariant.

Regarding the old category of locales, *B. Banaschewski* writes in ["Uniform completion in pointfree topology", in: S. E. Rodabaugh et al. (eds.), Topological and algebraic structures in fuzzy sets. Dordrecht: Kluwer. Trends Log. Stud. Log. Libr. 20, 19–56 (2003; Zbl 1034.06008)]: "... for some authors, the proper subject of pointfree topology is the category of *locales*, ... Whatever the reason for this, it remains a fact that any actual construction in this area is carried out in terms of frames and their homomorphisms ...".

The text under review demonstrates that with the new category, actual calculation can also be carried out in terms of locales and localic maps.

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Classification :

∗06D22 Frames etc.
06-02 Research monographs (ordered structures)
54B30 Categorical methods in general topology
54C20 Extension of maps on topological spaces
54D35 Compactifications
54E15 Uniform structures and generalizations