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Locales. (English)

Pedicchio, Maria Cristina (ed.) et al., Categorical foundations. Special topics in order, topology, algebra, and sheaf theory. Cambridge: Cambridge University Press. Encycl. Math. Appl. 97, 49-101 (2004). ISBN 0-521-83414-7/hbk

The elementary and historically motivating question of the theory of locales is this: how much information about a topological space T is contained in the complete lattice $\mathcal{O}T$ of all its open subsets when treated as merely a complete lattice by ignoring the fact that members of $\mathcal{O}T$ consist of points. On the other hand, since a continuous map $f: S \to T$ between spaces induces a lattice morphism $h: \mathcal{O}T \to \mathcal{O}S$ defined by $h(U) = f^{-1}(U)$, another question arises: how much about f is retained by h. This explains another name of the subject: *pointfree topology.* However, studying locales is not merely solving problems of how to formulate 'pointed' concepts and related results without mentioning points (in passing, Hausdorffness is a good example of how to do so). One advantage of working with locales is that a pointfree context may necessitate a choice-free argument, thereby providing a constructive proof. From that reason locales are far from being 'generalized topological spaces' only.

More advantages are provided in the introductory part of the article under review, from which we cite: "In many situations, certain spaces are non-trivial only by virtue of some choice principle, whereas their lattices of open sets already have previous existence, before such assumptions. This means that, in some sense, we always see the lattice of opens, but to see their points may require some additional tool in the form of some choice principle." The constructive approach to topology is probably best illustrated by the Tychonoff product theorem for locales.

The paper under review is a very interesting and useful expository article on selected topics in the theory of locales (or frames). More than half of the bibliography items have been published (or written) after *Peter Johnstone*'s classical monograph [Stone spaces, Cambridge Univ. Press (1982; Zbl 0499.54001)].

Section 1 (Spaces, frames, and locales) describes the pointfree way of thinking that led to consider a frame X (as a substitute of a space), i.e., a complete lattice in which finite meets distribute over arbitrary joins: $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$ for all $a \in X$ and $S \subset X$, as well as a frame homomorphism $h: Y \to X$ (as a substitute of a continuous function) as a map that preserves all joins and all finite meets. The obtained category is denoted by Frm. Sending a topological space T to $\mathcal{O}T$ and f to f^{-1} yields a contravariant functor $\mathsf{Fop} \to \mathsf{Frm}$. The dual category Loc has the same objects as Frm while morphisms (= localip maps) act in the oposite direction, i.e., one now has a covariant functor $\mathsf{Lc}: \mathsf{Top} \to \mathsf{Loc}$. The latter functor is shown to have the well-known) right adjoint functor $\mathsf{Pt}: \mathsf{Loc} \to \mathsf{Top}$ which acts (on objects) by taking a locale X to the space of its 'points', i.e., to $\mathsf{Pt}(X) = (\{h : X \to \{0,1\} \text{ in } \mathsf{Frm}\}, \{\Sigma_a : a \in X\})$ where $\Sigma_a = \{h : h(a) = 1\}$, while, given a localic map $f: \mathsf{Pt}(X) \to \mathsf{Pt}(Y)$, $\mathsf{Pt}(f)(h) = h \circ f^*$ where f^* denotes the corresponding frame homomorphism from Y to X.

The remaining sections are: 2. Sublocales; 3. Limits and colimits; 4. Some subcategories

of locales; 5. Open and closed maps; 6. Compact locales and compactifications; 7. Locally compact locales.
Sections are divided into subsections, most of which end with exercises.

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Classification :

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