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Insertion and extension results for pointfree complete regularity. (English) Zbl 1284.06020

Since the advent of lower and upper semicontinuity in locales (to the development of which the current authors – either jointly, solely, or in collaboration with other authors – have contributed most significantly), there have been insertion-type characterisations of higher separation axioms. An example is that a locale $L$ is normal precisely if whenever $f \leq g$, where $f$ is an upper semicontinuous real function on $L$ and $g$ lower semicontinuous, then a continuous real function on $L$ can be inserted between them [J. Picado, Topology Appl. 153, No. 16, 3203-3218 (2006; Zbl 1104.06007)]. Corresponding to this theorem is an extension theorem saying that a closed sublocale of a normal locale is $C$-embedded [J. Gutiérrez García, J. Pure Appl. Algebra 213, No. 6, 1064–1074 (2009; Zbl 1187.06005)]. As is the case with other works of these authors (individual, joint, or with other collaborators), this paper is lucidly written. It starts with a well-motivated question. In [J. Gutiérrez García and T. Kubiak, Appl. Gen. Topol. 8, No. 2, 239–242 (2007; Zbl 1152.54020)] a function $f : X \to [0,1]$ is called compact-like if $f^{-1}[t,1]$ is compact for every $t \in (0,1]$. It is then proved that a topological space $X$ is completely regular if and only if for any functions $f, g : X \to [0,1]$ with $f$ compact-like, $g$ lower semicontinuous, and $f \leq g$, there is a continuous function $h : X \to [0,1]$ such that $f \leq h \leq g$. Naturally, then, one may ask if such an insertion-type characterisation holds for locales. The main theme of the paper addresses this question. A full answer is not known, but a partial one covering a large class of locales is available. By first characterising regular and completely regular spaces in terms of compact sublocales (of the spaces viewed as locales), they define what they call $c$-regular and completely $c$-regular locales. Every (completely) regular locale is (completely) $c$-regular. In fact, for a wide class of locales (including the spatial ones and the zero-dimensional ones) (complete) regularity is equivalent to (complete) $c$-regularity, but it is not known if this holds for all locales. The insertion theorem (after defining in a natural way upper and lower compact-like real functions on a locale) is then proved. It says that if $L$ is a completely $c$-regular locale and $f \leq g$, for some upper compact-like $f$ and lower compact-like $g$, then a continuous real function on $L$ can be inserted between $f$ and $g$. The converse holds for locales in which compact sublocales are complemented. The accompanying extension theorem states that every closed sublocale of a completely $c$-regular locale is $C$-embedded.

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54C15 Retractions of topological spaces
54C30 Real-valued functions on topological spaces
54C45 $C$- and $C^*$-embedding

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