Zbl pre05898723

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Frames and locales. Topology without points. (English)

Frontiers in Mathematics. Berlin: Springer. xix, 398 p. EUR 59.95/net; SFR 80.00; $ 74.95; £ 53.99 (2012). ISBN 978-3-0348-0153-9/pbk; ISBN 978-3-0348-0154-6/ebook
http://dx.doi.org/10.1007/978-3-0348-0154-6

In 2008, the authors of the text under review published a monograph [Locales treated mostly in a covariant way. Textos de Matemática. Série B 41. Coimbra: Universidade de Coimbra, Departamento de Matemática. (2008; Zbl 1154.06007)] in which, as the title states, locales were treated "covariantly" in the sense that morphisms in **Loc** were, for the first time, functions in the real sense. It shall be recalled that, prior to that, morphisms in **Loc** were simply arrows obtained by reversing frame homomorphisms.

The current book continues that approach to the category of locales. It is however not a mere extension of the earlier monograph; indeed, it covers far more ground and includes topics which were not in the monograph. As any text in pointfree topology should do, after an elegantly written introduction in which a brief history of the subject is given, the book starts with a recount of sobriety and the T_D-axiom, which is then quickly followed by the definition of the category of locales, various facets of spectra, the usual adjunctions and several criteria for spatiality.

The foundation having been laid, sublocales are tackled next in the third chapter. First, they are introduced as "sub-locales" in the sense that any beginner would expect a sublocale to be defined, and then via the other alternative routes of frame congruences and nuclei. Within the same chapter, open and closed sublocales are given prominence, as are open and closed localic maps. The treatment of images and preimages follows that introduced in the authors' earlier monograph. This is one of the many instances where having legitimate functions for localic maps shows its advantages. A student still cutting his or her teeth in "the art of pointless thinking", as *P. T. Johnstone* puts it in ["The art of pointless thinking: a student's guide to the category of locales", Category theory at work, Proc. Workshop, Bremen/Ger. 1991, Res. Expo. Math. 18, 85–107 (1991; Zbl 0745.18003)], learns about preimages before having to grasp the highly non-trivial construction of products of locales. This is in contrast with the case where one's maps of locales are mere arrows in the guise of frame homomorphisms written backwards.

Products of locales, or, more precisely, coproducts of frames, are constructed in Chapter IV. Here the authors show that they are not averse to using "contravariant" methods when lucidity calls for that. Indeed, throughout the book the authors do change to the frame perspective where such a change brings transparency and clarity. There would be a lot of unnecessary obfuscation in trying to introduce products of locales "covariantly". The chapter culminates with an excellent comparison of products of locales and those of spaces, as well as a detailed discussion on epimorphisms in **Frm**.

Separation axioms are discussed in the chapter following limits, starting with subfitness and gradually ascending (or should that be descending?) to the next stronger axiom, all the way to complete regularity and normality. Every student of pointfree topology knows that dense frame homomorphisms are monomorphisms in **RegFrm**. Less known – which is the last proposition the authors prove in this chapter – is that monomorphisms in **RegFrm** are precisely the dense frame homomorphisms. To emphasize the remark made above about switching from **Loc** to **Frm**, it should be pointed out that the proposition alluded to here is stated in the former category, but the proof carried out in the latter.

Sublocales are revisited in Chapter VI, but this is no afterthought. The main gist of the chapter is a characterization of complemented sublocales, and locales all of whose sublocales are spatial. Two particular notions related to sublocales, not discussed in the book, which would have fitted well in this chapter are pointless parts of locales and rare sublocales. The book however is not the poorer for the omission because these are covered in articles that are easily accessible (see, for instance, *T. Plewe*'s ["Sublocale lattices", J. Pure Appl. Algebra 168, No.2-3, 309–326 (2002; Zbl 1004.18003)]). Apropos omissions, the book does not discuss coherent locales. The authors explain in the introduction that there is not much one can say about these locales which is not already in [*P. T. Johnstone*, Stone spaces. Cambridge Studies in Advanced Mathematics, 3. Cambridge etc.: Cambridge University Press. (1982; Zbl 0499.54001)]. An enthusiast of coherent locales should of course not ignore *B. Banaschewski*'s ["Radical ideals and coherent frames", Commentat. Math. Univ. Carol. 37, No.2, 349–370 (1996; Zbl 0853.06014)].

Compactness and local compactness are treated next, together with what is called supercompactness. The existence (and construction of) the compact completely regular reflection of a locale L – the localic Stone-Čech compactification – is part of this chapter.

Collectively, Chapters VIII, X and XII form a comprehensive account of the notions of nearness and uniformity, from basic concepts to completeness and the construction of the completion, and also functoriality properties. Although the treatment is comprehensive and self-contained, a beginning student of structured frames should also study *B. Banaschewski*'s ["Uniform completion in pointfree topology", Dordrecht: Kluwer Academic Publishers. Trends Log. Stud. Log. Libr. 20, 19–56 (2003; Zbl 1034.06008)]. Whereas Chapter VIII deals with the symmetric side of things (nearness and uniformity), Chapter XII delves more on the asymmetric side. This was touched on in the authors' 2008 monograph, but not to the same extent as in the current book. Chapter X deals with a weaker form of completion called Cauchy completion.

In Chapter IX the authors introduce paracompact locales, and use material from the immediate predecessor chapter to give the very elegant characterization that paracompactness is equivalent to admitting a complete uniformity. Also included (at the end of the chapter) is the construction of the paracompact reflection of a locale. In view of the special interest in Boolean locales, a statement within this chapter informing the reader that Boolean locales are paracompact would perhaps have been beneficial for those not yet fully au fait with the subject.

Diameters and metric locales are in Chapter XI, an appropriately chosen locale (no pun intended) for this topic because, after all, a locale admits a metric diameter precisely if it is metrizable; where the latter is understood to mean that it admits a countably generated uniformity. This is a very substantive chapter, covering, among other things, contractive frame homomorphisms between metric frames, the category they form and

other related categories.
Chapter XIII is about connectedness and local connectedness. It is a chapter which, as the authors do point out, could have been placed earlier after products were introduced. It contains, inter alia, what the authors call a "weird example" which epitomizes the peculiar behaviour of connected locales, to borrow a phrase from *I. Kříž* and *A. Pultr*'s ["Peculiar behaviour of connected locales", Cah. Topologie Géom. Différ. Catég. 30, No.1, 25–43 (1989; Zbl 0668.06007)].
The penultimate chapter is about real functions on a locale. It treats continuous and semicontinuous functions on a locale. Not only that; mere real-valued functions (which are not necessarily continuous or semicontinuous) are also discussed. It is mentioned within the chapter that extended real functions, with possibly infinite values, can be defined on a locale. Indeed, the forthcoming article ["Extended real functions in point-free topology", J. Pure Appl. Algebra, 216, 905–922 (2012)] by *B. Banaschewski, J. Gutiérrez García* and *J. Picado* does exactly that. Considering the vastness of the subject of real functions on a locale, this is a somewhat concise chapter the main aim of which appears to have been to introduce the basic concepts involved. A reading of this chapter must, as the authors hasten to caution, be supplemented with a thorough study of [*B. Banaschewski*, The real numbers in pointfree topology. Textos de Matemática. Série B. 12. Coimbra: Universidade de Coimbra, Departamento de Matemática, (1997; Zbl 0891.54009)].
The final chapter is on localic groups. After the basics are taken care of and the category of localic groups is constructed, the Closed Subgroup Theorem is proved. Every localic group sports two uniformities, the left and the right, which come for free with the group structure. The last proposition proved in the chapter is that every group homomorphism between localic group is uniform relative to either of the uniformities.
At the end of the text there are two appendices; one on partially ordered sets and the other on categories. They cover rudiments of these topics that are needed in the book. In summary, this book is an erudite account of the current status of pointfree topology, written beautifully in the authors' inimitable style. It is both easily accessible to a complete beginner, and an excellent source of reference for the seasoned pointfree practitioner.

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Keywords : Fames; locales; localic maps
Classification :
- ∗06D22 Frames etc.
- 18B30 Categories of topological spaces
- 54A05 Topological spaces and generalizations
- 54C30 Real-valued functions on topological spaces
- 54D30 Compactness
- 54E17 Nearness spaces