ON SEMICONTINUITY OF REAL FUNCTIONS: AN ALGEBRAIC DESCRIPTION

LESSON SUMMARY

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Coimbra 2008

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(...) it is my strong conviction that the only decisive test of the viability of an idea, or of a new vision of the world, is that of time. The fertility of an idea is to be judged by the quality of its offspring, and not through honors.

A. GROTHENDIECK

1. Introduction: a lecture on topology, about semicontinuity, in the non-classical context of frames and locales

In spaces like $\mathbb{R}$ or $\mathbb{R}^2$ there are plenty of continuous maps with real values. But there are non-trivial topological spaces that do not admit continuous real-valued functions other than the constant ones. The abundance of real continuous functions in a space X can be assessed by the existence of functions that indeed separate all subsets that can possibly be separated[1] (in that case one says that X is a space ≪*with plenty of continuous real functions*≫ [4]). The (separation) lemma of Urysohn, one of the fundamental classical results of point-set topology, characterizes such topological spaces: they are precisely the *normal spaces*[2].

In terms of characteristic functions, Urysohn's Lemma means precisely that in any normal space, whenever $\chi_F \leq \chi_A$ for a closed F and an open A, there exists a continuous function $h : X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_A$. The (insertion) theorem of Katětov-Tong[3] stresses this characterization by

[1]Two subsets U and V of X are *separable* if there is a continuous map $f : X \to \mathbb{R}$ such that $f(U) = \{1\}$ and $f(V) = \{0\}$; of course, this is only possible if the closures of U and V are disjoint.

[2]At first sight it may appear surprising that the class of such spaces is so vaste, containing for example all compact Hausdorff spaces.

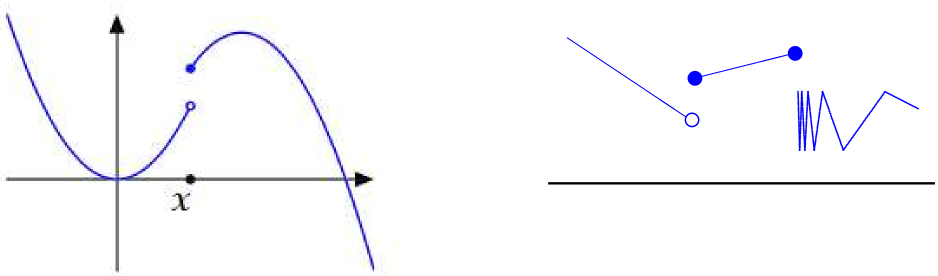
[3]Originally announced by Hing Tong in 1948 (the proof was however only published in 1952 [39]), Miroslav Katětov shares the name of the theorem because of his independent version, with an improved proof [25]. Such result is part of the classical theory of general topology, with roots in Hahn [17] and Dieudonné [5], that proved the theorem in the particular cases of metrizable spaces and paracompact spaces, respectively.

replacing χ_F and χ_A, respectively, by an arbitrary upper semicontinuous function and an arbitrary lower semicontinuous function:

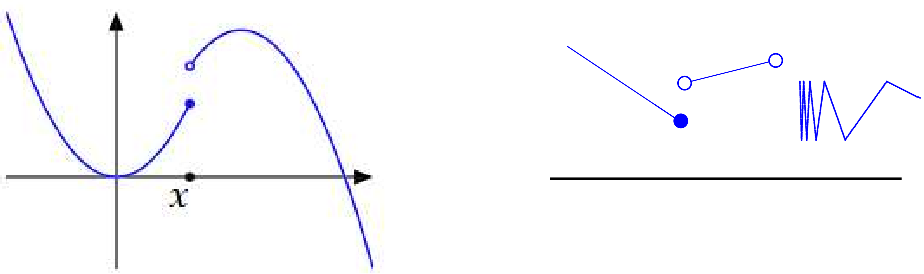
> *A topological space X is normal if and only if, for any upper semicontinuous $f : X \longrightarrow \mathbb{R}$ and any lower semicontinuous $g : X \longrightarrow \mathbb{R}$ satisfying $f \leq g$, there is a continuous function $h : X \longrightarrow \mathbb{R}$ such that $f \leq h \leq g$.*

Similarly, the (extension) theorem of Tietze, another important classical theorem, may be also obtained in a very elegant way as a particular case of the Katětov-Tong Theorem. This justifies the importance of this theorem and of the concept of semicontinuity in Real Analysis.

It was Baire that in 1899 introduced the notion of semicontinuity for real-valued functions (with domain $\mathbb{R}$). More generally, a real function f defined in a topological space $(X, \mathcal{O}X)$ is *upper* (resp. *lower*) *semicontinuous* if, for any $x \in X$ and $r \in \mathbb{R}$ satisfying $f(x) < r$ (resp. $f(x) > r$), there is a neighbourhood $U \subseteq X$ of x such that $f(y) < r$ (resp. $f(y) > r$) for every $y \in U$.



Upper semicontinuous functions



Lower semicontinuous functions

Equivalently, f is upper (resp. lower) semicontinuous if and only if for each real r the set $\{x \in X \mid f(x) \geq r\}$ (resp. $\{x \in X \mid f(x) \leq r\}$) is closed.

Obviously, for each closed subset F of X, χ_F is upper semicontinuous, and for each open subset A of X, χ_A is lower semicontinuous.

Denoting by $\mathcal{T}_l$ the *lower topology* of $\mathbb{R}$, generated by intervals $]-\infty, q[$, with $q \in \mathbb{Q}$, we may further say that $f : X \to \mathbb{R}$ is upper semicontinuous if and only if $f : (X, \mathcal{O}X) \to (\mathbb{R}, \mathcal{T}_l)$ is continuous. Analogously, considering the *upper topology* of $\mathbb{R}$, generated by intervals $]p, +\infty[$, $p \in \mathbb{Q}$, that we shall denote by $\mathcal{T}_u$, $f : X \to \mathbb{R}$ is lower semicontinuous if and only if $f : (X, \mathcal{O}X) \to (\mathbb{R}, \mathcal{T}_u)$ is continuous.

Here the points of the spaces in question — the elements of X in $(X, \mathcal{O}X)$ and the real numbers in $(\mathbb{R}, \mathcal{T}_l)$ or $(\mathbb{R}, \mathcal{T}_u)$ — seem to be relevant for the description of the two concepts. Is it that so? Actually no, as we shall see in the sequel.

In [2], Banaschewski wrote:

> ≪*The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.*
>
> (...) *the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.*≫

Our goal, with this lesson, is to show how we can extend this study to the case of arbitrary real functions. We shall see how several facts of classical topology related with real numbers and semicontinuous functions are, in fact, consequences of more general results of pointfree topology, where they have their logical antecedents.

This is thus not a lesson about topological spaces and classical point-set topology, but about locales (and frames) and the corresponding pointfree topology.

2. The new setting

2.1. Locales

The first thing to say about *locales* is that they behave like topological spaces. The theory of locales is defined in order to clone inside it topological spaces: one speaks about *sublocales* (cf. subspaces) and, in particular, of *closed, open* and *dense sublocales* (cf. closed, open and dense subspaces). One speaks about *continuous maps between locales* (cf. continuous maps between topological spaces) and, in particular, of *proper* and *open maps* (cf. proper and open maps between spaces). One speaks about *compact locales* and, analogously, many other separation axioms have their versions in locales: e.g, one speaks of *compact Hausdorff locales*, *regular locales*, *normal locales*, etc.

This analogy between the theory of locales and the theory of topological spaces is not quite exact; otherwise, the two theories would be indistinguishable and so locale theory would be redundant.

What exists is a translating device between the two theories: each topological space X defines naturally a locale $\mathcal{O}(X)$ (specifically, his topology). And given a locale X there exists a topological space $\Sigma(X)$ naturally associated to X. More precisely, there is a categorical adjunction between the category Top of topological spaces and continuous maps and the category Loc of locales, defined by the *open functor* $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ and the *spectral functor* $\Sigma : \mathsf{Loc} \to \mathsf{Top}$.

Suppose we have a space X and that we translate it into a locale $\mathcal{O}X$ and that, next, we transform it back into a space $\Sigma\mathcal{O}(X)$. Will we recover the given space X (up to an homeomorphism) ? Similarly, starting from a locale X, is $\mathcal{O}\Sigma(X)$ isomorphic to X?

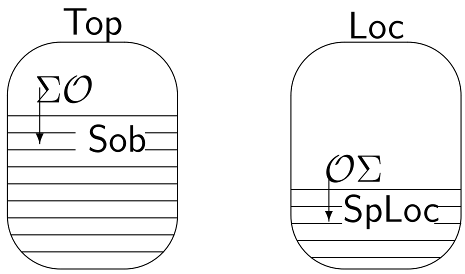
Both responses are, in general, negative, that is, the translating device is not exact. However, if we restrict our attention to the topological spaces X

for which $\Sigma\mathcal{O}(X)$ is homeomorphic to X (i.e., the so-called *sober spaces*[4], introduced by Grothendieck) and to locales X for which $\mathcal{O}\Sigma(X)$ is isomorphic to X (the so-called *spatial locales*), the restrictions of those translations are exact. In other words, the theories of sober spaces and spatial locales are equivalent (more precisely, the functors Σ and $\mathcal{O}$ define an equivalence between the corresponding categories Sob and SpLoc).

Now the following question is unavoidable: is Sob a significant subcategory of Top? In other words, is the class of sober spaces sufficiently big to contain the most common topological spaces? The response, fortunately for locale theory, is positive: the sobriety condition relies somewhere between axioms T_0 and T_2 (being incomparable with T_1). As Johnstone wrote in [23],

> ≪(...) *in effect, one sacrifices a small amount of pathology (non-sober spaces) in order to achieve a category that is more smoothly and purely "topological" than the category of spaces itself.*≫

So the category Loc of locales contains a subcategory, SpLoc, which is an equivalent copy of the subcategory Sob of Top:



This is a good reason to take seriously the study of locales: in practice, when studying topological spaces, we are almost always considering sober spaces and so we might as well be working within the category of locales.

There is however another important reason: the categorical properties of Loc are different from Top, with advantages to the former in many situations. For example, the embedding of sober spaces inside locales is a full embedding; it does not preserve products, in general, which implies some discrepancy between the two products. As it was originally observed by J. Isbell in 1972

[4]A topological space X is *sober* if for every meet-irreducible open subset U there is exactly one $x \in X$ such that $U = X \setminus \overline{\{x\}}$ (an open $U \neq X$ is called *meet-irreducible* whenever $U = U_1 \cap U_2$, with U_1 and U_2 open, implies $U = U_1$ or $U = U_2$).

[18], those differences make the theory of locales very interesting: e.g., the paracompactness property for locales is productive, the property of Lindelöf for regular locales is also productive. Another wonderful surprise: every localic subgroup of a localic group is closed (*localic groups* are to topological groups as locales are to topological spaces). (As it is well-known, all those assertions are false in the classical setting.)

There are other compelling reasons for considering the category of locales as the right framework within which to do topology: the study of locales is, in a logical sense, purer than the study of topological spaces. Proofs in locales require usually less axioms than the corresponding proofs in classical topology[5].

What do we gain with this, besides the simple formal aspect? One gains the possibility of doing topology in the general constructive context of topos theory [30], a theory that gave a new impetus to the intuitionistic approach (of Brouwer and Heyting) to mathematics.

Toposes are mathematical universes. Some are Boolean (i.e., satisfy the law of excluded middle), however there are many non-Boolean toposes that naturally occur in some areas of mathematics that attest the existence of relevant mathematical universes where the law of excluded middle does not hold. If we want to guarantee that our mathematics can be carried in any of these universes we have to ensure that it does not depend on the law of excluded middle. In many occasions, the dependence of a topological proof from this law vanishes when we translate it in a proof for locales. Locale theory is, therefore, compatible with the philosophy of intuitionists and constructivists[6].

Further, in pointfree topology one can usually avoid another axiom that has a weaker link to reality: the axiom of choice. Of course the axiom of choice is, and has always been, freely used in the current practice of "classical" mathematics. In his general formulation it is even equivalent to a great number of mathematical theorems. For example, Tychonoff Theorem (of classical topology) is equivalent to the axiom of choice. Because of this and

[5]And, as stated by Occham in his principle of parsimony, why doing with more what can be done with less? ("*Frustra fit per plura quod potest fieri per pauciora.*")

[6]André Joyal [24] was the first to emphasize the advantages of such (pointfree) approach to topology in the constructive context of topos theory.

other consequences, it is understandable that some counter-intuitive results that follow from the axiom of choice (like, for instance, the Banach-Tarski Paradox about decompositions) are ignored. The development of a topology free of the axiom of choice seems an impossible task regarding that dependence: if we want to have Tychonoff Theorem (and of course we need to have it if we want our topology to be useful) we need the axiom of choice. So we have no alternative, unless we change our definition of topology.

This is precisely what we do when we move to the category of locales. Changing slightly the definition of space we introduce a new category in which we can obtain our topological results. And there Tychonoff Theorem can be proved independently of any choice principle [20]. Pointfree topology is indeed "choice-free" in its nature[7].

In summary, locales have characteristics that go beyond their simple interest as generalized topological spaces. In many situations, certain spaces are only non-trivial in virtue of some choice principle, while their lattices of opens have already previous existence, independently of such principle. This means that, in a sense, we always see the lattice of open sets, while to see its points we need an additional tool in the form of some choice principle [35]. This idea was concisely expressed by Banaschewski [1] with the following slogan:

$$\frac{\begin{array}{c}\textit{choice-free localic argument}\\ + \\ \textit{adequate choice principle}\end{array}}{\textit{classical result in topological spaces}}$$

Of course the question of knowing how far locale theory is really topology remains. One of the main goals of locale theory is to clone the ideas, concepts and results of classical topology in its language since the translating device mentioned above does not solve the problem completely. In this lesson we will describe how this cloning can be effected with the topic of semicontinuity.

[7] See [M. Escardo, *Tychonoff Theorem and AC in locales*, Top. Com., Vol. 8, N. 1], in `http://at.yorku.ca/t/o/p/d/55.htm`.

2.2. Frames

What motivates the definition of a locale?

Given a topological space $(X, \mathcal{O}X)$, the lattice $(\mathcal{O}X, \subseteq)$ of open sets is complete, since any union of open sets is an open set; of course the *infinite distribution law*

$$A \wedge \bigvee_{i \in I} B_i = \bigvee_{i \in I} (A \wedge B_i)$$

holds in $\mathcal{O}X$ since the operations $\wedge$ (being a finite meet) and $\bigvee$ coincide with the usual set-theoretical operations of $\cap$ (intersection) and $\bigcup$ (union), respectively[8]. Moreover, if

$$f : (X, \mathcal{O}X) \rightarrow (Y, \mathcal{O}Y)$$

is a continuous map, then f^{-1} defines an aplication of $\mathcal{O}Y$ into $\mathcal{O}X$ that clearly preserves the operations $\wedge$ and $\bigvee$. Therefore, defining a *frame* as a complete lattice L satisfying the infinite distribution law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i),$$

and defining a *frame homomorphism* $h : L \rightarrow M$ as a map from L in M such that $h(\wedge_{i \in F} a_i) = \wedge_{i \in F} h(a_i)$ for every finite F (in particular, for $F = \varnothing$, $h(1) = 1$) and $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$ for every I (in particular, for $I = \varnothing$, $h(0) = 0$), we have the category Frm of frames[9] and a contravariant functor

[8] On the other hand, infinite meets $\bigwedge_{i \in I} A_i$ are given by the interior $(\bigcap_{i \in I} A_i)^\circ$ of the intersection $\bigcap_{i \in I} A_i$.

[9] By the Adjoint Functor Theorem, a complete lattice satisfies the infinite distribution law if and only if it is an Heyting Algebra. Consequently, frames are precisely complete Heyting algebras (they have thus *relative pseudocomplements* defined by

$$a \rightarrow b := \bigvee \{c \in L \mid a \wedge c \leq b\}$$

and, in particular, *pseudocomplements*, given by $a^* := a \rightarrow 0$). However, frame homomorphisms are not, in general, homomorphisms of complete Heyting algebras because they do not necessarily preserve the Heyting operator $\rightarrow$.

$$\begin{tikzcd}[row sep=small]
\mathcal{O}: \mathsf{Top} \arrow[r] & \mathsf{Frm} \\
(X,\mathcal{O}X) \arrow[r, mapsto] \arrow[dd, "f"'] & \mathcal{O}X \\
 & \\
(Y,\mathcal{O}Y) \arrow[r, mapsto] & \mathcal{O}Y \arrow[uu, "f^{-1}"']
\end{tikzcd}$$

Because of contravariance, to keep the original geometrical motivation it is necessary to introduce the dual category $\mathsf{Frm}^{\mathrm{op}}$, turning the functor $\mathcal{O}$ covariant. This is the way the category Loc is introduced: it is precisely the category $\mathsf{Frm}^{\mathrm{op}}$. So, a locale X is the same thing as a frame, but we shall use the notation $L = \mathcal{O}X$ to refer to X as an object of Frm. The corresponding morphisms diverge; localic morphisms are defined abstractly, as morphisms of frames acting in the opposite direction. If $f : X \to Y$ is a morphism of locales, the corresponding frame homomorphism will be denoted by $f^* : \mathcal{O}Y \to \mathcal{O}X$. These notations will make clear whether we are referring to an object or morphism as being in Loc or in Frm.

This reveals another important aspect of locale theory, with fundamental importance in its development: the dual category of Loc is an algebraic category, with all the nice properties and tools available in any category of algebras[10], allowing the development of localic topology in a pure algebraic way, in the mirror image of Loc. This aspect will be decisive in the treatment of semicontinuity.

Intermezzo: a brief historical note

The first mathematician to consider the notion of open set as the basic concept for the study of continuity was Felix Hausdorff in 1914. Using the family (lattice) of open sets, Marshall Stone (1936) established topological representations of Boolean algebras and distributive lattices, which paved the ground for the famous dualities of Stone and Priestley. H. Wallman (1938) applied

[10] Being Frm an *algebraic* category with *presentations by equations* (i.e., its objects are described by a proper class of operations and equations), it has free objects and quotients are described by congruences, which allows *presentations* of frames *by generators and relations* [21].

tools from lattice theory to obtain the nowadays called Wallman compactification, and in the 40's McKinsey and Tarski studied the "*algebra of topology*", that is, topology from an algebraic point of view.

But the fundamental change occurred at the end of the 50's with the works of Charles Ehresmann (and one of his students, Jean Bénabou), in Paris, and Seymour Papert, in Cambridge: complete lattices with the infinite distribution law deserved to be studied by its own, rather than a simple tool for studying topological spaces. Ehresman introduced the name "*local lattice*" to designate them.

This new approach to topology (noiwadays referred to as *Pointfree Topology*) fulfils the idea that the points of a space should be considered as secondary to their open sets and, according to that, deals with abstract "lattices of open sets". It is worth pointing that already in Portugal, in 1946, A. Pereira Gomes wrote [32]:

> ≪*In Abstract Algebra, as in General Topology, one finds examples of an evolution that leaves the* <u>*points*</u> *with a secondary role in favor of the more generic notion of* <u>*part*</u> *(figure, subset, element of a structure).*≫

But it was certainly with the publication by John Isbell, in 1972, of the article [18], that the real importance of the theme emerged and made clear the need for a specific terminology for the dual category of the category of frames[11]. Isbell designated the objects of the dual category by *locales* and wrote in *Zentralblatt für Mathematik*:

> ≪(...) *topology is better modelled in the category of locales than in topological spaces or another of their variants.*≫

Afterwards, many topological properties were extended by C. H. Dowker, D. Papert (Strauss), P. T. Johnstone, B. Banaschewski, A. Pultr and others to those "generalized spaces". In [22] Peter Johnstone presents an excellent survey of the motivations and goals of locale theory (see, also, [23]). His book "*Stone Spaces*" [21] describes in detail, and with precision, the influence of

[11] According to Johnstone [21], this was made even more evident later on in the non-published work of André Joyal [24].

the work of Stone in modern mathematics, in general, and in the theory of locales and frames, in particular.

2.2. Sublocale lattices and the frame of reals

One of the fundamental differences between Top and Loc relies on their lattices of subobjects. In fact, sublocale lattices are much more complicated than their topological counterparts (complete atomic Boolean algebras): they are in general *co-frames* (i.e., complete lattices satisfying the distribution law $S \vee \bigwedge_I T_i = \bigwedge_I (S \vee T_i)$, dual to the distribution law that characterizes frames). Even the sublocale lattice of a topology $\mathcal{O}X$ can be much larger than the Boolean algebra of the subspaces of X (e.g., $\mathbb{Q}$ has $2^{\mathfrak{c}}$ many non-isomorphic sublocales [19]).

Let X be a locale. The *sublocales* $j : Y \rightarrowtail X$ of X, that is, the regular monomorphisms in Loc with codomain X (or still, the surjective frame homomorphisms $j^* : \mathcal{O}X \twoheadrightarrow \mathcal{O}Y$) can be described in several ways (cf. [35]). Here we shall use the description of [34]:

A subset S of $L = \mathcal{O}X$ is a *sublocale* of X if:

(1) For each $A \subseteq S$, $\bigwedge A \in S$.

(2) For any $a \in L$ and $s \in S$, $a \to s \in S$.

Since any intersection of sublocales is a sublocale, the set $\mathfrak{S}X$ of all sublocales of X is a complete lattice. This is a co-frame, in which $\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i$, $\bigvee_{i\in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in I} S_i\}$, $0 = \{1\}$ and $1 = L$. It will be convenient to work with the dual lattice $(\mathfrak{S}X)^{\mathrm{op}}$, that we shall denote by $\mathfrak{S}L$.

Each sublocale S is itself a frame with $\wedge$ and $\to$ as in L (the top of S coincides with the one of L but the bottom 0_S may differ from the one of L). It determines a quotient $c_S : L \twoheadrightarrow S$, given by $c_S(x) := \bigwedge\{s \in S \mid x \leq s\}$.

In spite of $\mathfrak{S}L$ not being in general a Boolean algebra, it contains fortunately many complemented elements. For example, for each $a \in L$, the sets

$$\mathfrak{c}(a) := {\uparrow} a \quad \text{and} \quad \mathfrak{o}(a) := \{a \to b \mid b \in L\}$$

define sublocales of X, complemented to each other. The former are the so-called *closed sublocales*, while the latter are the *open sublocales*. The corresponding frame quotients are given by, respectively,

$$c_{\mathfrak{c}(a)}(x) := a \vee x \quad \text{and} \quad c_{\mathfrak{o}(a)}(x) := a \to x.$$

The following is a list of some of the most significative properties of $\mathfrak{S}L$ [35]:

(S_1) $\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$ is subframe of $\mathfrak{S}L$ isomorphic to L; the isomorphism $L \to \mathfrak{c}L$ is given by $a \mapsto \mathfrak{c}(a)$. In particular, $\mathfrak{c}(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \mathfrak{c}(a_i)$ and $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$.

(S_2) Let $\mathfrak{o}(L)$ denote the subframe of $\mathfrak{S}L$ generated by $\{\mathfrak{o}(a) \mid a \in L\}$. The map $L \to \mathfrak{o}(L)$ defined by $a \mapsto \mathfrak{o}(a)$ is a dual lattice embedding. In particular, we have $\mathfrak{o}(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} \mathfrak{o}(a_i)$ and $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \vee \mathfrak{o}(b)$.

(S_3) $\mathfrak{c}(a) \leq \mathfrak{o}(b) \Leftrightarrow a \wedge b = 0$.

(S_4) $\mathfrak{o}(a) \leq \mathfrak{c}(b) \Leftrightarrow a \vee b = 1$.

(S_5) The *closure* $\overline{S} := \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \leq S\}$ and the *interior* $S^\circ := \bigwedge\{\mathfrak{o}(a) \mid S \leq \mathfrak{o}(a)\}$ of a sublocale S satisfy the following proprerties[12]: $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$, $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$, $\neg\,\overline{S^*} = S^\circ$.

Since frame theory is algebraic in nature and constructive in its methods, we need to construct an appropriate frame of reals that might be defined in any topos with a natural number. We have thus to give up of working in the usual (Euclidean) topology of $\mathbb{R}$ and choose the frame of reals $\mathfrak{L}(\mathbb{R})$, that can be defined in a very easy way [24, 2].

In fact, since Frm is an algebraic category (in particular, there exist free frames and frame quotients are described by congruences), we have at our disposal the familiar procedure from traditional algebra of *presentation* of objects by *generators and relations*: it suffices to consider the quotient of the free frame in the given set of generators, modulo the congruence generated by pairs (u, v) for the given relations $u = v$.

[12]The symbol $\neg$ denotes the complementation operator.

It is therefore very natural (and very useful) to introduce the reals from the totally ordered set $\mathbb{Q}$ of rational numbers, in the following way[13] [24, 2]:

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, and by the following relations:

(R_1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R_2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R_3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R_4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

Let

$$(p, -) := \bigvee_{q \in \mathbb{Q}} (p, q) \quad \text{and} \quad (-, q) := \bigvee_{p \in \mathbb{Q}} (p, q).$$

With $(p, -)$ and $(-, q)$ taken as the primitive notions, $\mathfrak{L}(\mathbb{R})$ may be equivalently defined as the frame generated by these elements, subject to relations:

(R'_1) $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$.

(R'_2) $(p, -) \vee (-, q) = 1$ whenever $p < q$.

(R'_3) $(p, -) = \bigvee_{r>p}(r, -)$.

(R'_4) $(-, q) = \bigvee_{s<q}(-, s)$.

(R'_5) $\bigvee_{p \in \mathbb{Q}}(p, -) = 1$.

(R'_6) $\bigvee_{q \in \mathbb{Q}}(-, q) = 1$.

By imposing only relations (R'_1)-(R'_4) one gets the frame $\mathfrak{L}(\overline{\mathbb{R}})$, the localic counterpart of $\overline{\mathbb{R}} = [-\infty, +\infty]$.

We need further to introduce the following subframes of $\mathfrak{L}(\mathbb{R})$ and $\mathfrak{L}(\overline{\mathbb{R}})$:

[13] Classically, this is just the usual topology of the real line, but from a constructive point of view these two notions differ (see [2, 10]). Note also that this is a definition independent from any notion of real number.

$$\begin{aligned}
\mathfrak{L}_u(\mathbb{R}) &= \langle\{(p,-)\,(p\in\mathbb{Q}) \mid (\mathrm{R}3') \text{ and } (\mathrm{R}5')\}\rangle\,,\\
\mathfrak{L}_l(\mathbb{R}) &= \langle\{(-,q)\,(q\in\mathbb{Q}) \mid (\mathrm{R}4') \text{ and } (\mathrm{R}6')\}\rangle\,,\\
\mathfrak{L}_u(\overline{\mathbb{R}}) &= \langle\{(p,-)\,(p\in\mathbb{Q}) \mid (\mathrm{R}3')\}\rangle\,,\\
\mathfrak{L}_l(\overline{\mathbb{R}}) &= \langle\{(-,q)\,(q\in\mathbb{Q}) \mid (\mathrm{R}4')\}\rangle\,.
\end{aligned}$$

A map from the generating set of $\mathfrak{L}(\mathbb{R})$ into L defines a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$ if and only if it transforms relations (R_1)-(R_4) of $\mathfrak{L}(\mathbb{R})$ into identities in L (the same can be said in a similar way for the other frames of reals defined above).

3. Semicontinuity in locales

3.1. Genesis

It is now natural to look to frame homomorphisms $h : \mathfrak{L}(\mathbb{R}) \to L$ as the *continuous real functions* on L (for the motivation, consult [2]). One gets this way the ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ of continuous real functions on L. It is then tempting (as Li and Wang did in [29]) to define an upper (resp. lower) semicontinuous real function on L as a frame homomorphism

$$h : \mathfrak{L}_l(\mathbb{R}) \to L \text{ (resp. } h : \mathfrak{L}_u(\mathbb{R}) \to L). \tag{3.1.1}$$

However, as we showed in [33], with these definitions the Katětov-Tong Theorem does not hold in its full generality, contrary to what is stated in [29]. In fact, some condition needs to be imposed on the f and g in question in order to ensure the existence of a continuous h such that $f \le h \le g$ ([33], Theorem 4.6).

Why is that so? Because when applied to spatial frames, the definition (3.1.1) does not describe completely the classical concept of semicontinuity. To observe that recall that continuous maps $X \to Y$, for sober Y, are in bijection with frame homomorphisms $\mathcal{O}Y \to \mathcal{O}X$. More precisely [36]:

A space Y is T_0 if and only if for each frame homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$ there is **at most** *one continuous map $f : X \to Y$ such that $h = \mathcal{O}(f)$.*

A space Y is weakly sober[14] *if and only if for each frame homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$ there is* **at least** *one continuous map $f : X \to Y$ such that $h = \mathcal{O}(f)$.*

[14] The sobriety condition previously referred to is the conjunction of two requirements, namely the T_0 condition and the so-called *weak sobriety* condition [36] "for each meet-irreducible $U \subseteq X$ there exists an $x \in X$ such that $U = X \setminus \overline{\{x\}}$".

Consequently, since $(\mathbb{R}, \mathcal{T}_u)$ and $(\mathbb{R}, \mathcal{T}_l)$ are both T_0 but non-sober, the correspondences

$$X \xrightarrow{f} (\mathbb{R}, \mathcal{T}_u) \in \mathsf{Top} \quad \overset{\mathcal{O}}{\rightsquigarrow} \quad \mathcal{T}_u \xrightarrow{\mathcal{O}(f)} \mathcal{O}X \in \mathsf{Frm}$$

and

$$X \xrightarrow{g} (\mathbb{R}, \mathcal{T}_l) \in \mathsf{Top} \quad \overset{\mathcal{O}}{\rightsquigarrow} \quad \mathcal{T}_l \xrightarrow{\mathcal{O}(g)} \mathcal{O}X \in \mathsf{Frm}$$

are injective but not surjective. Digging more deeply, one gets the following (where $\pi_x : \mathcal{O}X \to \{0, 1\}$ is defined by $\pi_x(U) = 1$ iff $x \in U$):

Proposition 3.1.1. ([12]) *Upper semicontinuous maps $f : X \to \mathbb{R}$ are in a bijective correspondence (via $\mathcal{O}$) with frame homomorphisms $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ for which the set $\{q \in \mathbb{Q} \mid \pi_x(h(-, q)) = 1\}$ is bounded below for every $x \in X$.*

The following elementary fact suggests to look to semicontinuity from a bitopological point of view[15]:

Proposition 3.1.2. *Given a topological space $(X, \mathcal{O}X)$, let $\mathcal{C}X$ be the topology on X generated by all closed subsets of $(X, \mathcal{O}X)$. Then, for each $f : X \to \mathbb{R}$, the following conditions are equivalent:*

(i) *f is upper semicontinuous.*

(ii) *The map $f : (X, \mathcal{O}X, \mathcal{C}X) \to (\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ is bicontinuous.*

This proposition ensures that

$$\mathsf{Top}\big((X, \mathcal{O}X), (\mathbb{R}, \mathcal{T}_l)\big) \simeq \mathsf{BiTop}\big((X, \mathcal{O}X, \mathcal{C}X), (\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)\big).$$

But, by the dual adjunction [3]

$$\mathsf{BiTop} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{BiFrm},$$

[15] Recall that a *bitopological space* is a triple $(X, \mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are arbitrary topologies on X. The category BiTop of bitopological spaces has as morphisms the *bicontinuous maps* $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{T}_1', \mathcal{T}_2')$, that is, maps $f : X \to Y$ that are simultaneously continuous from $(X, \mathcal{T}_1)$ into $(Y, \mathcal{T}_1')$ and from $(X, \mathcal{T}_2)$ into $(Y, \mathcal{T}_2')$. A *biframe* is a triple (L_0, L_1, L_2) where L_0 is a frame and L_1 and L_2 are subframes of L_0 that generate L_0, that is, each $x \in L_0$ is a join of finite meets of elements in $L_1 \cup L_2$. The corresponding homomorphisms $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ are the frame homomorphisms $h : L_0 \to M_0$ such that $h(L_i) \subseteq M_i$ $(i \in \{1, 2\})$. We shall denote the corresponding category by BiFrm.

similar to the one between Top and Frm, we have the natural isomorphism

$$\mathsf{BiTop}\big((X,\mathcal{T}_1,\mathcal{T}_2),\Sigma(L,L_1,L_2)\big) \xrightarrow{\sim} \mathsf{BiFrm}\big((L,L_1,L_2),\mathcal{O}(X,\mathcal{T}_1,\mathcal{T}_2)\big).$$

Combining this, for

$$(L,L_1,L_2) = (\mathfrak{L}(\mathbb{R}),\mathfrak{L}_l(\mathbb{R}),\mathfrak{L}_u(\mathbb{R}))$$

and

$$(X,\mathcal{T}_1,\mathcal{T}_2) = (X,\mathcal{O}X,\mathcal{C}X),$$

with the isomorphism[16] $\Sigma\big(\mathfrak{L}(\mathbb{R}),\mathfrak{L}_l(\mathbb{R}),\mathfrak{L}_u(\mathbb{R})\big) \simeq (\mathbb{R},\mathcal{T}_l,\mathcal{T}_u)$, we get

$$\mathsf{BiTop}\big((X,\mathcal{O}X,\mathcal{C}X),(\mathbb{R},\mathcal{T}_l,\mathcal{T}_u)\big) \xrightarrow{\sim} \mathsf{BiFrm}\big((\mathfrak{L}(\mathbb{R}),\mathfrak{L}_l(\mathbb{R}),\mathfrak{L}_u(\mathbb{R})),\mathcal{O}(X,\mathcal{O}X,\mathcal{C}X)\big).$$

On the other hand, $\mathcal{O}(X,\mathcal{O}X,\mathcal{C}X) = (\mathcal{O}X \vee \mathcal{C}X,\mathcal{O}X,\mathcal{C}X)$ is isomorphic to the biframe $(\mathfrak{S}(\mathcal{O}X),\mathfrak{c}(\mathcal{O}X),\mathfrak{o}(\mathcal{O}X))$. Therefore

$$\mathsf{Top}\big((X,\mathcal{O}X),(\mathbb{R},\mathcal{T}_l)\big) \simeq \mathsf{BiFrm}\big((\mathfrak{L}(\mathbb{R}),\mathfrak{L}_l(\mathbb{R}),\mathfrak{L}_u(\mathbb{R})),(\mathfrak{S}(\mathcal{O}X),\mathfrak{c}(\mathcal{O}X),\mathfrak{o}(\mathcal{O}X))\big).$$

Moreover, for each frame L, one has [12]:

Proposition 3.1.3. *The correspondence $\Phi : h \longmapsto h_{|\mathfrak{L}_l(\mathbb{R})}$ establishes a bijection between*

$$\mathsf{BiFrm}\Big((\mathfrak{L}(\mathbb{R}),\mathfrak{L}_l(\mathbb{R}),\mathfrak{L}_u(\mathbb{R})),(\mathfrak{S}L,\mathfrak{c}L,\mathfrak{o}L)\Big)$$

and

$$\Big\{f : \mathfrak{L}_l(\mathbb{R}) \to L \in \mathsf{Frm} \mid \bigvee_{p\in\mathbb{Q}} \mathfrak{o}(f(-,p)) = 1\Big\}.$$

This suggests immediately the following definitions [12]:

(sc$_1$) An *upper semicontinuous real function* on a frame L is a frame homomorphism $f : \mathfrak{L}_l(\mathbb{R}) \to L$ satisfying $\bigvee_{q\in\mathbb{Q}} \mathfrak{o}(f(-,q)) = 1$.

(sc$_2$) A *lower semicontinuous real function* on a frame L is a frame homomorphism $g : \mathfrak{L}_u(\mathbb{R}) \to L$ satisfying $\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(g(p,-)) = 1$.

[16] Note that contrarily to spaces $(\mathbb{R},\mathcal{T}_l)$ and $(\mathbb{R},\mathcal{T}_u)$, the bitopological space $(\mathbb{R},\mathcal{T}_l,\mathcal{T}_u)$ is sober.

It is now time to introduce notations for the several classes of functions in question. We shall consider the classes

$$\overline{\mathsf{usc}}(L) := \mathsf{Frm}(\mathfrak{L}_l(\overline{\mathbb{R}}), L)$$

and

$$\mathsf{usc}(L) := \{f : \mathfrak{L}_l(\mathbb{R}) \to L \mid \bigvee_{q\in\mathbb{Q}} \mathfrak{o}(f(-,q)) = 1\}$$

partially ordered by

$$f_1 \leq f_2 \equiv f_2(-,q) \leq f_1(-,q) \text{ for every } q \in \mathbb{Q},$$

and the classes

$$\overline{\mathsf{lsc}}(L) := \mathsf{Frm}(\mathfrak{L}_u(\overline{\mathbb{R}}), L)$$

and

$$\mathsf{lsc}(L) =: \{g : \mathfrak{L}_u(\mathbb{R}) \to L \mid \bigvee_{p\in\mathbb{Q}} \mathfrak{o}(g(p,-)) = 1\}$$

partially ordered by

$$g_1 \leq g_2 \equiv g_1(p,-) \leq g_2(p,-) \text{ for every } p \in \mathbb{Q}.$$

We shall further consider $\overline{\mathsf{c}}(L) := \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L)$ and $\mathsf{c}(L) := \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$, partially ordered by

$$\begin{aligned} h_1 \leq h_2 \quad &\equiv \quad {h_1}_{|\mathfrak{L}_u(\overline{\mathbb{R}})} \leq {h_2}_{|\mathfrak{L}_u(\overline{\mathbb{R}})} \\ &\Leftrightarrow \quad {h_2}_{|\mathfrak{L}_l(\overline{\mathbb{R}})} \leq {h_1}_{|\mathfrak{L}_l(\overline{\mathbb{R}})}. \end{aligned}$$

Finally, the following relations [33] generalize the usual way of comparing elements of $\mathsf{LSC}(X, \overline{\mathbb{R}})$ and $\mathsf{USC}(X, \overline{\mathbb{R}})$, overcoming the fact that members of $\overline{\mathsf{lsc}}(L)$ and $\overline{\mathsf{usc}}(L)$ have distinct domains:
For every $f \in \overline{\mathsf{usc}}(L)$ and $g \in \overline{\mathsf{lsc}}(L)$ we define

$$\begin{aligned} f \leq g \quad &\equiv \quad f(-,q) \vee g(p,-) = 1 \text{ whenever } p < q \in \mathbb{Q}, \\ g \leq f \quad &\equiv \quad g(p,-) \wedge f(-,p) = 0 \text{ for every } p \in \mathbb{Q}. \end{aligned}$$

We can generate functions in those classes with the help of the so called *scales* [16]. A family $\mathcal{A} := \{a_r \mid r \in \mathbb{Q}\} \subseteq L$ is called *ascending* (resp. *descending*) if $r \leq s$ implies $a_r \leq a_s$ (resp. $a_r \geq a_s$). An ascending (resp. descending) family is called an

- *ascending u-scale* (resp. *descending u-scale*) if $\bigvee_{r\in\mathbb{Q}} a_r = 1$ (resp. $\bigvee_{r\in\mathbb{Q}} a_r^* = 1$),
- *ascending l-scale* (resp. *descending l-scale*) if $\bigvee_{r\in\mathbb{Q}} a_r^* = 1$ (resp. $\bigvee_{r\in\mathbb{Q}} a_r = 1$),
- *ascending scale* (resp. *descending scale*) if it is simultaneously an ascending (resp. descending) u-scale and an ascending (resp. descending) l-scale and $a_r^* \vee a_s = 1$ (resp. $a_r \vee a_s^* = 1$) for every $r < s$.

Lemma 3.1.4. ([16]) (a) *Let $\mathcal{A} := \{a_r \mid r \in \mathbb{Q}\}$ be an ascending family of L. Then:*

(1) *The formula $f(-,q) := \bigvee_{s<q} a_s$ determines a function $f \in \overline{\mathsf{usc}}(L)$; if $\mathcal{A}$ is a u-scale, then $f \in \mathsf{usc}(L)$.*

(2) *The formula $g(p,-) := \bigvee_{r>p} a_r^*$ determines a function $g \in \overline{\mathsf{lsc}}(L)$; if $\mathcal{A}$ is an l-scale, then $g \in \mathsf{lsc}(L)$.*

(3) *If $\mathcal{A}$ satisfies $a_r^* \vee a_s = 1$ for every $r < s$ then the formulas*

$$h(-,q) := \bigvee_{s<q} a_s \quad \text{and} \quad h(p,-) := \bigvee_{r>p} a_r^*$$

determine a function $h \in \bar{\mathsf{c}}(L)$; if $\mathcal{A}$ is a scale then $h \in \mathsf{c}(L)$.

(b) *Let $\mathcal{A} := \{a_r \mid r \in \mathbb{Q}\}$ be a descending family of L. Then:*

(1) *The formula $f(-,q) := \bigvee_{s<q} a_s^*$ determines a function $f \in \overline{\mathsf{usc}}(L)$; if $\mathcal{A}$ is a u-scale, then $f \in \mathsf{usc}(L)$.*

(2) *The formula $g(p,-) := \bigvee_{r>p} a_r$ determines a function $g \in \overline{\mathsf{lsc}}(L)$; if $\mathcal{A}$ is an l-scale, then $g \in \mathsf{lsc}(L)$.*

(3) *If $\mathcal{A}$ satisfies $a_r \vee a_s^* = 1$ for every $r < s$ then the formulas*

$$h(-,q) := \bigvee_{s<q} a_s^* \quad \text{and} \quad h(p,-) := \bigvee_{r>p} a_r$$

determine a function $h \in \bar{\mathsf{c}}(L)$; if $\mathcal{A}$ is a scale then $h \in \mathsf{c}(L)$.

The definitions (sc_1) and (sc_2) have proved to be the right approach to develop the concept of semicontinuity in locale theory, as the articles [12, 13, 14, 15] show, but the fact that the three classes of functions $\mathsf{c}(L)$, $\mathsf{usc}(L)$ and $\mathsf{lsc}(L)$ (or $\overline{\mathsf{c}}(L)$, $\overline{\mathsf{usc}}(L)$ and $\overline{\mathsf{lsc}}(L)$) have distinct domains is quite unpleasant: it would be expectable to have a function continuous if and only if it is simultaneously upper and lower semicontinuous (and here continuous functions are not even a subclass of semicontinuous functions!). How can we remedy this?

The set

$$\mathsf{F}(X, \mathbb{R})$$

of all functions $X \to \mathbb{R}$ (not necessarily continuous or semicontinuous) is clearly in bijection with the set $\mathsf{Top}\big((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T})\big)$, whatever topology $\mathcal{T}$ we put on the reals. In particular, for the usual (Euclidean) topology $\mathcal{T}_e$ of $\mathbb{R}$, we have

$$\mathsf{F}(X, \mathbb{R}) \simeq \mathsf{Top}\big((X, \mathcal{P}(X)), (\mathbb{R}, \mathcal{T}_e)\big).$$

Therefore, by the adjunction between Top and Frm,

$$\mathsf{F}(X, \mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}X)).$$

Since $\mathcal{P}X$ is the subspace lattice of X, when we move to Loc replacing X by an arbitrary frame L, we should replace $\mathcal{P}X$ by the sublocale lattice $\mathfrak{S}L$ of L. This conceptually justifies that we adopt the frame homomorphisms

$$\mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L$$

as the definition of *real function* in an arbitrary frame L[17]:

[17] It is interesting to note that, being the spectrum $\Sigma\mathfrak{L}(\mathbb{R})$ of $\mathfrak{L}(\mathbb{R})$ homeomorphic to $\mathbb{R}$, and given the bijective correspondence between real numbers and frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \{0, 1\}$, real numbers are, essentially, the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \{0, 1\}$. Thus, this definition indicates that, from the point of view of logic, the real functions on L should be seen as the "*$\mathfrak{S}L$-valued real numbers*".

(SC$_1$) A *real function* (resp. *extended real function*) on a frame L is a frame homomorphism

$$\mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L \text{ (resp. } \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{S}L).$$

We denote by $\mathsf{F}(L) = \mathsf{c}(\mathfrak{S}L)$ (resp. $\overline{\mathsf{F}}(L)$) the set of all real functions (resp. extended real functions) on L.

(SC$_2$) A real function (resp. extended real function) F on a frame L is *upper semicontinuous* if

$$F(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}L \text{ (resp. } F(\mathfrak{L}_l(\overline{\mathbb{R}})) \subseteq \mathfrak{c}L),$$

i.e., each sublocale $F(-,q)$ is closed. We denote by $\mathsf{USC}(L)$ (resp. $\overline{\mathsf{USC}}(L)$) the set of all upper semicontinuous real functions (resp. extended upper semicontinuous real functions) on L.

(SC$_3$) A real function (resp. extended real function) G on a frame L is *lower semicontinuous* if

$$G(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L \text{ (resp. } G(\mathfrak{L}_u(\overline{\mathbb{R}})) \subseteq \mathfrak{c}L),$$

i.e., each sublocale $G(p,-)$ is closed. We denote by $\mathsf{LSC}(L)$ (resp. $\overline{\mathsf{LSC}}(L)$) the set of all lower semicontinuous real functions (resp. extended lower semicontinuous real functions) on L.

(SC$_4$) Finally, a real function (resp. extended real function) H on a frame L is *continuous* if

$$H(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L \text{ (resp. } H(\mathfrak{L}(\overline{\mathbb{R}})) \subseteq \mathfrak{c}L),$$

i.e., each sublocale $H(p,q)$ is closed. We denote by $\mathsf{C}(L)$ (resp. $\overline{\mathsf{C}}(L)$) the set of all continuous real functions (resp. extended continuous real functions) on L.

To compare functions in these classes it suffices to introduce the following order relation in $\overline{\mathsf{F}}(L)$ (cf. page 18):

$$\begin{aligned} F \le G \quad &\equiv \quad F(p,-) \le G(p,-) \text{ for every } p \in \mathbb{Q} \\ &\Leftrightarrow \quad G(-,q) \le F(-,q) \text{ for every } q \in \mathbb{Q}. \end{aligned}$$

With this new setting we have solved our problem:

$$
\begin{tikzcd}[column sep=small]
 & \mathsf{F}(L) & \\
\mathsf{USC}(L) \arrow[ur] & & \mathsf{LSC}(L) \arrow[ul] \\
 & \mathsf{C}(L) = \mathsf{USC}(L) \cap \mathsf{LSC}(L) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{\mathsf{F}}(L) & \\
\overline{\mathsf{USC}}(L) \arrow[ur] & & \overline{\mathsf{LSC}}(L) \arrow[ul] \\
 & \overline{\mathsf{C}}(L) = \overline{\mathsf{USC}}(L) \cap \overline{\mathsf{LSC}}(L) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\tag{3.1.2}
$$

Of course $\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L) = \mathsf{c}(L)$ and $\overline{\mathsf{C}}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L) = \overline{\mathsf{c}}(L)$. The following results ensure that $\mathsf{USC}(L) \simeq \mathsf{usc}(L)$, $\mathsf{LSC}(L) \simeq \mathsf{lsc}(L)$, $\overline{\mathsf{USC}}(L) \simeq \overline{\mathsf{usc}}(L)$ and $\overline{\mathsf{LSC}}(L) \simeq \overline{\mathsf{lsc}}(L)$.

Proposition 3.1.5. *Let L be a frame. Each $f \in \overline{\mathsf{usc}}(L)$ induces a function $\Psi_u(f) \in \overline{\mathsf{USC}}(L)$, defined by*

$$
\begin{aligned}
\Psi_u(f)(-,q) &:= \mathfrak{c}(f(-,q)) \\
\Psi_u(f)(p,-) &:= \bigvee_{r>p} \mathfrak{o}(f(-,r)).
\end{aligned}
$$

The map $\Psi_u : \overline{\mathsf{usc}}(L) \to \overline{\mathsf{USC}}(L)$ satisfies the following:

(a) *Ψ_u is an order-isomorphism between $\overline{\mathsf{usc}}(L)$ and $\overline{\mathsf{USC}}(L)$.*

(b) *$\Psi_{u|\mathsf{usc}(L)}$ is an order-isomorphism between $\mathsf{usc}(L)$ and $\mathsf{USC}(L)$.*

Proof: Let $f \in \overline{\mathsf{usc}}(L)$. Then $\{\mathfrak{o}(f(-,r))\}_{r\in\mathbb{Q}}$ is a descending family of $\mathfrak{S}L$ in the conditions of (b3) (Lemma 3.1.4). Thus, the formulas

$$\Psi_u(f)(p,-) := \bigvee_{r>p} \mathfrak{o}(f(-,r))$$

and

$$\Psi_u(f)(-,q) := \bigvee_{s<q} \mathfrak{c}(f(-,s)) = \mathfrak{c}(\bigvee_{s<q} f(-,s)) = \mathfrak{c}(f(-,q))$$

determine a function $\Psi_u(f) \in \overline{\mathsf{c}}(\mathfrak{S}L)$, which is clearly a function of $\overline{\mathsf{USC}}(L)$.

(a) We show now that Ψ_u is a bijection. For that we prove that, for each $F \in \overline{\mathsf{USC}}(L)$, there exists a unique $\Phi_u(F) \in \overline{\mathsf{usc}}(L)$ such that $\Psi_u(\Phi_u(F)) = F$.

Applying the isomorphism $\mathfrak{c} : L \to \mathfrak{c}L$, it is easy to verify that the map $\Phi_u(F) : \mathfrak{L}_l(\overline{\mathbb{R}}) \to L$ defined by $\Phi_u(F)(-, q) := \mathfrak{c}^{-1}(F(-, q))$ is a frame homomorphism that satisfies $\Psi_u(\Phi_u(F))(-, q) = \mathfrak{c}(\Phi_u(F)(-, q)) = F(-, q)$ for every $q \in \mathbb{Q}$. On the other hand, since $F(r, -) \leq F(-, r)^* = \mathfrak{o}(\Phi_u(F)(-, r))$, then, for each $p \in \mathbb{Q}$,

$$\Psi_u(\Phi_u(F))(p, -) = \bigvee_{r>p} \mathfrak{o}(\Phi_u(F)(-, r)) \geq \bigvee_{r>p} F(r, -) = F(p, -).$$

Moreover, since $1 = F(p, -) \vee F(-, q) = F(p, -) \vee \mathfrak{c}(\Phi_u(F)(-, q))$ for $q > p$, then $\mathfrak{o}(\Phi_u(F)(-, q)) \leq F(p, -)$. Hence

$$\Psi_u(\Phi_u(F))(p, -) = \bigvee_{q>p} \mathfrak{o}(\Phi_u(F)(-, q)) \leq F(p, -).$$

Therefore, $\Psi_u(\Phi_u(F)) = F$.

Finally, for each $f \in \overline{\mathsf{usc}}(L)$, since $\Psi_u(f)(-, q) = \mathfrak{c}(f(-, q))$ for every $q \in \mathbb{Q}$, it is obvious that $\Phi_u(\Psi_u(f)) = f$. The correspondence Ψ_u is thus an isomorphism since, for each $f_1, f_2 \in \overline{\mathsf{usc}}(L)$, we have

$$\begin{aligned} f_1 \leq f_2 &\Leftrightarrow f_2(-, q) \leq f_1(-, q) \text{ for every } q \in \mathbb{Q} \\ &\Leftrightarrow \mathfrak{c}(f_2(-, q)) \leq \mathfrak{c}(f_1(-, q)) \text{ for every } q \in \mathbb{Q} \\ &\Leftrightarrow \Psi_u(f_2)(-, q) \leq \Psi_u(f_1)(-, q) \text{ for every } q \in \mathbb{Q} \\ &\Leftrightarrow \Psi_u(f_1) \leq \Psi_u(f_2). \end{aligned}$$

(b) When $f \in \mathsf{usc}(L)$, it is obvious that $\Psi_u(f) \in \mathsf{USC}(L)$. Moreover, for each $F \in \mathsf{USC}(L)$, $\Phi_u(F) \in \mathsf{usc}(L)$. Indeed, for each $p \in \mathbb{Q}$,

$$0 = F(p, -) \wedge F(-, p) = F(p, -) \wedge \mathfrak{c}(\Phi_u(F)(-, p)),$$

so that $\mathfrak{o}(\Phi_u(F)(-, p)) \geq F(p, -)$. Consequently

$$\bigvee_{p\in\mathbb{Q}} \mathfrak{o}(\Phi_u(F)(-, p)) \geq \bigvee_{p\in\mathbb{Q}} F(p, -) = 1. \qquad \square$$

In a similar way we get:

Proposition 3.1.6. *Let L be a frame. Each $g \in \overline{\mathsf{lsc}}(L)$ induces a function $\Psi_l(g) \in \overline{\mathsf{LSC}}(L)$, given by*

$$\Psi_l(g)(p,-) \;:=\; \mathfrak{c}(g(p,-))$$

$$\Psi_l(g)(-,q) \;:=\; \bigvee_{s<q} \mathfrak{o}(g(s,-)).$$

The map $\Psi_l : \overline{\mathsf{lsc}}(L) \to \overline{\mathsf{LSC}}(L)$ satisfies the following:

(a) *Ψ_l is an order-isomorphism between $\overline{\mathsf{lsc}}(L)$ and $\overline{\mathsf{LSC}}(L)$.*

(b) *$\Psi_{l|\mathsf{lsc}(L)}$ is an order-isomorphism between $\mathsf{lsc}(L)$ and $\mathsf{LSC}(L)$.* □

Proposition 3.1.7. *Let L be a frame. Each $h \in \overline{\mathsf{c}}(L)$ induces a function $\Psi(h) \in \overline{\mathsf{C}}(L)$, given by*

$$\Psi(h)(p,-) \;=\; \mathfrak{c}(h(p,-)) \;:=\; \Psi_l(h)(p,-)$$

$$\Psi(h)(-,q) \;=\; \mathfrak{c}(h(-,q)) \;:=\; \Psi_u(h)(q,-).$$

The map $\Psi : \overline{\mathsf{c}}(L) \to \overline{\mathsf{C}}(L)$ satisfies the following:

(a) *Ψ is an order-isomorphism between $\overline{\mathsf{c}}(L)$ and $\overline{\mathsf{C}}(L)$.*

(b) *$\Psi_{|\mathsf{c}(L)}$ is an order-isomorphism between $\mathsf{c}(L)$ and $\mathsf{C}(L)$.* □

Observation 3.1.8. Defining the *characteristic map* χ_S, for any complemented sublocale S of L, by

$$\chi_S(-,q) := \begin{cases} 0, & \text{if } q \le 0, \\ S, & \text{if } 0 < q \le 1, \\ 1, & \text{if } q > 1, \end{cases} \quad \text{and} \quad \chi_S(p,-) := \begin{cases} 1, & \text{if } p < 0, \\ S^*, & \text{if } 0 \le p < 1, \\ 0, & \text{if } p \ge 1, \end{cases}$$

it is easy to see that:

(a) $\chi_S \in \mathsf{USC}(L)$ if and only if S is a closed sublocale.

(b) $\chi_S \in \mathsf{LSC}(L)$ if and only if S is an open sublocale.

(c) $\chi_S \in \mathsf{C}(L)$ if and only if S is a clopen sublocale.

Let $H \in \mathsf{C}(L)$. If we denote by h the corresponding function of $\mathsf{c}(L)$ given by Proposition 3.1.7 then:

(a) $\chi_{\mathfrak{c}(a)} \leq H \Leftrightarrow h(-,0) = 0$ and $h(-,1) \leq a$.

(b) $\chi_{\mathfrak{o}(a)} \leq H \Leftrightarrow h(-,0) = 0$ and $h(-,1) \leq a^*$.

(c) $H \leq \chi_{\mathfrak{c}(a)} \Leftrightarrow h(0,-) \leq a^*$ and $h(1,-) = 0$.

(d) $H \leq \chi_{\mathfrak{o}(a)} \Leftrightarrow h(0,-) \leq a$ and $h(1,-) = 0$.

In conclusion, with the introduction of definitions (SC_1)-(SC_4) we have now the same freedom as we had in the classical context, being able to speak about arbitrary (not necessarily continuous) real functions. In particular we have the desired identity

$$\frac{\begin{array}{c}\textit{upper semicontinuity}\\ +\\ \textit{lower semicontinuity}\end{array}}{\textit{continuity}}$$

Further, diagrams (3.1.2) make sense in locale theory to the following citation from Gillman and Jerison [11]:

> «*The set $C(X)$ of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers.*
>
> (...) *In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty).*
>
> (...) *The partial ordering on $\mathbb{R}^X$ is defined by: $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$.*
>
> (...) *The set of all continuous functions from the topological space X into the space $\mathbb{R}$ is denoted $C(X)$.*
>
> (...) *Therefore $C(X)$ is a commutative ring, a subring of $\mathbb{R}^X$.*»

3.2. Regularization

A basic fact from Real Analysis asserts that every real function $f : X \to \mathbb{R}$ on a topological space X, not necessarily lower semicontinuous, admits a lower semicontinuous regularization $f_* : X \to \overline{\mathbb{R}}$, given by the lower limit of f:

$$f_*(x) := \liminf_{y \to x} f(y) = \bigvee \{ \bigwedge f(U) \mid x \in U \in \mathcal{O}X \} \qquad [26].$$

This is the largest lower semicontinuous minorant of f:

$$f_* = \bigvee \{ g \in \mathsf{LSC}(X, \overline{\mathbb{R}}) \mid g \leq f \}.$$

For each $p \in \mathbb{Q}$ we have

$$f_*^{-1}(]p, +\infty[) = \bigcup_{r>p} (f^{-1}([r, +\infty[))^\circ = X \setminus \bigcap_{r>p} \overline{f^{-1}(]-\infty, r[)},$$

which means that the lower regularization f_* takes values in $\mathbb{R}$ if and only if it has a lower semicontinuous minorant; equivalently, if and only if

$$\bigcup_{r \in \mathbb{Q}} f_*^{-1}(]r, +\infty[) = X,$$

that is,

$$\bigcap_{r \in \mathbb{Q}} \overline{f^{-1}(]-\infty, r[)} = \varnothing.$$

Once we know already how to deal with generic real functions, we may now try to approach the above construction in the localic setting[18].

We start with the lower regularization. The construction can be performed in a surprising easy and transparent way:

Let $H \in \overline{\mathsf{F}}(L)$. The family $\{\overline{H(r,-)} \mid r \in \mathbb{Q}\}$ is descending so, by Lemma 3.1.4, formulas

$$H^\circ(p,-) := \bigvee_{r>p} \overline{H(r,-)}$$

$$H^\circ(-,q) := \bigvee_{s<q} \neg\, \overline{H(s,-)}$$

determine a frame homomorphism $H^\circ : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{S}L$, that we call the *lower regularization* of H. It satisfies, among others, the following properties [16]:

[18] Since symbols $(\cdot)_*$ and $(\cdot)^*$ have a special meaning in frames, we use $(\cdot)^\circ$ and $(\cdot)^-$, that have the advantage of making explicit the similarities between the lower and upper regularizations and the interior and closure operators in Topology (for instance, $(\chi_A)^\circ = \chi_{A^\circ}$ and $(\chi_A)^- = \chi_{\overline{A}}$).

Proposition 3.2.1. *For each $H \in \overline{\mathsf{F}}(L)$ we have:*

(a) $H^\circ \in \overline{\mathsf{LSC}}(L)$.

(b) $H^\circ \leq H$.

(c) *If $G \in \overline{\mathsf{LSC}}(L)$ and $G \leq H$ then $G \leq H^\circ$. Therefore,*

$$H^\circ = \bigvee \{G \in \overline{\mathsf{LSC}}(L) \mid G \leq H\}.$$

(d) $\overline{\mathsf{LSC}}(L) = \{H \in \overline{\mathsf{F}}(L) \mid H = H^\circ\}$.

(e) $\mathsf{LSC}(L) = \{H \in \mathsf{F}(L) \mid H = H^\circ\}$.

(f) $H^{\circ\circ} = H^\circ$.

Proof: (a) It is obvious, by the definition of H°, that $H^\circ(p, -) \in \mathfrak{c}L$.
(b) For each $p \in \mathbb{Q}$,

$$H^\circ(p, -) = \bigvee_{r>p} \overline{H(r, -)} \leq \bigvee_{r>p} H(r, -) = H(p, -).$$

(c) If $G \in \overline{\mathsf{LSC}}(L)$ and $G \leq H$ then, for each $p \in \mathbb{Q}$ and each $r > p$, $G(r, -) \in \mathfrak{c}L$ and $G(r, -) \leq H(r, -)$. Thus $G(r, -) \leq \overline{H(r, -)}$ and, consequently,

$$H^\circ(p, -) = \bigvee_{r>p} \overline{H(r, -)} \geq \bigvee_{r>p} G(r, -) = G(p, -).$$

(d) It is an immediate consequence of the previous assertions.

(e) It is an immediate consequence of (d).

(f) It is an immediate consequence of the previous assertions. $\square$

Analogously, we can define the *upper regularization* of $H \in \overline{\mathsf{F}}(L)$:

$$H^-(-, q) \;:=\; \bigvee_{s<q} \overline{H(-, s)},$$

$$H^-(p, -) \;:=\; \bigvee_{r>p} \neg\, \overline{H(-, r)}.$$

Proposition 3.2.2. ([16]) *For each* $H \in \overline{\mathsf{F}}(L)$ *we have:*

(a) $H^- \in \overline{\mathsf{USC}}(L)$.

(b) $H \leq H^-$.

(c) *If* $F \in \overline{\mathsf{USC}}(L)$ *and* $H \leq F$ *then* $H^- \leq F$. *Therefore*

$$H^- = \bigwedge \{F \in \overline{\mathsf{USC}}(L) \mid H \leq F\}.$$

(d) $\overline{\mathsf{USC}}(L) = \{H \in \overline{\mathsf{F}}(L) \mid H = H^-\}$.

(e) $\mathsf{USC}(L) = \{H \in \mathsf{F}(L) \mid H = H^-\}$.

(f) $H^{--} = H^-$. $\square$

In particular, for each sublocale S of L:

$$\chi_S{}^\circ(p, -) = \begin{cases} 1 & \text{if } p < 0, \\ \overline{S^*} & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases} \qquad \chi_S{}^\circ(-, q) = \begin{cases} 0 & \text{if } q \leq 0, \\ \neg\,\overline{S^*} & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1, \end{cases}$$

$$\chi_S{}^-(-, q) = \begin{cases} 0 & \text{if } q \leq 0, \\ \overline{S} & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1, \end{cases} \qquad \chi_S{}^-(p, -) = \begin{cases} 1 & \text{if } p < 0, \\ \neg\,\overline{S} & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1. \end{cases}$$

Therefore,

$$\chi_S{}^- = \chi_{\overline{S}} \quad \text{and} \quad \chi_S{}^\circ = \chi_{\neg\,\overline{S^*}} = \chi_{S^\circ}$$

(since $\neg\,\overline{S^*} = S^\circ$, by property ($\mathrm{S}_3$) of 2.2).

3.3. Insertion

As we observed in the Introduction, the theorems about the existence of continuous real functions in normal spaces rank among the fundamental results of point-set topology. They can be classified in three types:

- separation theorems (like the Urysohn Lemma),
- extension theorems (like the Tietze Theorem),
- insertion theorems (like the Katětov-Tong Theorem).

The latter are the most important since they imply the former two as corollaries.

We are now ready to show that all these results are, ultimately, results about locales, from which they readily follow as particular cases.

We begin by the pointfree version of Katětov-Tong insertion theorem, that holds for *normal frames*, that is, frames in which, whenever $a \vee b = 1$, there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$ (or, equivalently, there exists $u \in L$ such that $a \vee u = 1 = b \vee u^*$).

It is not hard to show that a frame L is normal if and only if

> *for any countable subsets $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ of L satisfying $a_i \vee (\bigwedge_{j\in\mathbb{N}} b_j) = 1$ and $b_i \vee (\bigwedge_{j\in\mathbb{N}} a_j) = 1$, for every $i \in \mathbb{N}$, there exists $u \in L$ such that $a_i \vee u = 1$ and $b_i \vee u^* = 1$ for every $i \in \mathbb{N}$* [33].

Based on this characterization it is then possible to show [33] that, for each enumeration $\{\alpha_i \mid i \in \mathbb{N}\}$ of $\mathbb{Q}$, whenever $F \in \mathsf{USC}(L)$ and $G \in \mathsf{LSC}(L)$ satisfy $F \leq G$, there exists $\{u_{\alpha_i}\}_{i\in\mathbb{N}} \subseteq L$ such that

$$(q > \alpha_i) \;\Rightarrow\; (F(-,q) \vee \mathfrak{c}(u_{\alpha_i}) = 1), \tag{3.3.1}$$
$$(p < \alpha_i) \;\Rightarrow\; (G(p,-) \vee \mathfrak{c}(u^*_{\alpha_i}) = 1), \tag{3.3.2}$$
$$(\alpha_{j_1} < \alpha_{j_2}) \;\Rightarrow\; (u_{\alpha_{j_1}} \vee u^*_{\alpha_{j_2}} = 1). \tag{3.3.3}$$

Using this fact it is possible to show the Katětov-Tong Theorem:

Theorem 3.3.1. [Insertion: Katětov-Tong] ([33, 12])

For each frame L, the following assertions are equivalent:

(i) *L is normal.*

(ii) *For every $F \in \overline{\mathsf{USC}}(L)$ and $G \in \overline{\mathsf{LSC}}(L)$ satisfying $F \leq G$, there is a function $H \in \overline{\mathsf{C}}(L)$ such that $F \leq H \leq G$.*

(iii) *For every $F \in \mathsf{USC}(L)$ and $G \in \mathsf{LSC}(L)$ satisfying $F \leq G$, there is a function $H \in \mathsf{C}(L)$ such that $F \leq H \leq G$.*

Proof: (i)⇒(ii): Let $F \in \overline{\mathsf{USC}}(L)$ and $G \in \overline{\mathsf{LSC}}(L)$ such that $F \leq G$. By normality, there exists $\{u_{\alpha_i}\}_{i\in\mathbb{N}} \subseteq L$ satisfying conditions (3.3.1), (3.3.2) and (3.3.3). The latter means that $\{u_{\alpha_i}\}_{i\in\mathbb{N}}$ is a descending family of L satisfying condition (b3) of Lemma 3.1.4. Consequently, the formulas

$$h(-,q) := \bigvee_{\alpha_i<q} u^*_{\alpha_i} \quad \text{and} \quad h(p,-) := \bigvee_{\alpha_i>p} u_{\alpha_i}$$

define a function $h \in \overline{\mathsf{c}}(L)$. The corresponding function $H = \Psi(h) \in \overline{\mathsf{C}}(L)$ satisfies $F \leq H \leq G$:

Indeed, condition (3.3.1) means that, for each $q \in \mathbb{Q}$ and each $\alpha_i < q$, $F(-,q) \geq \mathfrak{o}(u_{\alpha_i})$. Then $F(-,q) \geq \overline{\mathfrak{o}(u_{\alpha_i})} = \mathfrak{c}(u^*_{\alpha_i})$, therefore

$$F(-,q) \geq \bigvee_{\alpha_i<q} \mathfrak{c}(u^*_{\alpha_i}) = \mathfrak{c}(\bigvee_{\alpha_i<q} u^*_{\alpha_i}) = H(-,q).$$

Similarly, by condition (3.3.2) one has $G(p,-) \geq \mathfrak{o}(u^*_{\alpha_i}) \geq \overline{\mathfrak{o}(u^*_{\alpha_i})}$ whenever $\alpha_i > p$. Thus, for each $p \in \mathbb{Q}$,

$$G(p,-) \geq \bigvee_{\alpha_i>p} \overline{\mathfrak{o}(u^*_{\alpha_i})} = \bigvee_{\alpha_i>p} \mathfrak{c}(u^{**}_{\alpha_i}) \geq \bigvee_{\alpha_i>p} \mathfrak{c}(u_{\alpha_i}) = \mathfrak{c}(\bigvee_{\alpha_i>p} u_{\alpha_i}) = H(p,-).$$

(ii)⇒(iii): Let $F \in \mathsf{USC}(L)$ and $G \in \mathsf{LSC}(L)$ such that $F \leq G$. It suffices to show that the function $H \in \overline{\mathsf{C}(L)}$ given by hypothesis belongs to $\mathsf{C}(L)$, that is, $\bigvee_{q\in\mathbb{Q}} H(-,q) = 1 = \bigvee_{p\in\mathbb{Q}} H(p,-)$, which is easy:

$$\bigvee_{q\in\mathbb{Q}} H(-,q) \geq \bigvee_{q\in\mathbb{Q}} G(-,q) = G\left(\bigvee_{q\in\mathbb{Q}}(-,q)\right) = G(1) = 1,$$

$$\bigvee_{p\in\mathbb{Q}} H(p,-) \geq \bigvee_{p\in\mathbb{Q}} F(p,-) = F\left(\bigvee_{p\in\mathbb{Q}}(p,-)\right) = F(1) = 1.$$

(iii)⇒(i): Suppose $a \vee b = 1$ in L. Then $\mathfrak{o}(b) \leq \mathfrak{c}(a)$, that is, $\chi_{\mathfrak{c}(a)} \leq \chi_{\mathfrak{o}(b)}$. By hypothesis, there exists $H \in \mathsf{C}(L)$ such that $\chi_{\mathfrak{c}(a)} \leq H \leq \chi_{\mathfrak{o}(b)}$. Consider the closed sublocale $H\left(\frac{1}{2},-\right)$, which is equal to $\mathfrak{c}(u)$ for some $u \in L$. Then

$$1=\chi_{\mathfrak{c}(a)}(-,\frac{3}{4})\vee\chi_{\mathfrak{c}(a)}(\frac{1}{2},-)\leq\chi_{\mathfrak{c}(a)}(-,\frac{3}{4})\vee H(\frac{1}{2},-)=\mathfrak{c}(a)\vee\mathfrak{c}(u)=\mathfrak{c}(a\vee u).$$

This guarantees that $a \vee u = 1$. On the other hand, $H(-,\frac{1}{2}) \leq \neg H(\frac{1}{2},-) = \mathfrak{o}(u)$ and, therefore, since $H(-,\frac{1}{2})$ is closed, $H(-,\frac{1}{2}) \leq \overline{\mathfrak{o}(u)} = \mathfrak{c}(u^*)$. Consequently,

$$\mathfrak{c}(b\vee u^*)=\mathfrak{c}(b)\vee\mathfrak{c}(u^*)\geq\chi_{\mathfrak{o}(b)}(\frac{1}{4},-)\vee H(-,\frac{1}{2})\geq\chi_{\mathfrak{o}(b)}(\frac{1}{4},-)\vee\chi_{\mathfrak{o}(b)}(-,\frac{1}{2})=1,$$

which shows that $b \vee u^*$ is also equal to 1. This proves the normality of L.

□

If we apply Theorem 3.3.1 to the topology $\mathcal{O}X$ of a normal space X, the implication "(i) ⇒ (iii)" provides the non-trivial implication of the classical Katětov-Tong Theorem([25, 39]):

Let $f : X \to \mathbb{R}$ be an upper semicontinuous function and $g : X \to \mathbb{R}$ a lower semicontinuous function such that $f \leq g$. The family

$$\{f^{-1}(]-\infty,q[) \mid q \in \mathbb{Q}\}$$

is a descending u-scale, while

$$\{g^{-1}(]p,+\infty[) \mid p \in \mathbb{Q}\}$$

is a descending l-scale. Consequently, by Lemma 3.1.4, the formulas

$$\begin{aligned} F(-,q) &:= \mathfrak{c}(f^{-1}(]-\infty,q[)) \quad \text{and} \quad F(p,-) := \bigvee_{r>p} \mathfrak{o}(f^{-1}(]-\infty,r[)) \\ G(p,-) &:= \mathfrak{c}(g^{-1}(]p,+\infty[)) \quad \text{and} \quad G(-,q) := \bigvee_{s<q} \mathfrak{o}(g^{-1}(]s,+\infty[)) \end{aligned}$$

establish functions $F, G : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}(\mathcal{O}X)$. The condition $f \leq g$ implies $f^{-1}(]-\infty,q[) \supseteq g^{-1}(]-\infty,q[)$ for every $q \in \mathbb{Q}$, therefore $F \leq G$. Consider $H \in \mathsf{C}(\mathcal{O}X)$ provided by Theorem 3.3.1, and the corresponding continuous map $h : X \to \mathbb{R}$ defined by

$$h(x) \in]p,q[\text{ if and only if } x \in \mathfrak{c}^{-1}(H(p,q)).$$

It is then obvious that $f \leq h \leq g$.

Other insertion theorems were meanwhile obtained for other classes of frames [13, 14, 15]. In the sequel we present briefly these results. The first is, in some sense, a dual version of the previous theorem and generalizes the Stone (insertion) theorem [38].

Theorem 3.3.2. [Insertion: Stone] ([28, 14])

For each frame L, the following assertions are equivalent[19]*:*

(i) *L is extremally disconnected*[20].

(ii) *For every $F \in \mathsf{USC}(L)$ and $G \in \mathsf{LSC}(L)$ satisfying $G \leq F$, there exists $H \in \mathsf{C}(L)$ such that $G \leq H \leq F$.*

The next is the monotone version of Katětov-Tong Theorem and generalizes the (monotone insertion) theorem of Kubiak [27].

Let

$$\mathsf{UL}(L) := \{(F, G) \in \mathsf{USC}(L) \times \mathsf{LSC}(L) \mid F \leq G\}$$

be partially ordered by the order inherited from $\mathsf{F}(L)^{\mathrm{op}} \times \mathsf{F}(L)$, i.e.,

$$(F_1, G_1) \leq (F_2, G_2) \ \equiv\ F_2 \leq F_1 \text{ and } G_1 \leq G_2.$$

Theorem 3.3.3. [Monotone insertion: Kubiak] ([13])

For each frame L, the following assertions are equivalent:

(i) *L is monotonically normal*[21].

(ii) *There is a monotone map $\Delta : \mathsf{UL}(L) \to \mathsf{C}(L)$ such that $F \leq \Delta(F, G) \leq G$ for every pair $(F, G) \in \mathsf{UL}(L)$.*

[19] Some more conditions equivalent to (i) are:
(a) $\mathsf{C}(L) = \{F^\circ \mid F \in \mathsf{USC}(L)\}$; (b) $\mathsf{C}(L) = \{G^- \mid G \in \mathsf{LSC}(L)\}$; (c) If $F \in \mathsf{USC}(L)$, $G \in \mathsf{LSC}(L)$ and $G \leq F$, then $G^- \leq F^\circ$ [14].

[20] A frame L is *extremally disconnected* if $a^* \vee a^{**} = 1$ for every $a \in L$. These frames are precisely those in which the second De Morgan law $(\bigwedge_{i \in I} a_i)^* = \bigvee_{i \in I} a_i^*$ holds; by that reason, they are also referred to as *De Morgan frames*.

[21] The definition of normality may be rephrased in the following way: let $\mathcal{D}_L = \{(a, b) \in L \times L \mid a \vee b = 1\}$; a frame L is normal if and only if there exists a map $\Delta : \mathcal{D}_L \to L$ satisfying $a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$. Equipping $\mathcal{D}_L$ with the partial order $(\leq^{\mathrm{op}}, \leq)$ inherited from $L^{\mathrm{op}} \times L$, L is called *monotonically normal* [13] if it is normal and Δ is monotone.

Writing briefly $\mathbf{0} \leq F$ whenever $F(-,0) = 0$ and $F \leq \mathbf{1}$ whenever $F(1,-) = 0$, let $\mathsf{C}^*(L)$ denote the class $\{H \in \mathsf{C}(L) \mid \mathbf{0} \leq H \leq \mathbf{1}\}$ of all *bounded* continuous real functions on L (one may define, in a similar way, the classes $\mathsf{USC}^*(L)$ and $\mathsf{LSC}^*(L)$). The next theorem is the pointfree version of the (insertion) theorem of Michael [31].

Theorem 3.3.4. [Bounded insertion: Michael] ([15])

For each frame L, the following assertions are equivalent:

(i) *L is perfectly normal*[22].

(ii) *L is normal and for each $G \in \mathsf{LSC}^*(L)$ there exists $H \in \mathsf{C}(L)$ such that $\mathbf{0} \leq H \leq G$ and $H(0,-) = G(0,-)$.*

Let $F \leq G \in \mathsf{F}(L)$. Let us denote by $\iota(F,G)$ the join

$$\bigvee_{r \in \mathbb{Q}} F(-,r) \wedge G(r,-).$$

The notation $F < G$ means $\iota(F,G) = 1$. Clearly, $\mathbf{0} < G$ if and only if $G(0,-) = 1$. The last insertion theorem in our list is the pointfree version of the (insertion) theorem of Dowker [6].

Theorem 3.3.5. [Strict insertion: Dowker] ([15])

For each frame L, the following assertions are equivalent:

(i) *L is countably paracompact*[23].

(ii) *L is normal and for each $G \in \mathsf{LSC}^*(L)$ satisfying $0 < G$ there exists $H \in \mathsf{C}(L)$ such that $\mathbf{0} < H < G$.*

---

[22] A frame L is called *perfectly normal* [15] if, for each $a \in L$, there exists a countable subset $B \subseteq L$ such that $a = \bigvee B$ and $b \prec a$ (i.e., $b^* \vee a = 1$) for every $b \in B$.

[23] A frame L is said to be *countably paracompact* [8] if every countable cover of L has a locally finite refinement.

3.4. Separation

Let L be a normal frame and consider $a, b \in L$ satisfying $a \vee b = 1$. By property (S_4) of 2.2, $\mathfrak{o}(b) \leq \mathfrak{c}(a)$. Therefore, $\chi_{\mathfrak{c}(a)} \leq \chi_{\mathfrak{o}(b)}$. Consequently, applying Theorem 3.3.1 we get $H \in \mathsf{C}(L)$ such that

$$\chi_{\mathfrak{c}(a)} \leq H \leq \chi_{\mathfrak{o}(b)}.$$

Let h denote the corresponding function from $\mathsf{c}(L)$. By Observation 3.1.8, it follows immediately the non-trivial implication of the following corollary:

Corollary 3.4.1. [Separation: Urysohn]

A frame L is normal if and only if, for every $a, b \in L$ satisfying $a \vee b = 1$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((-,0) \vee (1,-)) = 0$, $h(0,-) \leq a$ and $h(-,1) \leq b$.

The statement of 3.4.1 is precisely the statement of the (separation) lemma of Urysohn for frames [7] (cf. [2], Prop. 5), that extends the famous Urysohn Lemma of point-set topology.

We can deduce in a similar way, from Theorem 3.3.2, the frame counterpart of the (separation) lemma of Stone:

Corollary 3.4.2. [Separation: Stone] ([14])

A frame L is extremally disconnected if and only if, for every $a, b \in L$ satisfying $a \wedge b = 0$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((-,0) \vee (1,-)) = 0$, $h(0,-) \leq a^*$ and $h(-,1) \leq b^*$.

If we do the same with Theorem 3.3.4 we get the pointfree version of a separation result due to Vedenissoff [40]:

Corollary 3.4.3. [Bounded separation: Vedenissoff] ([15])

A frame L is perfectly normal if and only if, for every $a, b \in L$ satisfying $a \vee b = 1$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((-,0) \vee (1,-)) = 0$, $h(0,-) = a$ and $h(-,1) = b$.

3.5. Extension

For any sublocale S of L, let $c_S : L \twoheadrightarrow S$ denote the corresponding frame quotient, given by $c_S(x) := \bigwedge\{s \in S \mid x \leq s\}$. We say that $\widetilde{H} \in \mathsf{C}(L)$ is a *continuous extension* of $H \in \mathsf{C}(S)$ whenever the following diagram commutes

$$\begin{tikzcd}
 & \mathfrak{c}(L) \arrow[r, leftrightarrow, "\mathfrak{c}"] & L \arrow[d, "c_S"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathfrak{c}(S) \arrow[r, leftrightarrow, "\mathfrak{c}"'] & S
\end{tikzcd}$$

i.e., $c_S \circ \mathfrak{c} \circ \widetilde{H} = \mathfrak{c} \circ H$.

As it is shown in [33], from Theorem 3.3.1 it also follows the (extension) Theorem of Tietze for frames [41]:

Corollary 3.5.1. [Extension: Tietze]

For each frame L, the following assertions are equivalent:

(i) *L is normal.*

(ii) *For any closed sublocale S of L and any $H \in \mathsf{C}(S)$, there exists a continuous extension $\widetilde{H} \in \mathsf{C}(L)$ of H.*

Dually, from Theorem 3.3.2 it follows:

Corollary 3.5.2. [Extension: Stone] ([14])

For each frame L, the following assertions are equivalent:

(i) *L is extremally disconnected.*

(ii) *For any open sublocale S of L and any $H \in \mathsf{C}(S)$, there exists a continuous extension $\widetilde{H} \in \mathsf{C}(L)$ of H.*

In order to formulate the monotone version of 3.5.1 we introduce the following notation: given a sublocale S of L, a function $\mathcal{E}_S : \mathsf{C}(S) \to \mathsf{C}(L)$ is said to be an *extender* if $\mathcal{E}_S(H)$ is a continuous extension of H for every $H \in \mathsf{C}(S)$.

Let S be a closed sublocale of L and let $H \in \mathsf{C}(S)$. Define $\widehat{H}^u \in \mathsf{USC}(L)$ and $\widehat{H}^l \in \mathsf{LSC}(L)$ by

$$\widehat{H}^u(-,q) := \begin{cases} 0 & \text{if } q \leq 0, \\ H(-,q) & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1, \end{cases} \quad \widehat{H}^u(p,-) := \begin{cases} 1 & \text{if } p < 0, \\ \bigvee_{r>p} \neg H(-,r) & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases}$$

$$\widehat{H}^l(p,-) := \begin{cases} 1 & \text{if } p < 0, \\ H(p,-) & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases} \quad \widehat{H}^l(-,q) := \begin{cases} 0 & \text{if } q \leq 0, \\ \bigvee_{s<q} \neg H(s,-) & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1. \end{cases}$$

It is easy to check that $\widehat{H}^u \leq \widehat{H}^l$ i.e., $\widehat{H} := (\widehat{H}^u, \widehat{H}^l) \in \mathsf{UL}(L)$. We have then the pointfree counterpart of the (extension) Theorem of Stares [37]:

Corollary 3.5.3. [Monotone extension: Stares] ([13])

For each frame L, the following are equivalent:

(i) *L is monotonically normal.*

(ii) *For each closed sublocale S of L there exists an extender $\mathcal{E}_S : \mathsf{C}^*(S) \to \mathsf{C}^*(L)$ such that, for any $S_1, S_2 \in \mathfrak{S}L$ and $H_i \in \mathsf{C}^*(S_i)$ $(i = 1, 2)$ satisfying $\widehat{H}_1 \leq \widehat{H}_2$, $\mathcal{E}_{S_1}(H_1) \leq \mathcal{E}_{S_2}(H_2)$.*

Finally:

Corollary 3.5.4. [Bounded extension] ([15]) *For each frame L, the following are equivalent:*

(i) *L is perfectly normal.*

(ii) *For every closed sublocale S of L and any $H \in \mathsf{C}^*(S)$, there exists a continuous extension $\widetilde{H} \in \mathsf{C}^*(L)$ of H such that $\widetilde{H}(0,1) \geq S$.*

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