Interior-preserving covers, spectrum covers and semicontinuous real functions in frames

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THE SETTING

THE CATEGORY OF QUASI-UNIFORM FRAMES

[J. P., 1995]

Entourages

$E \in L \oplus L$ $\quad\equiv\quad$ $E \subseteq (L \times L, \leq)$

$$(x,y) \leq (z,w) \in E \Rightarrow (x,y) \in E$$

$$\{x\} \times S \subseteq E \Rightarrow (x, \bigvee S) \in E$$

$$S \times \{x\} \subseteq E \Rightarrow (\bigvee S, x) \in E$$

such that $\displaystyle\bigvee_{(x,x)\in E} x = 1.$

$$x \oplus y := \downarrow (x,y) \cup \{(0,a),(a,0) \mid a \in L\}$$

$$E \circ F := \bigvee \{x \oplus y \mid \exists z \neq 0 : x \oplus z \subseteq E, z \oplus y \subseteq F\}$$

Objects $\quad (L,\mathcal{E}) \quad \mathcal{E} \neq \emptyset$ filter of $(Ent(L),\subseteq)$

(Q1) $\forall\, E \in \mathcal{E} \quad \exists F \in \mathcal{E} : F \circ F \subseteq E$

(Q2) $\forall\, x \in L \quad x = \bigvee\{y \in L \mid \underbrace{y \overset{\overline{\mathcal{E}}}{\lhd_1} x}\}$

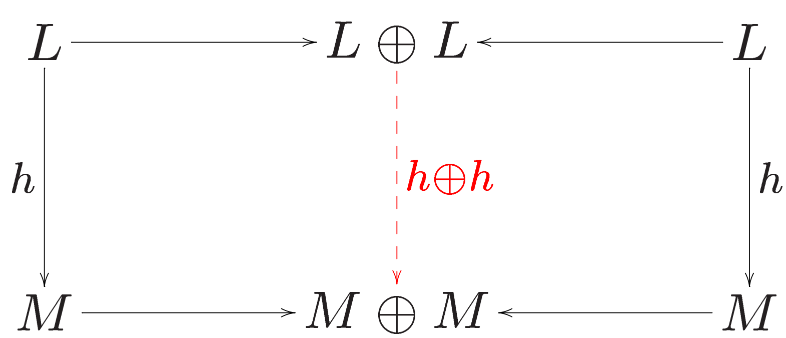
$\exists\, E \in \overline{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}^{-1} : E \circ (y \oplus y) \subseteq x \oplus x$

$y \overset{\overline{\mathcal{E}}}{\lhd_2} x \;\equiv\; \exists\, E \in \overline{\mathcal{E}} : (y \oplus y) \circ E \subseteq x \oplus x$

$$(L,\mathcal{E}) \in QUFrm \xrightarrow{\mathcal{L}_i(\mathcal{E}) := \{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd_i} x\}\}} (L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$$

is a biframe

Morphisms $\quad h : (L,\mathcal{E}_1) \longrightarrow (M,\mathcal{E}_2)$



$$E \in \mathcal{E}_1 \;\overset{h \oplus h}{\longmapsto}\; (h \oplus h)(E) \in \mathcal{E}_2$$

THE SUBOBJECT LATTICE

Locale X

$\mathcal{S}(X)$ is a CO-FRAME

$\updownarrow \cong$

$(\mathfrak{C}L)^{op}$ dual of the congruence lattice of $L = \mathcal{O}X$

$$\left.\begin{array}{ll} \nabla_a := \{(x,y) \mid x \vee a = y \vee a\} & \text{CLOSED} \\ \Delta_a := \{(x,y) \mid x \wedge a = y \wedge a\} & \text{OPEN} \end{array}\right\} \text{complem.}$$

$\nabla L := \{\nabla_a \mid a \in L\}$ subframe of $\mathfrak{C}L$

$L \to \nabla L : a \mapsto \nabla_a$ isomorphism

$\Delta L := < \{\Delta_a \mid a \in L\} >$ subframe of $\mathfrak{C}L$

$L \to \Delta L : a \mapsto \Delta_a$ dual poset embedding

$(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe SKULA BIFRAME of L

MOTIVATING EXAMPLE

[W. Hunsaker, J.P., 2002]

Frame L

$(\mathfrak{C}L, \nabla L, \Delta L)$

$E_a := (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a) \qquad (a \in L)$

The E_a $(a \in L)$ generate a quasi-uniformity $\mathcal{F}$ on $\mathfrak{C}L$

$$(\mathfrak{C}L, \mathcal{F}) \rightsquigarrow \mathcal{L}_1(\mathcal{F}) = \nabla L \cong L$$

compatible

The Frith quasi-uniformity $\mathcal{F}$

FLETCHER CONSTRUCTION IN FRM

[Kyungpook Math. J. (*to appear*)]

$E_a := (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a)$

$\mathcal{A}$: family of "nice" covers of L

$$R_A := \bigcap_{a \in A} E_a \in Ent(\mathfrak{C}L)? \qquad A \in \mathcal{A}$$

$$\Updownarrow$$

$$\bigvee_{A_1 \cup A_2 = A} \Big(\bigwedge_{A_1} \nabla_a \wedge \bigwedge_{A_2} \Delta_a\Big) = 1$$

Fletcher covers

$\mathcal{E}_{\mathcal{A}}$:= the filter generated by all R_A

PROP. If $\bigcup \mathcal{A}$ generates L then $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}}) = \nabla L$, and we get a transitive quasi-unif. compatible with L.

"NICE" COVERS

TOP ⟷ *dual* ⟷ **FRM**

$\{A_i \mid i \in I\}$ — open cover — $\bigwedge_I \Delta_{a_i} = 0 \Leftrightarrow \bigvee_I a_i = 1$

$int(\bigcap_J A_i) = \bigcap_J A_i$ — int.-pres. — $int(\bigvee_J \Delta_{a_i}) = \bigvee_J \Delta_{a_i}$

$$\| $$

$$\Delta_{\bigwedge_J a_i}$$

$\Updownarrow$ (TOP) $\Downarrow$, $\not\Uparrow$ non-scattered locales (FRM)

$cl(\bigcup_J (X \setminus A_i)) = \bigcup_J (X \setminus A_i)$ — cl.-pres. — $cl(\bigwedge_J \nabla_{a_i}) = \bigwedge_J \nabla_{a_i}$

$$\| $$

$$\nabla_{\bigwedge_J a_i}$$

$\{X \setminus A_i \mid i \in I\}$ — cl. co-cover — $\bigvee_I \nabla_{a_i} = 1 \Leftrightarrow \bigvee_I a_i = 1$

CLOSURE-PRESERVING COVERS

EXAMPLES OF CLOSURE-PRES. COVERS

Locally finite covers $A \subseteq L$

- $\exists$ cover $C \subseteq L$ s.t., for every $c \in C$,

  $A_c := \{a \in A \mid a \wedge c \neq 0\} \in \mathcal{P}_{fin}(A)$.

Point-finite covers [Dowker, Strauss, 1973]

$A \subseteq L$

- $\forall\, x \in L, \quad x = \bigwedge_{F \in \mathcal{P}_{fin}(A)} (x \vee \bigvee (A \setminus F))$.

Well-monotone covers $A \subseteq L$,

- well-ordered by the partial order $\leq$ of L.

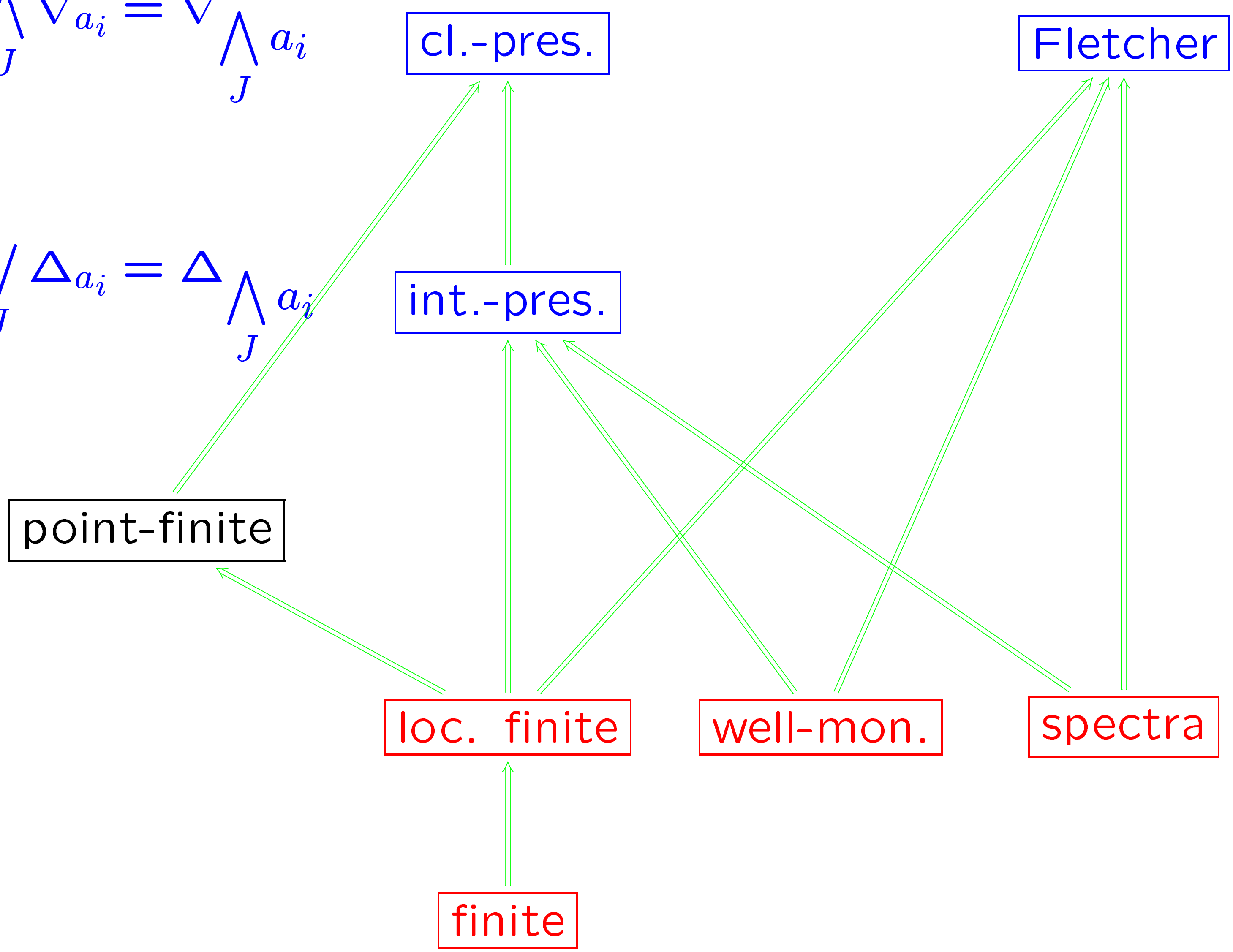
Spectrum covers $A = \{a_n \mid n \in \mathbb{Z}\} \subseteq L$

- $a_n \leq a_{n+1}$
- $\bigvee_{n \in \mathbb{Z}} \Delta_{a_n} = 1$ (in particular, $\bigwedge_{n \in \mathbb{Z}} a_n = 0$).

$$\bigvee \Big(\bigwedge_{A_1} \nabla_a \wedge \bigwedge_{A_2} \Delta_a \Big) = 1$$

$$\bigwedge_J \nabla_{a_i} = \nabla_{\bigwedge_J a_i}$$

$$\bigvee_J \Delta_{a_i} = \Delta_{\bigwedge_J a_i}$$



EXAMPLES OF COMPATIBLE QUASI-UNIF.

Subbase	Quasi-unif.
$\{R_A \mid A$ finite cover$\}$	$\mathcal{F}$
$\{R_A \mid A$ locally finite cover$\}$	$\mathcal{LF}$
$\{R_A \mid A$ point-finite Fletcher cover$\}$	$\mathcal{PF}$
$\{R_A \mid A$ closure-pres. Fletcher cover$\}$	$\mathcal{FT}$
$\{R_A \mid A$ well-monotone cover$\}$	$\mathcal{W}$
$\{R_A \mid A$ spectrum cover$\}$	$\mathcal{SC}$

THE QUASI-UNIFORMITY $\mathcal{Q}$ OF THE REALS

FRAME OF REALS $\mathfrak{L}(\mathbb{R})$ [B. Banaschewski]

$\mathfrak{L}(\mathbb{R}) = Frm\langle (p,q)(p,q \in \mathbb{Q}) \mid$

$$(1)\ (p,q) \wedge (r,s) = (p \vee r, q \wedge s)$$

$$(2)\ p \leq r < q \leq s \Rightarrow (p,q) \vee (r,s) = (p,s)$$

$$(3)\ \bigvee \{(r,s) \mid p < r < s < q\} = (p,q)$$

$$(4)\ \bigvee \{(p,q) \mid p,q \in \mathbb{Q}\} = 1\rangle.$$

$$(-,q) := \bigvee_{p \in \mathbb{Q}} (p,q) \qquad (p,-) := \bigvee_{q \in \mathbb{Q}} (p,q)$$

$\mathfrak{L}_u(\mathbb{R}) = \langle (-,q) \mid q \in \mathbb{Q} \rangle.$

$\mathfrak{L}_l(\mathbb{R}) = \langle (p,-) \mid p \in \mathbb{Q} \rangle.$

$(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}))$ BIFRAME OF REALS

$$Q_n := \bigvee_{0<q-p<\frac{1}{n}} (-,q) \oplus (p,-) \qquad (n \in \mathbb{N})$$

- $Q_n \in Ent(\mathfrak{L}(\mathbb{R}))$
- $Q_{n+1} \subseteq Q_n$
- $Q_{2n} \circ Q_{2n} \subseteq Q_n$

Let $\mathcal{Q}$ be the filter generated by $\{Q_n \mid n \in \mathbb{N}\}$

PROPOSITION. $(\mathfrak{L}(\mathbb{R}), \mathcal{Q})$ is a quasi-unif. frame whose underlying biframe is the biframe of reals.

SEMICONTINUOUS REAL FUNCTIONS

upper semicontinuous real function on L: homomorphism $h : \mathfrak{L}_u(\mathbb{R}) \to L$.

For any space X, there is a bijection

$$\boxed{\mathsf{Frm}(\mathfrak{L}_u(\mathbb{R}), \mathcal{O}X) \longrightarrow \mathsf{Top}(X, \mathbb{R}_u)}$$

$$h \longmapsto \tilde{h}$$

$$(\tilde{h}(x) < p \text{ iff } x \in h(-, p),\ \forall\, p \in \mathbb{Q})$$

lower semicontinuous real function on L: homomorphism $h : \mathfrak{L}_l(\mathbb{R}) \to L$.

THE SEMICONTINUOUS QUASI-UNIFORMITY

$\mathcal{S} =$ all u.s.c. $h : \mathfrak{L}_u(\mathbb{R}) \to L \cong \nabla L$ s.t. $\bigvee_{p \in \mathbb{Q}} \neg h(-,p) = 1$

each h extends to a biframe morphism

$$\begin{array}{rcl} \overline{h} : \mathfrak{L}(\mathbb{R}) & \longrightarrow & \mathfrak{C}L \\ (-,p) & \longmapsto & h(-,p) \\ (p,-) & \longmapsto & \bigvee_{q>p} \neg h(-,q) \end{array}$$

$$E_{h,n} := \overline{h} \oplus \overline{h}(Q_n) = \bigvee_{0<q-p<\frac{1}{n}} \overline{h}(-,q) \oplus \overline{h}(p,-)$$

$(h \in \mathcal{S},\ n \in \mathbb{N})$

This is a subbase for a compatible quasi-uniformity $\mathcal{E}_{\mathcal{S}}$ on $\mathfrak{C}L$. Notation: $\mathcal{SC}$

$\mathcal{SC}$ is the coarsest quasi-uniformity on $\mathfrak{C}L$ for which each biframe map $h : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ is a uniform hom. $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{SC})$.

For each $a \in L$,

$$h_a^u : \mathfrak{L}_u(\mathbb{R}) \to L$$
$$(-, p) \mapsto \begin{cases} 1 \text{ if } 1 < p \\ a \text{ if } 0 < p \leq 1 \\ 0 \text{ if } p \leq 0 \end{cases}$$

Let $\mathcal{S}$ be a collection of u.s.c. $h : \mathfrak{L}_u(\mathbb{R}) \to \nabla L$ s.t. $\bigvee_{p \in \mathbb{Q}} \neg h(-, p) = 1$, containing all h_a^u.

$$E_{h_a^u, n} \circ (\nabla_a \oplus \nabla_a) \subseteq \nabla_a \oplus \nabla_a,$$
$$(\Delta_a \oplus \Delta_a) \circ E_{h_a^u, n} \subseteq \Delta_a \oplus \Delta_a$$

PROPOSITION. $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for a compatible quasi-uniformity on $\mathfrak{C}L$.

EXAMPLE. $\mathcal{S} = \{h_a^u \mid a \in L\}$

$$E_{h_a^u, n} = (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a)$$

Hence $\boxed{\mathcal{E}_{\mathcal{S}} = \mathcal{F}}$

THEOREM. Let $\mathcal{A}$ be the collection of all spectrum covers of L. Then $\mathcal{E}_{\mathcal{A}} = \mathcal{SC}$.

$$\{R_A \mid A \in \mathcal{A}\} \quad \longleftrightarrow \quad \{E_{h,m} \mid h \in \mathcal{S}, m \in \mathbb{N}\}$$

$$A = \{a_n \mid n \in \mathbb{Z}\} \subseteq L \quad \rightsquigarrow \quad h_A : \mathfrak{L}_u(\mathbb{R}) \longrightarrow L$$

$$(-, p) \longmapsto a_{[p]}$$

$$\boxed{E_{h_A,1} \subseteq R_A}$$

$$A_{h,m} = \{h(-, \tfrac{n}{2m}) \mid n \in \mathbb{Z}\} \leftarrow\!\!\rightsquigarrow \quad h \in \mathcal{S}, m \in \mathbb{N}$$

$$\boxed{R_{A_{h,m}} \subseteq E_{h,m}}$$

COROLLARY. $\mathcal{SC}$ is a transitive quasi-uniformity.

BOUNDEDNESS

u.s.c. $h : \mathfrak{L}_u(\mathbb{R}) \to L$ is bounded if $h(-, p) = 1$,

some $p \in \mathbb{Q}$.

THEOREM. For every totally bounded quasi-unif. $\mathcal{E}$ on $\mathfrak{C}L$, there exists a family $\mathcal{S}$ of bounded u.s.c. $h : \mathfrak{L}_u(\mathbb{R}) \to L$ such that $\mathcal{E} = \mathcal{E}_{\mathcal{S}}$.

$$h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}), \mathfrak{L}_l(\mathbb{R})) \longrightarrow (L_0, L_1, L_2) \in \mathsf{BiFRm}$$

$$\Updownarrow$$

TRAIL PAIR (t_1, t_2) in (L_0, L_1, L_2)

trail in L_0 [B. B.]	descending tr. in L_0 [B.B.]
$t_1 : \mathbb{Q} \to L_0$	$t_2 : \mathbb{Q} \to L_0$
$p < q \Rightarrow t_1(p) \prec t_1(q)$	$p < q \Rightarrow t_2(q) \prec t_2(p)$
$\bigvee_{p \in \mathbb{Q}} t_1(p) = 1 = \bigvee_{p \in \mathbb{Q}} t_1(p)^*$	$\bigvee_{p \in \mathbb{Q}} t_2(p) = 1 = \bigvee_{p \in \mathbb{Q}} t_2(p)^*$
continuous	continuous
$\bigvee_{p<q} t_1(p) = t_1(q)$	$\bigvee_{p>q} t_2(p) = t_2(q)$

$$t_1(\mathbb{Q}) \subseteq L_1,\ t_2(\mathbb{Q}) \subseteq L_2$$

$$p < q \Rightarrow t_1(q) \vee t_2(p) = 1$$

$$\forall\, p \in \mathbb{Q},\quad t_1(p) \wedge t_2(p) = 0$$