Point-finiteness and semicontinuity in pointfree topology

Jorge Picado

Mathematics Department - University of Coimbra

PORTUGAL

point-finite covers

[Dowker and Strauss, *Paracompact frames and closed maps*, 1975]

upper semicontinuous real functions: $f : \mathfrak{L}_u(\mathbb{R}) \to L$

[Li and Wang, *Localic Katětov-Tong insertion theorem*, 1997]

FUNCTIONAL QUASI-UNIFORMITIES IN POINTFREE TOPOLOGY

[M.J. Ferreira, J. P., *On the construction of quasi-uniform structures in pointfree topology*, 2004]

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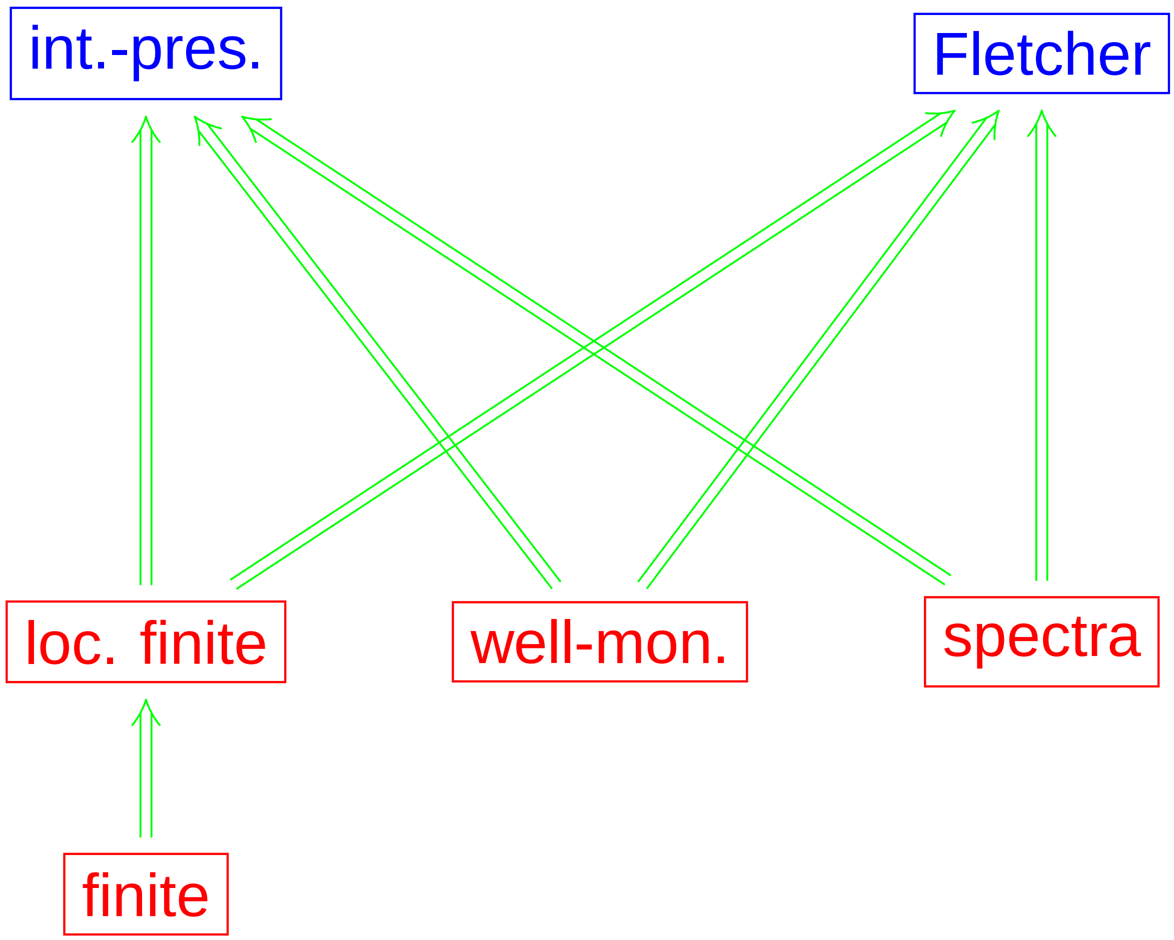
(also useful: characterizations of normality, paracompactness ...)

COVERS IN FRM



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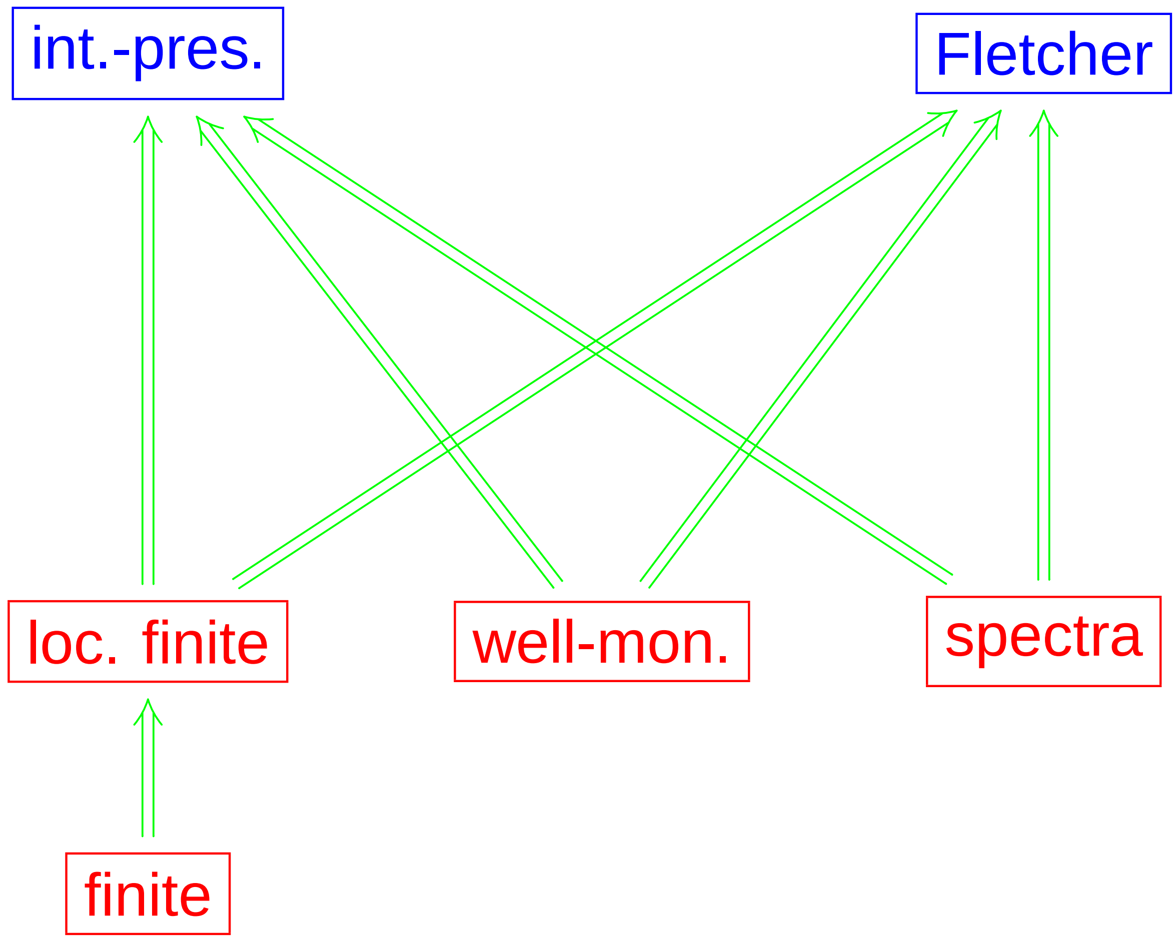

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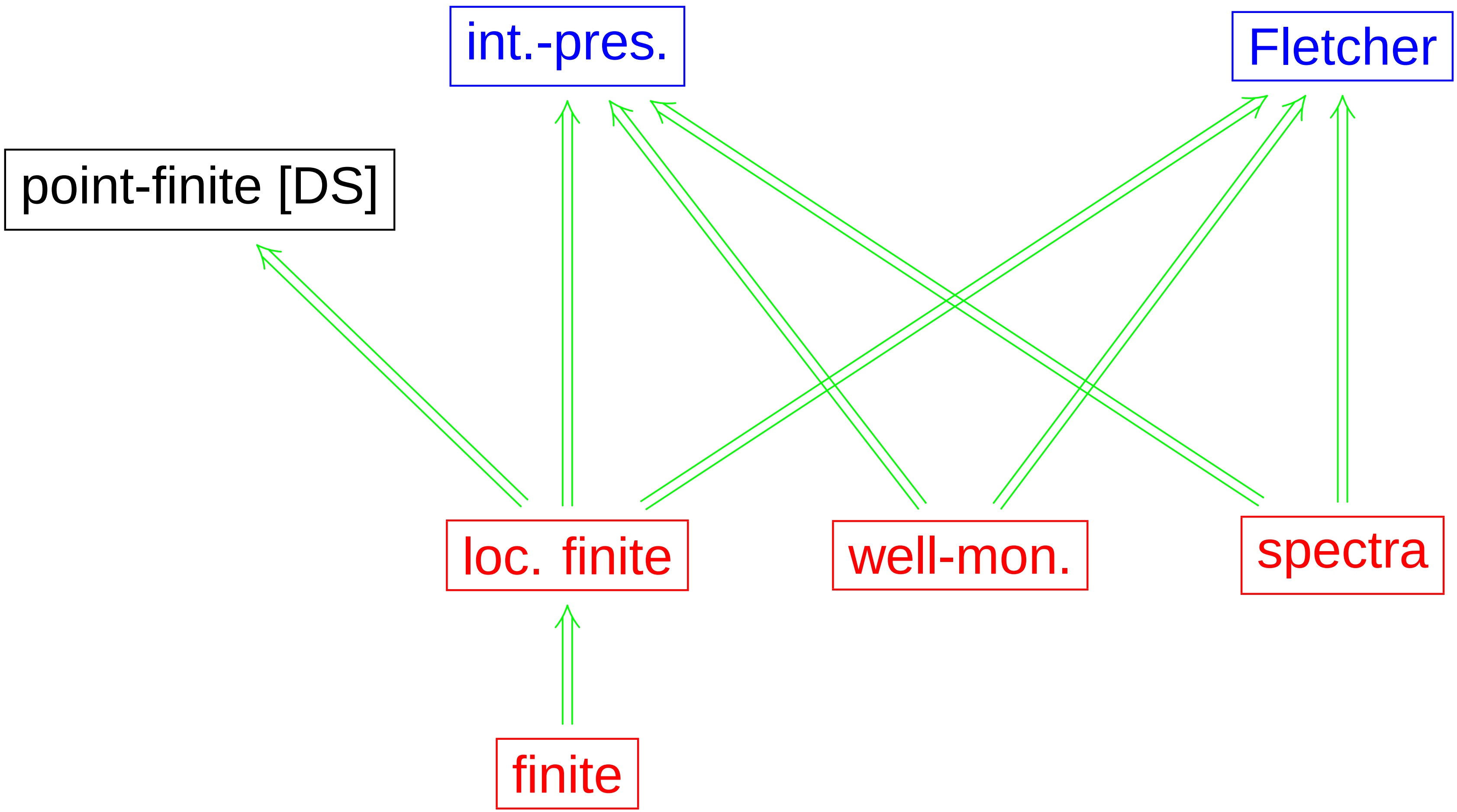

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$\bigcap_F (S \cup A_F) = S \cup \bigcap_F A_F$

[Lefschetz, 1942]

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every point-finite cover is shrinkable [Lefschetz]

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$$\boxed{\Leftrightarrow \forall C \in \mathcal{C} \ \exists \text{ finite } I_C : C \wedge \bigvee_J A_j = C \wedge \bigvee_{J \cap I_C} A_j \text{ for all } J \subseteq I.}$$

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Now we have

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$$\text{int}(\bigwedge_J A_i) = \bigwedge_J \text{int} A_i$$
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Proof: $\displaystyle\bigwedge_{F\in\mathcal{P}_f(\mathcal{A})} A_F \overset{?}{=} 0$

$$\bigwedge_{F\in\mathcal{P}_f(\mathcal{A})} A_F = \bigvee_{C\in\mathcal{C}} C \wedge \bigwedge_{F\in\mathcal{P}_f(\mathcal{A})} A_F = \bigvee_{C\in\mathcal{C}} (C \wedge \bigwedge_{F\in\mathcal{P}_f(\mathcal{A})} A_F)$$

(open families are distributive)

and, for every $C \in \mathcal{C}$, $C \wedge \displaystyle\bigwedge_{F\in\mathcal{P}_f(\mathcal{A})} A_F \leq C \wedge A_{I_C} = 0.$

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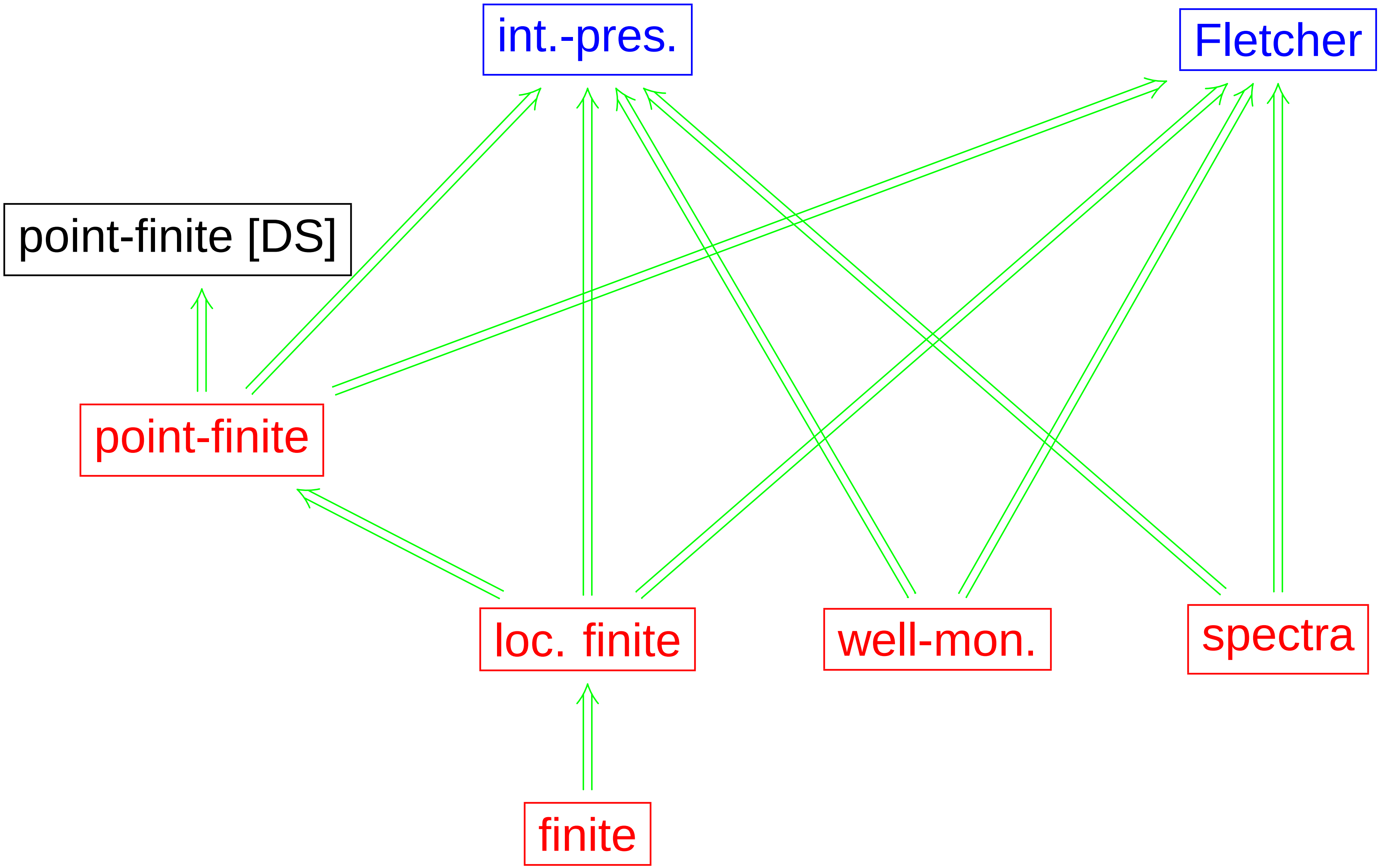

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PROPOSITION [Lefschetz's Theorem]. A frame L is normal iff for every p.-f. cover $\{a_i\}_I$ of L there exists a cover $\{b_i\}_I$ of L such that $a_i \vee b_i^* = 1$ for all i. (i.e. iff each point-finite cover is shrinkable.)

SEMICONTINUITY

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$\mathfrak{L}_u(\mathbb{R}) = Frm\Big\langle (-,p)(p \in \mathbb{Q}) \mid$ (1),(2),(3) $\Big\rangle$ upper frame of reals

(1) $p \leq q \Rightarrow (-,p) \leq (-,q)$

(2) $\displaystyle\bigvee_{q<p} (-,q) = (-,p)$

(3) $\displaystyle\bigvee_{p \in \mathbb{Q}} (-,p) = 1.$

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but $$\mathsf{FRM}(\mathfrak{L}_u(\mathbb{R}), \mathcal{O}X) \cong \mathsf{TOP}(X, \Sigma\mathfrak{L}_u(\mathbb{R}))$$

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$\mathbb{R}_u := (\mathbb{R}, \mathcal{T}_u)$ is not sober: $\Sigma L \ncong \mathbb{R}_u$

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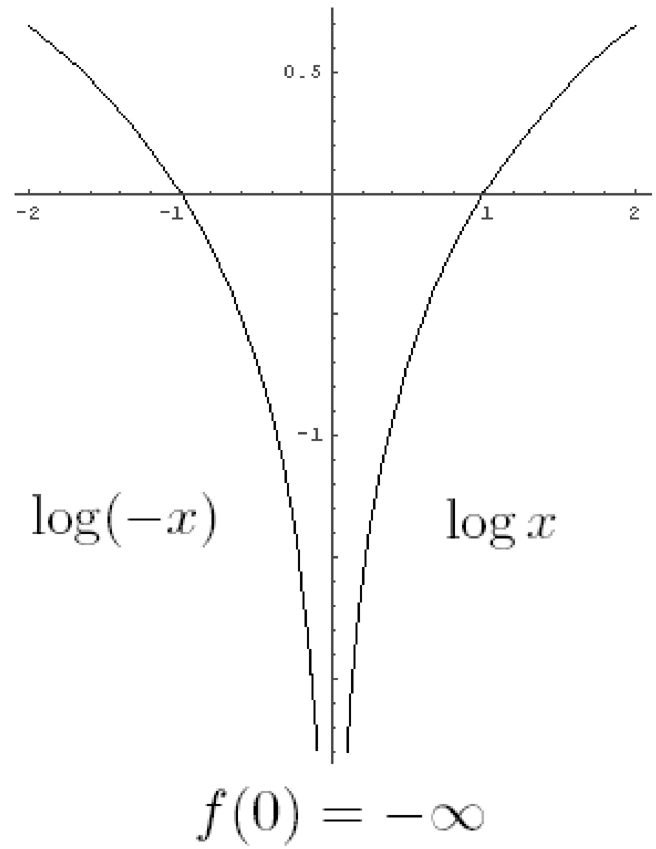

EXAMPLES

$$\mathsf{FRM}(\mathfrak{L}_u(\mathbb{R}), \mathcal{O}X) \cong \mathsf{TOP}(X, \Sigma\mathfrak{L}_u(\mathbb{R}))$$

$$\begin{aligned} h : \mathfrak{L}_u(\mathbb{R}) &\to \mathcal{O}X \\ (-,\alpha) &\mapsto X \end{aligned} \qquad \longmapsto \qquad \begin{aligned} f : X &\to \mathbb{R} \cup \{-\infty\} \\ x &\mapsto -\infty \end{aligned}$$

HOW TO DESCRIBE UPPER SEMICONTINUITY ALGEBRAICALLY?

$$f : X \to \mathbb{R} \text{ is u.s.c} \Leftrightarrow f : \underbrace{(X, \mathcal{O}X, \mathcal{C}X)}_{Sk(X)} \to (\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l) \in \text{BiTOP}$$

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$\Updownarrow$

$(\mathfrak{C}(\mathcal{O}X), \nabla(\mathcal{O}X), \Delta(\mathcal{O}X))$

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$$\mathsf{BiFRM}(\mathfrak{L}(\mathbb{R}), \mathfrak{C}L) \cong \left\{ f : \mathfrak{L}_u(\mathbb{R}) \to L \in \mathsf{FRM} \mid \bigvee_{p \in \mathbb{Q}} \Delta_{f(-,p)} = \mathbf{1} \right\}$$

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- $\mathcal{SC}(L)$ is induced by all u.s.c. real functions on L
- Katětov-Tong insertion theorem

CONCLUSION

u.s.c. real function on L: $f : \mathfrak{L}_u(\mathbb{R}) \to L \in$ FRM s.t. $\bigvee_{p \in \mathbb{Q}} \Delta_{f(-,p)} = \mathbf{1}$.

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u.s.c. real function on L: $f : \mathfrak{L}_u(\mathbb{R}) \to L \in$ FRM s.t. $\bigvee_{p \in \mathbb{Q}} \Delta_{f(-,p)} = \mathbf{1}$.

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PROPOSITION [Localic Katětov-Tong Insertion Theorem].

L is normal iff for every usc real function $f : \mathfrak{L}_u(\mathbb{R}) \to L$ and every lsc real function $g : \mathfrak{L}_l(\mathbb{R}) \to L$ with $f \leq g$ there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.