Localic real-valued functions: a general setting

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(joint work with J. Gutiérrez Garcia and T. Kubiak)

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Similarly, for $\text{lsc}(L)$ and $\text{LSC}(L)$ ...

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\begin{tikzcd}
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- By considering the algebraic operations of the function algebra $\mathcal{R}(\mathfrak{S}L)$, we obtain, in particular, a way of defining the sum of an upper semicontinuous functions with a lower semicontinuous one, etc.

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Then: (1) $H^\circ \in \overline{\mathsf{LSC}}(L)$.

(2) $H \in \mathsf{F}(L)$, $\bigvee_{p\in\mathbb{Q}} \overline{H(p,-)} = 1 \Rightarrow H^\circ \in \mathsf{LSC}(L)$.

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INSERTION THEOREMS

KATĚTOV-TONG INSERTION THEOREM:

The following conditions on a frame L are equivalent:

(1) L is normal.

(2) For every $F \in \mathsf{USC}(L)$ and every $G \in \mathsf{LSC}(L)$ with $F \leq G$, there exists $H \in \mathsf{C}(L)$ such that $F \leq H \leq G$.

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UL(L)$= \{(F, G) \in$ USC$(L) \times$ LSC$(L) : F \leq G\}$

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also versions for monotone normality, perfect normality, ...

SEMICONTINUITY IN POINTFREE TOPOLOGY

MOTIVATION:

J. Gutiérrez García and J. P.

On the algebraic representation of semicontinuity

J. PURE APPL. ALGEBRA 210 (2007) 299-306.

- J. Gutiérrez García, T. Kubiak and J. P.

Monotone insertion and monotone extension of frame homomorph.

To appear in JOURNAL OF PURE AND APPLIED ALGEBRA.

Lower and upper regularizations of frame semicont. real functions

SUBMITTED.

Pointfree forms of Michael and Dowker insertion theorems

IN PREPARATION.