Strict insertion of continuous real functions in pointfree topology

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MOTIVATION

Strict Insertion Theorem (DOWKER)

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≪(...) what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.≫ R. BALL & J. WALTERS-WAYLAND

[C- and C^*-quotients in pointfree topology, *Dissert. Math.* 412 (2002)]

POINTFREE SPACES

locales (or frames)

[categorical topology, topos theory, logic, ...]

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- Complete lattices L satisfying

$$\boxed{a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)}$$

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- $$\begin{array}{ccc} L & & L \\ \Big\downarrow f & & \Big\uparrow \\ M & & M \end{array}$$

preserves $\bigvee$ (incl. the bottom 0)

$\wedge$ (incl. the top 1)

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- $$L \xrightarrow{f} M \qquad\qquad L \longleftarrow M$$

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$\boxed{\text{Loc}}$

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  (spatial locales)

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$$\begin{tikzcd} (X, \mathfrak{O}X) \arrow[d, "f"'] \\ (Y, \mathfrak{O}Y) \end{tikzcd}$$

$\mathfrak{O}X$

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$$

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- complete Boolean algebras (spatial=atomic)

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- finite distributive lattices

$\vdots$

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MOTIVATES: $\boxed{F(L) := \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))}$

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INTERIOR

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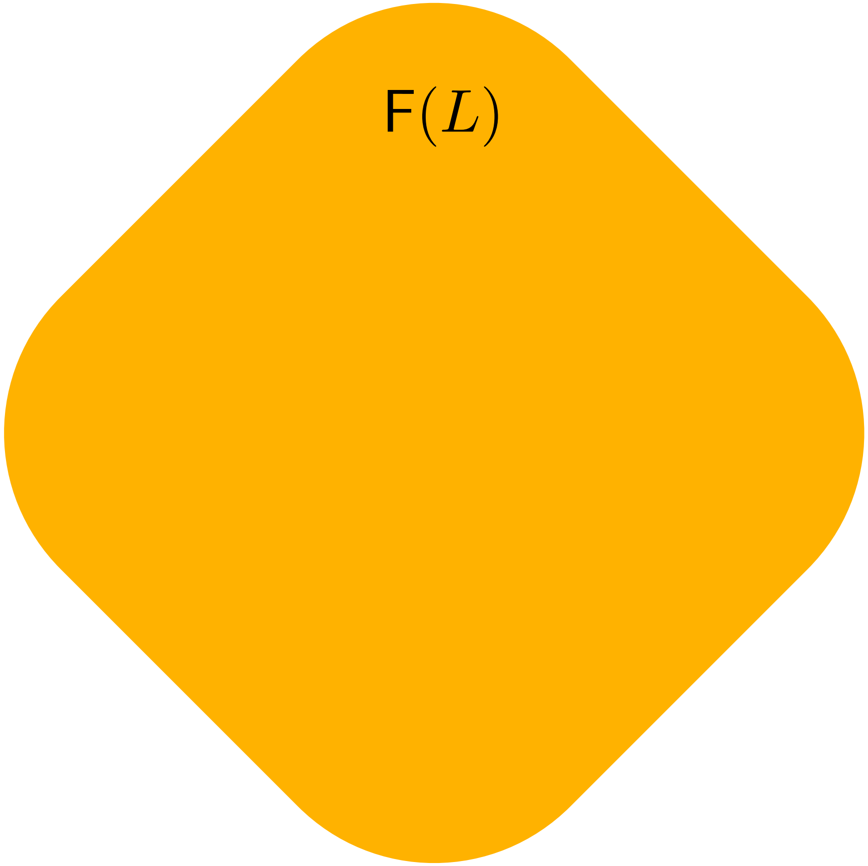
BERNHARD BANASCHEWSKI, **The real numbers in Pointfree Topology,** ***Textos de Matemática*, Vol. 12, Univ. Coimbra, 1998.**

REAL FUNCTIONS ON L

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$



SEMICONTINUITY AND CONTINUITY

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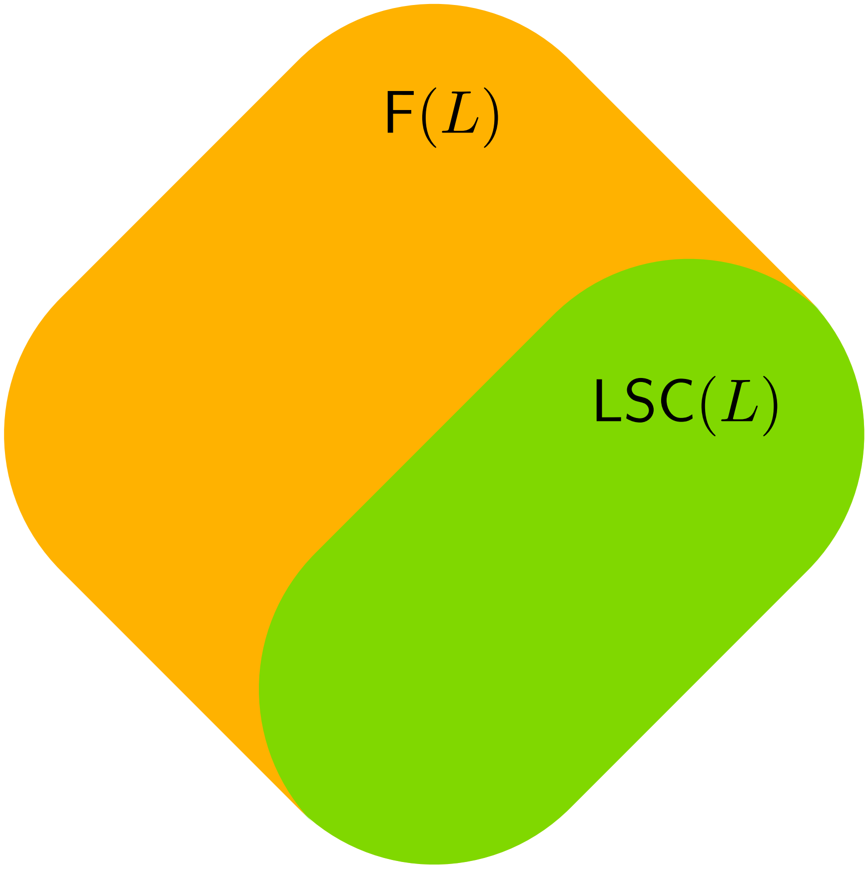

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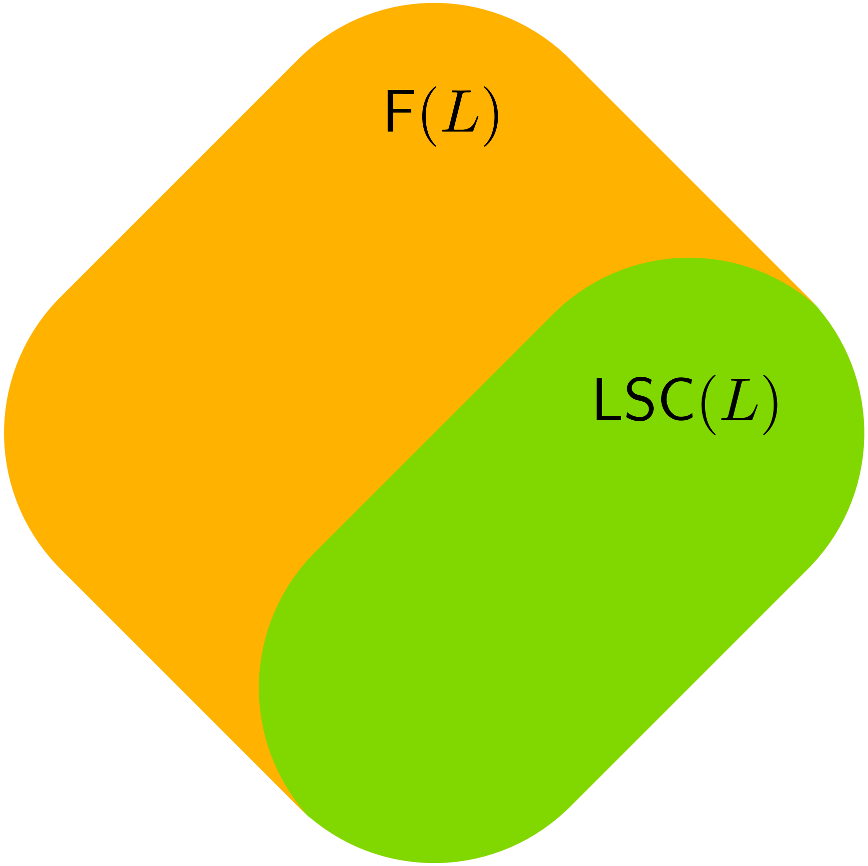

$f(p, \text{—}) \in \mathfrak{c}L$

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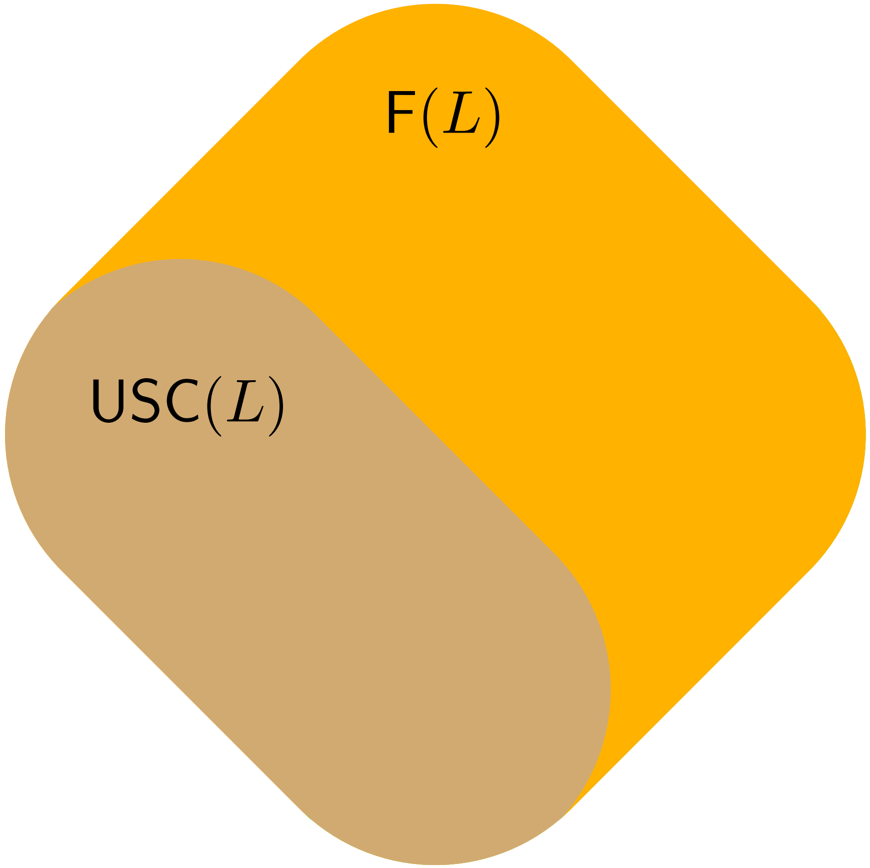

$f(p,-) \in \mathfrak{c}L$

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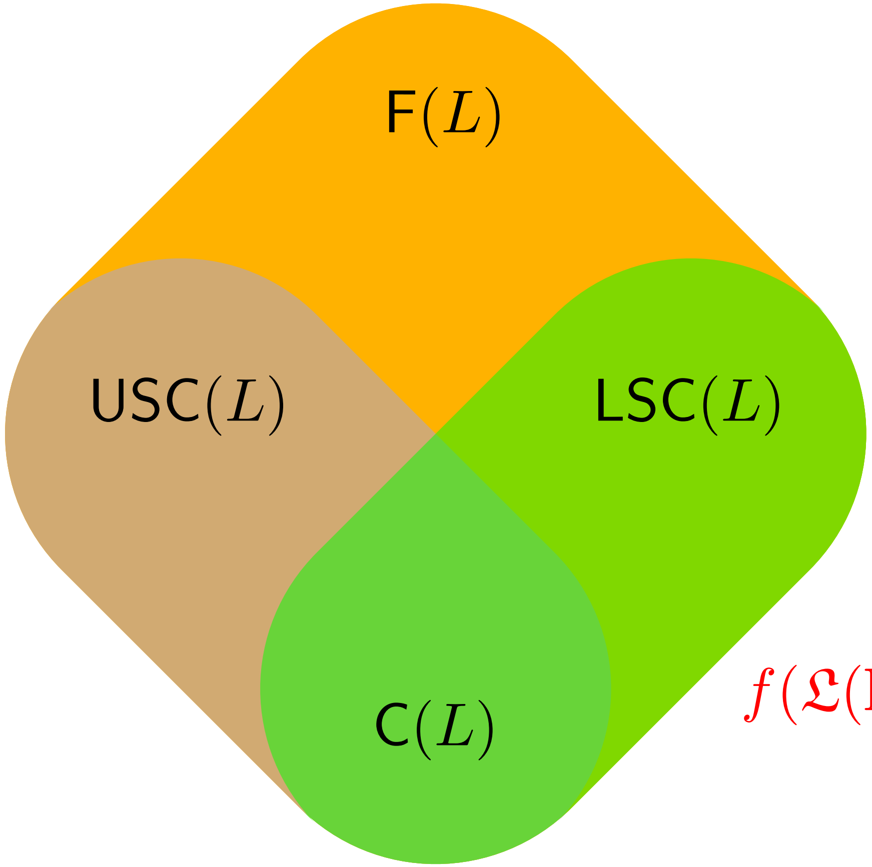

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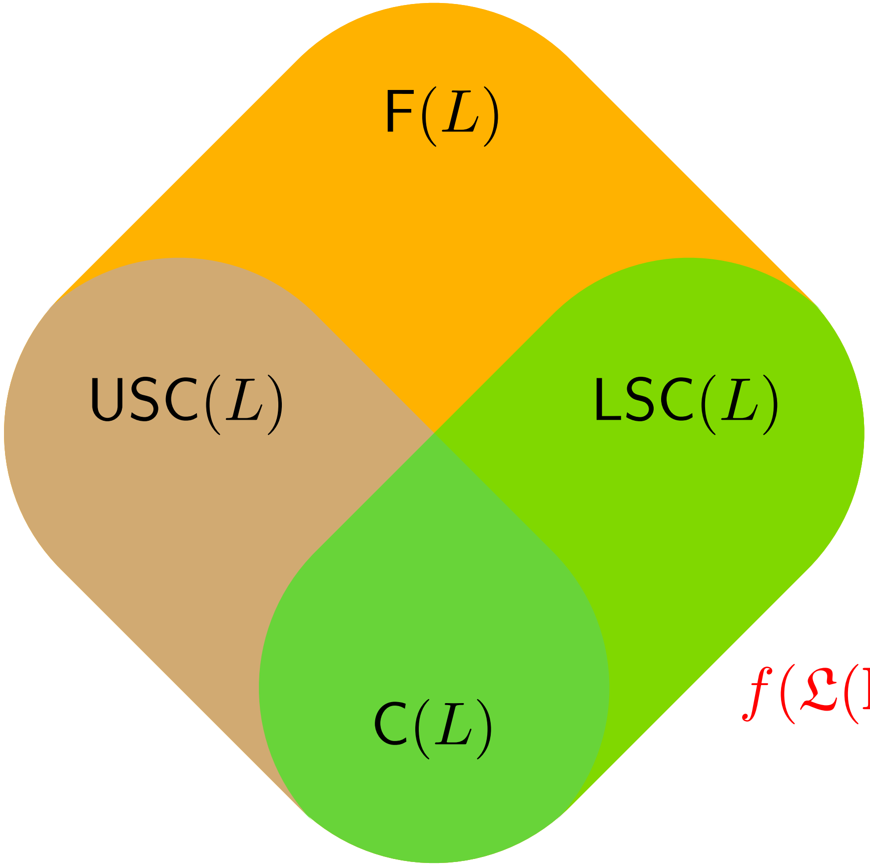

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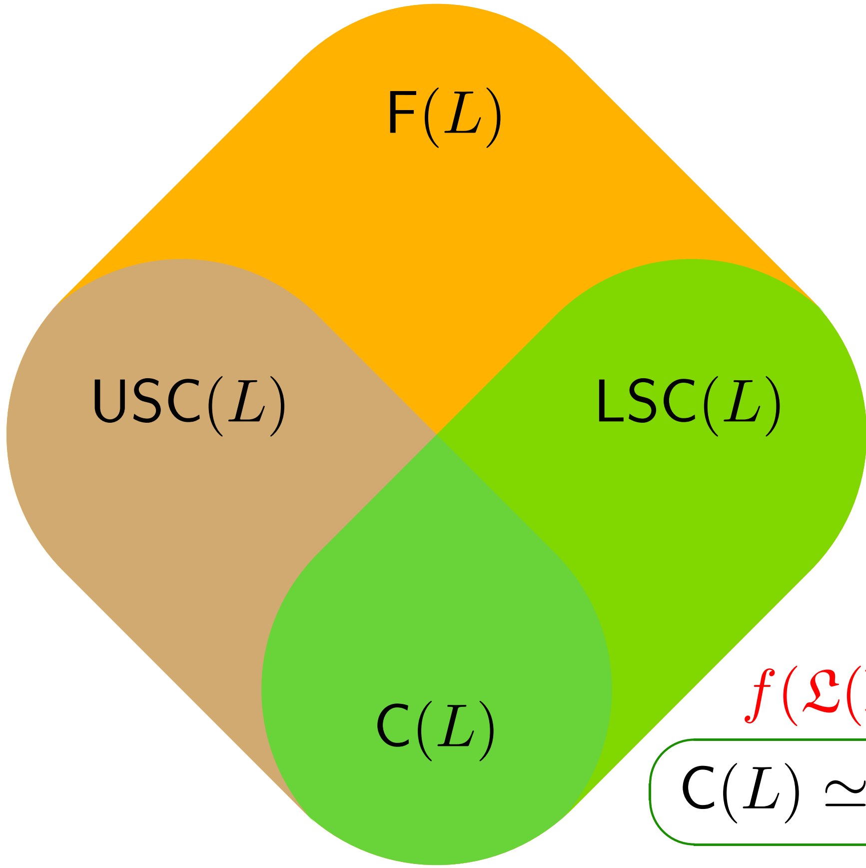

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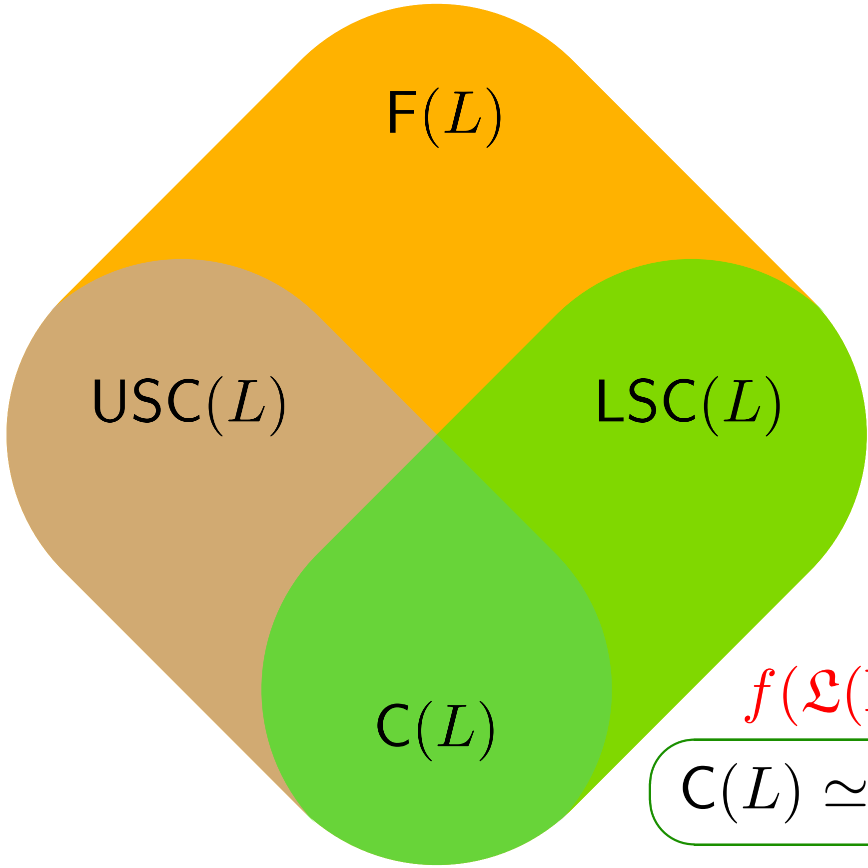

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$$f \leq g \;\equiv\; f(p, -) \leq g(p, -),\ \forall p \in \mathbb{Q}$$

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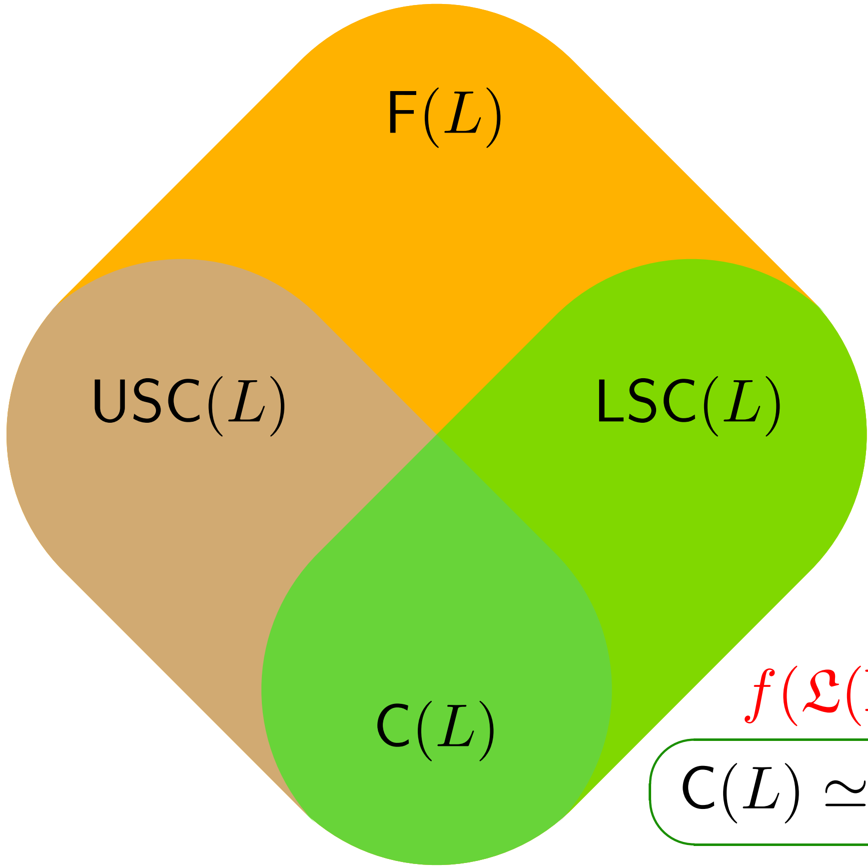

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J. Gutiérrez García, T. Kubiak & J. Picado
Localic real functions: a general setting, *J. Pure Appl. Algebra* 213 (2009) 1064-1074

DOWKER'S STRICT INSERTION

[C. H. Dowker, *Canad. J. Math.* 3 (1951) 219-224]

THEOREM. A topological space X is normal and countably paracompact

iff

for every $f, g : X \to \mathbb{R}$ with $f < g$, f usc and g lsc,

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- $f < g \;\Rightarrow\; f \le g$.
- $f_1 \le f_2,\, g_1 \le g_2 \;\Rightarrow\; \iota(f_2, g_1) \le \iota(f_1, g_2)$.
- $f < g$ iff $\mathbf{0} < g - f$.
- $f < g$ iff $\lambda \cdot f < \lambda \cdot g$ for every $0 < \lambda \in \mathbb{Q}$.

ALGEBRA IN $\mathsf{F}(L)$

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$\mathsf{F}(L)$	$\vee, \wedge$					
$\mathsf{LSC}(L)$						
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explicit formulas for $(f \diamond g)(p, —)$ and $(f \diamond g)(—, q)$

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explicit formulas for $(f \diamond g)(p, —)$ and $(f \diamond g)(—, q)$

$\mathsf{F}(L)$	$\vee, \wedge$	$-f$	$\lambda \cdot f$ ($\lambda > 0$)	$f + g$	$f \cdot g$	ℓ-ring
$\mathsf{LSC}(L)$	sublat.					
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J. GUTIÉRREZ GARCÍA & J. PICADO
Rings of real functions in Pointfree Topology, *submitted*

CONSTRUCTING REAL FUNCTIONS: SCALES

$$\mathcal{S} = \{S_p \mid p \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$$

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EXAMPLE: $f \cdot g$, $f, g \geq \mathbf{0}$

$f \cdot g \in \mathsf{F}(L)$ is the function generated by the scale

$$\boxed{S_p := \begin{cases} 1 & \text{if } p < 0 \\ \displaystyle\bigvee_{r>0} f(r, \text{—}) \wedge g(\frac{p}{r}, \text{—}) & \text{if } p \geq 0 \end{cases}}$$

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$$(f \cdot g)(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0 \\ \bigvee_{r>0} f(r,\text{—}) \wedge g(\frac{p}{r},\text{—}) & \text{if } q > 0 \end{cases}$$

THE THEOREM

THEOREM.
A locale L is normal and countably paracompact iff for every $f, g \in \mathsf{F}(L)$ with $f < g$, f usc and g lsc, there is an $h \in \mathsf{C}(L)$ such that $f < h < g$.

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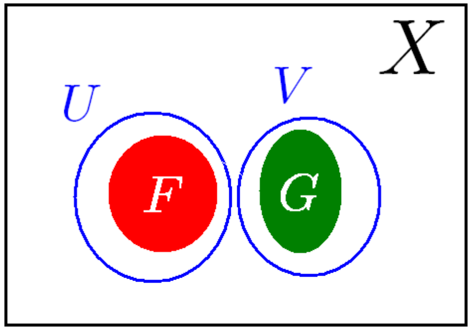

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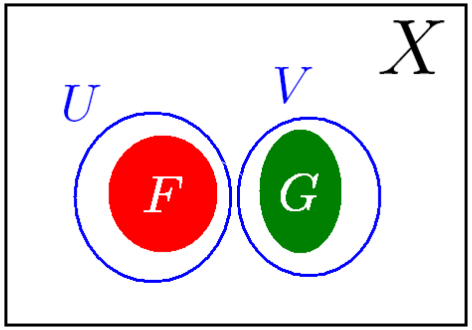

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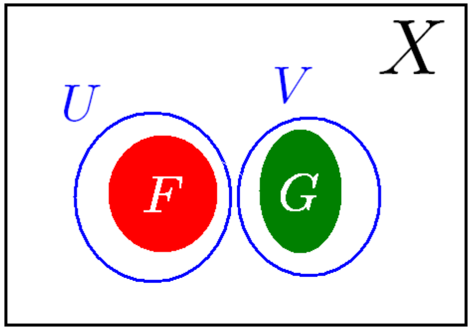

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COUNTABLE PARACOMPACTNESS:

every countable (open) cover has a locally finite (open) refinement.

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$$\Rightarrow f + \tfrac{k}{2} \leq \underbrace{h}_{\mathsf{C}(L)} \leq g - \tfrac{k}{2}$$ (by Katětov-Tong).

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X normal and countably paracompact

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scale in $\mathfrak{O}X$

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usc

PARTICULAR CASE: $L = \mathfrak{O}X$ X normal and countably paracompact

$$\widetilde{f} < \widetilde{g} \iff \iota(\widetilde{f}, \widetilde{g}) = 1 \iff \bigvee_{p \in \mathbb{Q}} \Big(\mathfrak{c}(f^{-1}(]-\infty, p[)) \wedge \mathfrak{c}(g^{-1}(]p, +\infty[)) \Big) = 1$$

$$\iff \mathfrak{c}\Big(\bigcup_{p \in \mathbb{Q}} \big(f^{-1}(]-\infty, q[) \cap g^{-1}(]p, +\infty[) \big) \Big) = 1$$

$$\iff \bigcup_{p \in \mathbb{Q}} \big(f^{-1}(]-\infty, q[) \cap g^{-1}(]p, +\infty[) \big) = X$$

$$\iff f(x) < g(x) \text{ for every } x \in X.$$

PARTICULAR CASE: $L = \mathfrak{O}X$

X normal and countably paracompact

$$\begin{aligned}
\widetilde{f} < \widetilde{g} \quad &\Leftrightarrow \quad \iota(\widetilde{f}, \widetilde{g}) = 1 \;\Leftrightarrow\; \bigvee_{p \in \mathbb{Q}} \big(\mathfrak{c}(f^{-1}(]-\infty, p[)) \wedge \mathfrak{c}(g^{-1}(]p, +\infty[))\big) = 1 \\
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&\Leftrightarrow \quad f(x) < g(x) \text{ for every } x \in X.
\end{aligned}$$

Hence there is $\widetilde{h} \in \mathsf{C}(\mathfrak{O}X)$ such that

$$\widetilde{f} < \widetilde{h} < \widetilde{g}$$

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$$\Updownarrow$$

$$h : X \to \mathbb{R}$$

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Hence there is $\widetilde{h} \in \mathsf{C}(\mathfrak{O}X)$ such that

$$\widetilde{f} < \widetilde{h} < \widetilde{g}$$

$$\Updownarrow$$

$h : X \to \mathbb{R}$ defined by

$$h(x) \in]p, q[\ \text{ iff } \ x \in \widetilde{h}(p, q).$$

MICHAEL'S STRICT INSERTION

[*Ann. Math.* 63 (1956)]

TFAE for a space X: (i) X is perfectly normal

(ii) $\underbrace{f}_{\text{USC}} \leq \underbrace{g}_{\text{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathsf{C}(X) : f \leq h \leq g$ and

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