The quantale of Galois connections

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PORTUGAL

GALOIS CONNECTIONS

[Birkhoff 1940, Ore 1944]

Posets A, B

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

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$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

- $f \in Ant(A, B)$ and $g \in Ant(B, A)$
- $\forall a \in A \; a \leq gf(a), \;\; \forall b \in B \; b \leq fg(b)$

$$\Rightarrow f(\bigvee S) = \bigwedge f(S), \; g(\bigvee S) = \bigwedge g(S).$$

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$f \in Gal(A,B)$ if $\exists\, f^+ : B \to A$ (necessarily unique) such that (f, f^+) is a Galois connection.

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$\Leftrightarrow f(\bigvee S) = \bigwedge f(S)$.

(when $A, B \in CLat$)

HOW TO COMPOSE GALOIS MAPS?

$(Gal(A, A^d), \circ)$ $\underbrace{\text{complete semigroup}}_{QUANTALE}$ of residuated maps

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$(Gal(A, A), ?)$ Galois maps

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HOW TO COMPOSE?

CATEGORICAL MOTIVATION

[Freyd, Scedrov 1990]

In any category with finite limits and images:

$$\begin{tikzcd} R_1 \arrow[r, tail, "m_1"] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \end{tikzcd}$$

$$\begin{tikzcd} R_2 \arrow[r, tail, "m_2"] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \end{tikzcd}$$

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The composition of the relations m_1 and m_2 is the image of f:

$$f(R_1 \circ R_2) \rightarrowtail^{m} X \times X$$

APPLICATION TO THE CATEGORY OF LOCALES

[Picado, Ferreira, *Quaest. Math.*, to appear]

Loc $\boxed{\text{POINTFREE TOPOLOGY}}$

Frm^d

frames: complete Heyting algebras
morphisms: preserve $\bigvee$, $\wedge$

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$f \in Gal(L, L)$ $\longleftrightarrow$ special binary relations on L

$g \circ f \in Gal(L, L)$ $\longleftarrow$

BINARY (TENSOR) PRODUCTS

Category of complete $\vee$-semilattices

The *tensor product* $A \otimes B$ of A and B:

G-ideals of $A \times B$

- $\downarrow R = R$
- $\{x\} \times U_2 \subseteq R \Rightarrow (x, \bigvee U_2) \in R$ (left)
- $U_1 \times \{y\} \subseteq R \Rightarrow (\bigvee U_1, y) \in R$ (right)

$$a \otimes b := \downarrow (a, b) \cup \downarrow (1, 0) \cup \downarrow (0, 1)$$

BINARY (TENSOR) PRODUCTS

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THE GALOIS COMPOSITION

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$$(g \circ f)(a) := \bigvee \{c \in C \mid (a, c) \in R_{f,g}\}.$$

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TOWARDS THE QUANTALE OF GALOIS CONNECTIONS

[*Alg. Univ.*, 2004]

STEP 1: THE QUANTALE OF ANTITONE MAPS

$f \in Ant(A, B), g \in Ant(B, C)$ $\qquad$ $A \xrightarrow{g \cdot f} C$ $\qquad$ $g \cdot f \in Ant(A, C)$

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- associative
- $g \cdot (\bigvee f_i) = \bigvee (g \cdot f_i)$
- $(\bigvee g_i) \cdot f = \bigvee (g_i \cdot f)$

on frames

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on frames

THEOREM: If L is a frame then $(Ant(L,L), \cdot)$ is a quantale

TOWARDS THE QUANTALE OF GALOIS CONNECTIONS [*Alg. Univ.*, 2004]

STEP 2: $Gal(L, L)$ IS A QUANTIC QUOTIENT of $Ant(L, L)$

$$f \in Ant(L, L)$$

$$j_0(f)(a) := \bigvee\{b \in L \mid \exists S \subseteq L : \bigvee S = a \text{ and } b \leq f(s)\ \forall s \in S\}.$$

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If L is a frame then $j_0(f) \in Ant(A, B)$

j_0 is a quantic prenucleus on $Ant(L, L)$

STEP 2: $Gal(L,L)$ IS A QUANTIC QUOTIENT of $Ant(L,L)$

Quantale $(Q,\cdot)$

quantic prenucleus $j_0 : Q \to Q$ $\begin{cases} \text{preclosure operator} \\ j_0(x)\cdot y \leq j_0(x\cdot y) \\ x\cdot j_0(y) \leq j_0(x\cdot y) \end{cases}$

TOWARDS THE QUANTALE OF GALOIS CONNECTIONS

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A GENERAL PROCEDURE ON QUANTALES

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$$j(x) := \wedge\{y \in Q_{j_0} \mid x \le y\} \qquad Q_{j_0} := Fix(j_0)$$

quantic nucleus $j : Q \to Q$ $\begin{cases} \text{closure operator} \\ j(x)\cdot j(y) \le j(x\cdot y) \end{cases}$

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For $a \cdot_j b := j(a\cdot b)$

$(Q_j, \cdot_j)$ is a quantale

$Q \xrightarrow{j} Q_j$

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$$j_0(f)(a) := \bigvee\{b \in L \mid \exists S \subseteq L : \bigvee S = a \text{ and } b \leq f(s)\ \forall s \in S\}$$

$$\Big\downarrow{\scriptstyle Fix(j_0)=Gal(L,L)}$$

$$j(f) = \bigwedge\{g \in Gal(L, L) \mid f \leq g\}$$

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$$\Big\downarrow{\scriptstyle Fix(j_0)=Gal(L,L)}$$

$$j(f) = \bigwedge\{g \in Gal(L,L) \mid f \leq g\}$$

Then

- $Fix(j) = Gal(L,L)$
- $g \cdot_j f = j(g \cdot f) = g \circ f$

THEOREM: $(Gal(L,L), \circ)$ is a quantale

RELATION QUANTALES

[Erné, Picado, *preprint*]

A, B complete lattices

$$Gal(A,B) \underset{G_1}{\overset{F_1}{\rightleftarrows}} A \otimes B \qquad (\simeq)$$

G-ideals

[Shmuely, 1974]

RELATION QUANTALES

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A, B complete lattices

$$
\begin{array}{ccc}
f & \xmapsto{F_1} & \{(a,b) \mid f(a) \geq b\} \\[2em]
\bigvee\{b \mid (a,b) \in R\} & \xleftarrow[G_1]{} & R
\end{array}
$$

RELATION QUANTALES

[Erné, Picado, *preprint*]

A, B complete lattices

$$
\begin{tikzcd}
Ant(A,B) \arrow[r, shift left, "F_1"] & \mathcal{D}(A \times B) \arrow[l, shift left, "G_1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
Gal(A,B) \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\simeq"] & A \otimes B \arrow[l, shift left, "G_1"]
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RELATION QUANTALES

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A, B complete lattices

$$\begin{tikzcd} Ant(A,B) \arrow[r, shift left, "F_1"] \arrow[d, equal] & \mathcal{D}(A\times B) \arrow[l, shift left, "G_1"] \\ Ant(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A\otimes B)^l \arrow[l, shift left] \arrow[u, "\subseteq"'] \end{tikzcd}$$

left G-ideals

$$\begin{tikzcd} Gal(A,B) \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\simeq"] & A\otimes B \arrow[l, shift left, "G_1"] \end{tikzcd}$$

G-ideals

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A, B complete lattices

$$\begin{tikzcd}
Ant(A,B) \arrow[r, shift left, "F_1"] & \mathcal{D}(A \times B) \arrow[l, shift left, "G_1"] \\
Ant(A,B) \arrow[u, equal] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes B)^l \arrow[l, shift left] \arrow[u, "\subseteq"'] \\
\mathcal{Z}\text{-}Ant(A,B) \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes B)^l_{\mathcal{Z}} \arrow[l, shift left] \arrow[u, "\subseteq"'] \\
Gal(A,B) \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & A \otimes B \arrow[l, shift left] \arrow[u, "\subseteq"']
\end{tikzcd}$$

left G-ideals

$f(\bigvee S) = \bigwedge f(S)$

$\forall S \in \mathcal{Z}$

$\mathcal{Z}$-right left G-ideals

G-ideals

RELATION QUANTALES

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A, B complete lattices

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\begin{tikzcd}
Ant(A,B) \arrow[r, shift left, "F_1"] \arrow[d, equal] & \mathcal{D}(A \times B) \arrow[l, shift left, "G_1"] \\
0\text{-}Ant(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"'] & (A \otimes B)^l_0 \arrow[l, shift left] \arrow[u, "\subseteq"]
\end{tikzcd}
$$

$\mathcal{Z} = \{\emptyset\}$ — 0-right left G-ideals

$$
\begin{tikzcd}
\mathcal{Z}\text{-}Ant(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"'] & (A \otimes B)^l_{\mathcal{Z}} \arrow[l, shift left] \\
Gal(A,B) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"'] & A \otimes B \arrow[l, shift left] \arrow[u, "\subseteq"]
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$f(\bigvee S) = \bigwedge f(S)$

$\forall S \in \mathcal{Z}$

$\mathcal{Z}$-right left G-ideals

G-ideals

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A, B complete lattices

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\begin{tikzcd}[column sep=large]
 & Ant(A,B) \arrow[r, shift left, "F_1"] \arrow[d, equal] & \mathcal{D}(A\times B) \arrow[l, shift left, "G_1"] & \\
\mathcal{Z}=\{\emptyset\} & 0\text{-}Ant(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A\otimes B)^l_0 \arrow[l, shift left] \arrow[u, "\subseteq"'] & 0\text{-right left G-ideals} \\
f(\bigvee S)=\bigwedge f(S) \ \ \forall S\in\mathcal{Z} & \mathcal{Z}\text{-}Ant(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] \arrow[u, "\subseteq"] & (A\otimes B)^l_{\mathcal{Z}} \arrow[l, shift left] \arrow[u, "\subseteq"', "\text{if } \emptyset\in\mathcal{Z}"] & \mathcal{Z}\text{-right left G-ideals} \\
 & Gal(A,B) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] \arrow[u, "\subseteq"] & A\otimes B \arrow[l, shift left] \arrow[u, "\subseteq"'] & \text{G-ideals}
\end{tikzcd}
$$

RELATION QUANTALES

[Erné, Picado, *preprint*]

A, B complete lattices

$$\boxed{A_0 = A - \{0\}, \quad B_0 = B - \{0\}}$$

$\mathcal{L}(A_0 \times B_0)$: (truncated) left G-ideals

$\mathcal{T}(A_0 \times B_0)$: (truncated) G-ideals

$\mathcal{Z}$-$\mathcal{T}(A_0 \times B_0)$: (truncated) $\mathcal{Z}$-right left G-ideals

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\mathcal{Z}\text{-}Ant(A,B) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes B)^l_{\mathcal{Z}} \arrow[l, shift left] \arrow[u, "\subseteq", "\text{if } \emptyset \in \mathcal{Z}"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & \mathcal{Z}\text{-}\mathcal{T}(A_0 \times B_0) \arrow[l, shift left] \arrow[u, "\subseteq"] \\
Gal(A,B) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & A \otimes B \arrow[l, shift left] \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & \mathcal{T}(A_0 \times B_0) \arrow[l, shift left] \arrow[u, "\subseteq"]
\end{tikzcd}
$$

RELATION QUANTALES

[Erné, Picado, *preprint*]

$R, S \subseteq A_0 \times A_0$

$$\boxed{R \cdot S := \{(a,c) \in A_0 \times A_0 \mid \exists\, b \in A_0 : (a,b) \in R \text{ and } (b,c) \in S\}}$$

(the usual product relation)

$$\boxed{(\mathcal{D}(A_0 \times A_0), \cdot) \text{ is a quantale}}$$

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$$L(R) := \downarrow \{(a, \bigvee U_2) \mid \emptyset \neq \{a\} \times U_2 \subseteq R\}$$

closure op.

$$T(R) := \downarrow\{(a, \bigvee U_2) \mid \emptyset \neq \{a\} \times U_2 \subseteq R\} \cup \downarrow\{(\bigvee U_1, b) \mid \emptyset \neq U_1 \times \{b\} \subseteq R\}$$

preclosure op.

$$\overline{T}(R) := \bigcap\{S \in \mathcal{T}(A_0 \times B_0) \mid R \subseteq S\}$$

closure op.

 [Erné, Picado, *preprint*]

THEOREM:

The following conditions on a complete lattice A are equivalent:

(1) A is pseudocomplemented.

(2) $\forall U \subseteq A \; (0 \neq a \leq \bigvee U \Rightarrow \exists\, b \neq 0 : \, b \in \, \downarrow\! a \cap \downarrow\! U)$.

$\vdots$

(8) L is a quantic nucleus on the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

(9) $\overline{T}$ is a quantic nucleus on the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

(10) $\mathcal{L}(A_0 \times A_0)$ is a quantic quotient of the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

(11) $\mathcal{T}(A_0 \times A_0)$ is a quantic quotient of the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

RELATION QUANTALES

[Erné, Picado, *preprint*]

THEOREM:

The following conditions on a complete lattice A are equivalent:

(1) A is $\mathcal{Z}$-pseudocomplemented.

(2) $\forall\, U \in \mathcal{Z}\ (0 \neq a \leq \bigvee U \Rightarrow \exists\, b \neq 0 :\ b \in\ \downarrow\! a \cap \downarrow\! U)$.

$\vdots$

(6) $\overline{T}^{\mathcal{Z}}$ is a quantic nucleus on the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

(7) $\mathcal{Z}$-$\mathcal{T}(A_0 \times A_0)$ is a quantic quotient of the quantale $(\mathcal{D}(A_0 \times A_0), \cdot)$.

RELATION QUANTALES

[Erné, Picado, *preprint*]

$$
\begin{tikzcd}
Ant(A,A) \arrow[r, shift left, "F_1"] \arrow[r, leftarrow, shift right, "G_1"'] \arrow[d, equal] & \mathcal{D}(A \times A) \arrow[r, shift left, "F_2"] \arrow[r, leftarrow, shift right, "G_2"'] & \mathcal{D}(A_0 \times A_0) \\
0\text{-}Ant(A,A) \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_0 \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & \mathcal{L}(A_0 \times A_0) \arrow[u, "\subseteq"] \\
\mathcal{Z}\text{-}Ant(A,A) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_{\mathcal{Z}} \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & \mathcal{Z}\text{-}\mathcal{T}(A_0 \times A_0) \arrow[u, "\subseteq"] \\
Gal(A,A) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & A \otimes A \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] \arrow[r, phantom, "\simeq"] & \mathcal{T}(A_0 \times A_0) \arrow[u, "\subseteq"]
\end{tikzcd}
$$

RELATION QUANTALES

[Erné, Picado, *preprint*]

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
Ant(A,A) \arrow[r, shift left, "F_1"] \arrow[d, equal] & \mathcal{D}(A \times A) \arrow[l, shift left, "G_1"] \arrow[r, shift left, "F_2"] & (\mathcal{D}(A_0 \times A_0), \cdot) \arrow[l, shift left, "G_2"] \\
0\text{-}Ant(A,A) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_0 \arrow[l, shift left] \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{D}(A_0 \times A_0), \cdot_L) \arrow[l, shift left] \arrow[u, "\subseteq"] \\
\mathcal{Z}\text{-}Ant(A,A) \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_{\mathcal{Z}} \arrow[l, shift left] \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{Z}\text{-}\mathcal{T}(A_0 \times A_0), \cdot_{\overline{T}^{\mathcal{Z}}}) \arrow[l, shift left] \arrow[u, "\subseteq"] \\
Gal(A,A) \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & A \otimes A \arrow[l, shift left] \arrow[u, "\subseteq"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{T}(A_0 \times A_0), \cdot_{\overline{T}}) \arrow[l, shift left] \arrow[u, "\subseteq"]
\end{tikzcd}
$$

RELATION QUANTALES

[Erné, Picado, *preprint*]

$$
\begin{tikzcd}[column sep=huge, row sep=large]
Ant(A,A) \arrow[r, shift left, "F_1"] \arrow[d, equal] & \mathcal{D}(A \times A) \arrow[l, shift left, "G_1"] \arrow[r, shift left, "F_2"] & (\mathcal{D}(A_0 \times A_0), \cdot) \arrow[l, shift left, "G_2"] \\
(0\text{-}Ant(A,A), \circ) \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_0 \arrow[l, shift left] \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{D}(A_0 \times A_0), \cdot_L) \arrow[l, shift left] \arrow[u, "\subseteq"'] \\
(\mathcal{Z}\text{-}Ant(A,A), \circ) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (A \otimes A)^l_{\mathcal{Z}} \arrow[l, shift left] \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{Z}\text{-}\mathcal{T}(A_0 \times A_0), \cdot_{\overline{T}^{\mathcal{Z}}}) \arrow[l, shift left] \arrow[u, "\subseteq"'] \\
(Gal(A,A), \circ) \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & A \otimes A \arrow[l, shift left] \arrow[u, "\subseteq"'] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & (\mathcal{T}(A_0 \times A_0), \cdot_{\overline{T}}) \arrow[l, shift left] \arrow[u, "\subseteq"']
\end{tikzcd}
$$