On the localic Katětov-Tong interpolation theorem

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THE SETTING

THE FRAME OF REALS $\overline{\mathfrak{L}}(\mathbb{R})$

[A. Joyal, P.T. Johnstone, B. Banaschewski]

$$\overline{\mathfrak{L}}(\mathbb{R}) = Frm\Big\langle (\alpha,\beta)(\alpha,\beta \in \mathbb{Q}) \mid (1)-(4) \Big\rangle$$

$$\mathfrak{L}(\mathbb{R}) \begin{cases} (1)\ (\alpha,\beta)\wedge(\gamma,\delta) = (\alpha\vee\gamma, \beta\wedge\delta) \\ (2)\ \alpha \le \gamma < \beta \le \delta \Rightarrow (\alpha,\beta)\vee(\gamma,\delta) = (\alpha,\delta) \\ (3)\ \bigvee\{(\gamma,\delta) \mid \alpha < \gamma < \delta < \beta\} = (\alpha,\beta) \end{cases}$$

$$(4)\ \bigvee\{(\alpha,\beta) \mid \alpha,\beta \in \mathbb{Q}\} = 1.$$

THE UPPER FRAME OF REALS $\overline{\mathfrak{L}}_u(\mathbb{R})$

$$(-,\alpha) := \bigvee_{\beta\in\mathbb{Q}} (\beta,\alpha)$$

$$\overline{\mathfrak{L}}_u(\mathbb{R}) = \Big\langle (-,\alpha) \mid \alpha\in\mathbb{Q} \Big\rangle_{\overline{\mathfrak{L}}(\mathbb{R})}$$

$$\mathfrak{L}_u(\mathbb{R}) = \Big\langle (-,\alpha) \mid \alpha\in\mathbb{Q} \Big\rangle_{\mathfrak{L}(\mathbb{R})}$$

THE LOWER FRAME OF REALS $\overline{\mathfrak{L}}_l(\mathbb{R})$

$$(\alpha,-) := \bigvee_{\beta\in\mathbb{Q}} (\alpha,\beta)$$

$$\overline{\mathfrak{L}}_l(\mathbb{R}) = \Big\langle (\alpha,-) \mid \alpha\in\mathbb{Q} \Big\rangle_{\overline{\mathfrak{L}}(\mathbb{R})}$$

$$\mathfrak{L}_l(\mathbb{R}) = \Big\langle (\alpha,-) \mid \alpha\in\mathbb{Q} \Big\rangle_{\mathfrak{L}(\mathbb{R})}$$

SEMICONTINUOUS REAL FUNCTIONS

$f : \overline{\mathfrak{L}}_u(\mathbb{R}) \longrightarrow L$ $\qquad$ $\overline{\mathcal{R}}_u(L)$

$(-, \alpha) \longmapsto f_\alpha$ $\qquad$ $\{f_\alpha\}_{\alpha \in \mathbb{Q}} \subseteq L$

$$\mathcal{R}_u(L) \begin{cases} (1)\ \alpha \leq \beta \Rightarrow f_\alpha \leq f_\beta & \text{(asc. chain)} \\ (2)\ \bigvee_{\beta < \alpha} f_\beta = f_\alpha & \text{(continuous)} \end{cases}$$

$$(3)\ \bigvee_{\alpha \in \mathbb{Q}} f_\alpha = 1 \qquad \text{(proper)}$$

$g : \overline{\mathfrak{L}}_l(\mathbb{R}) \longrightarrow L$ $\qquad$ $\overline{\mathcal{R}}_l(L)$

$(\alpha, -) \longmapsto g_\alpha$ $\qquad$ $\{g_\alpha\}_{\alpha \in \mathbb{Q}} \subseteq L$

$$\mathcal{R}_l(L) \begin{cases} (1)\ \alpha \leq \beta \Rightarrow g_\alpha \geq g_\beta & \text{(desc. chain)} \\ (2)\ \bigvee_{\beta > \alpha} g_\beta = g_\alpha & \text{(continuous)} \end{cases}$$

$$(3)\ \bigvee_{\alpha \in \mathbb{Q}} g_\alpha = 1 \qquad \text{(proper)}$$

$$\overline{\mathcal{R}}_u(L) \qquad \{f^1_\alpha\}_{\alpha\in\mathbb{Q}} = f_1 \;\leq\; f_2 = \{f^2_\alpha\}_{\alpha\in\mathbb{Q}}$$

$$\text{iff } f^1_\alpha \geq f^2_\alpha \quad \forall\alpha \in \mathbb{Q}$$

$$\overline{\mathcal{R}}_l(L) \qquad \{g^1_\alpha\}_{\alpha\in\mathbb{Q}} = g_1 \;\leq\; g_2 = \{g^2_\alpha\}_{\alpha\in\mathbb{Q}}$$

$$\text{iff } g^1_\alpha \leq g^2_\alpha \quad \forall\alpha \in \mathbb{Q}$$

$$f = \{f_\alpha\}_{\mathbb{Q}} \in \overline{\mathcal{R}}_u(L),\; g = \{g_\alpha\}_{\mathbb{Q}} \in \overline{\mathcal{R}}_l(L)$$

$$f \leq g \quad \equiv \quad f_\alpha \vee g_\beta = 1 \text{ whenever } \beta < \alpha$$

$$g \leq f \quad \equiv \quad f_\alpha \wedge g_\alpha = 0 \text{ for every } \alpha$$

REAL CONTINUOUS FUNCTIONS

$h : \overline{\mathfrak{L}}(\mathbb{R}) \to L$ $\rightsquigarrow$ (h_1, h_2)

$\overline{\mathcal{R}}(L)$

$h_1 \in \overline{\mathcal{R}}_u(L),\ h_2 \in \overline{\mathcal{R}}_l(L)$

such that $h_1 \leq h_2$ and $h_2 \leq h_1$

$$h_1(-,\alpha) := h(\textstyle\bigvee_{\beta<\alpha}(\beta,\alpha))$$

$$h_2(\alpha,-) := h(\textstyle\bigvee_{\beta>\alpha}(\alpha,\beta))$$

$$f \leq (h_1, h_2) \equiv f \leq h_1$$

$$\Leftrightarrow f \leq h_2$$

$$(h_1, h_2) \leq g \equiv h_1 \leq g$$

$$\Leftrightarrow h_2 \leq g$$

LOCALIC KATĚTOV-TONG THEOREM

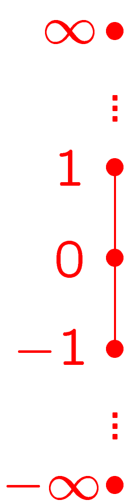
[Li and Wang, 1997]

L is normal iff $\forall\, f \in \overline{\mathcal{R}}_u(L),\ \forall\, g \in \overline{\mathcal{R}}_l(L)$

$$(f \leq g \Rightarrow\ \exists\, h \in \overline{\mathcal{R}}(L)\ :\ f \leq h \leq g)$$

COUNTEREXAMPLE

$\mathbb{Z} \cup \{\infty, -\infty\}$



$$f:\ \overline{\mathcal{L}}_u(\mathbb{R}) \to L \qquad \leq \qquad g:\ \overline{\mathcal{L}}_l(\mathbb{R}) \to L$$

$$(-,\alpha) \mapsto \min_{n \geq \alpha} n \in \mathbb{Z} \qquad\qquad (\alpha,-) \mapsto \infty$$

$$\nexists\, h \in \overline{\mathcal{R}}(L)\ :\ f \leq h \leq g$$

THEOREM 1.

L is normal iff $\forall\, f \in \mathcal{R}_u(L),\ \forall\, g \in \mathcal{R}_l(L)$

$$(f \leq g \Rightarrow\ \exists\, h \in \mathcal{R}(L)\ :\ f \leq h \leq g).$$

Proof: (Sketch) Let $\mathbb{Q} = \{\alpha_i \mid i \in \mathbb{N}\}$.

$$f \leq g \ \Rightarrow \begin{cases} (\forall \gamma > \alpha_1) \quad f_\gamma \vee (\bigwedge_{\delta<\alpha_1} g_\delta) = 1 \\ \\ (\forall \delta < \alpha_1) \quad (\bigwedge_{\gamma>\alpha_1} f_\gamma) \vee g_\delta = 1. \end{cases}$$

LEMMA 1. Let $f \in \mathcal{R}_u(L)$, $g \in \mathcal{R}_l(L)$, with $f \leq g$.

(a) $(\forall \alpha \in \mathbb{Q})\,(\forall \gamma > \alpha)\ f_\gamma \vee (\bigwedge_{\delta<\alpha} g_\delta) = \bigwedge_{\delta<\alpha} (f_\gamma \vee g_\delta) = 1$

(b) $(\forall \alpha \in \mathbb{Q})\,(\forall \delta < \alpha)\ (\bigwedge_{\gamma>\alpha} f_\gamma) \vee g_\delta = \bigwedge_{\gamma>\alpha} (f_\gamma \vee g_\delta) = 1$

LEMMA 2. L is normal iff

for every $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subseteq L$ satisfying

$$(\forall n \in \mathbb{N})\ \Big(a_n \vee (\bigwedge_{m\in\mathbb{N}} b_m) = 1 = (\bigwedge_{m\in\mathbb{N}} a_m) \vee b_n\Big)$$

there exists $u \in L$: $(\forall n \in \mathbb{N})\ (a_n \vee u = 1 = b_n \vee u^*)$.

$$\left.\begin{array}{l}\{f_\gamma\}_{\gamma>\alpha_1} \\ \{g_\delta\}_{\delta<\alpha_1}\end{array}\right\} \Rightarrow \begin{array}{c}(\exists\, u_{\alpha_1} \in L)(\forall\, \gamma > \alpha_1)(\forall\, \delta < \alpha_1) \\ (f_\gamma \vee u_{\alpha_1} = 1 = g_\delta \vee u^*_{\alpha_1})\end{array}$$

INDUCTIVELY: $\{u_{\alpha_i}\}_{i\in\mathbb{N}}$

$$x_{\alpha_i} := \bigvee_{\alpha_j<\alpha_i} u^*_{\alpha_j} \qquad y_{\alpha_i} := \bigvee_{\alpha_j>\alpha_i} u_{\alpha_j} \qquad (i \in I)$$

$$\boxed{h_1 := \{x_{\alpha_i}\}_{i\in\mathbb{N}} \qquad h_2 := \{y_{\alpha_i}\}_{i\in\mathbb{N}}}$$

is the solution

$$f \in \overline{\mathcal{R}}_u(L) \subseteq \mathcal{R}_u(L) \qquad g \in \overline{\mathcal{R}}_l(L) \subseteq \mathcal{R}_l(L)$$

$$h = (h_1, h_2) \in \mathcal{R}(L)$$

- $h_1 \in \overline{\mathcal{R}}_u(L) \Leftrightarrow (\forall \alpha \in \mathbb{Q})(\exists \beta \in \mathbb{Q})\Big(f_\alpha \wedge g_\beta \leq \bigvee_{\gamma \in \mathbb{Q}} h_1(-,\gamma)\Big)$

$\Uparrow$

(A$_1$) $\boxed{\bigvee_{\gamma \in \mathbb{Q}} \Delta_{g_\gamma} = 1}$

$\Uparrow$

(A$_2$) $\boxed{(\forall \alpha \in \mathbb{Q})(\exists \beta \in \mathbb{Q}) : f_\alpha \wedge g_\beta = 0}$

- $h_2 \in \overline{\mathcal{R}}_l(L) \Leftrightarrow (\forall \beta \in \mathbb{Q})(\exists \alpha \in \mathbb{Q})\Big(f_\alpha \wedge g_\beta \leq \bigvee_{\gamma \in \mathbb{Q}} h_2(\gamma,-)\Big)$

$\Uparrow$

(B$_1$) $\boxed{\bigvee_{\gamma \in \mathbb{Q}} \Delta_{f_\gamma} = 1}$

$\Uparrow$

(B$_2$) $\boxed{(\forall \beta \in \mathbb{Q})(\exists \alpha \in \mathbb{Q}) : f_\alpha \wedge g_\beta = 0}$

THEOREM 2. L is normal iff

$\forall\, f \in \overline{\mathcal{R}}_u(L)$, $\forall\, g \in \overline{\mathcal{R}}_l(L)$ satisfying (A$_i$) and (B$_i$)

$$(f \leq g \Rightarrow \exists\, h \in \overline{\mathcal{R}}(L) \,:\, f \leq h \leq g)$$

IMMEDIATE CONSEQUENCES

(1) CLASSICAL KATĚTOV-TONG THEOREM

$X :=$ normal space

Applied to $L := \mathcal{O}X$, the "⇒" part of Theorem 2, for $i = 2$, yields Katětov-Tong Interpolation Theorem.

(2) LOCALIC URYSOHN'S LEMMA

$L :=$ normal frame

Applied to $\chi_a^u \in \overline{\mathcal{R}}_u(L)$ and $\chi_b^l \in \overline{\mathcal{R}}_l(L)$, the "⇒" part of Theorem 2, for $i = 2$, yields localic Urysohn's Lemma.

(3) LOCALIC TIETZE'S EXTENSION THEOREM

CHARACTERISTIC FUNCTIONS

For each $a, b \in L$,

$$\chi_a^u : \overline{\mathfrak{L}}_u(\mathbb{R}) \to L \qquad (-,\alpha) \mapsto \begin{cases} 1 \text{ if } \alpha > 1 \\ a \text{ if } 0 < \alpha \leq 1 \\ 0 \text{ if } \alpha \leq 0 \end{cases} \qquad \chi_a^u \in \overline{\mathcal{R}}_u(L)$$

$$\chi_b^l : \overline{\mathfrak{L}}_l(\mathbb{R}) \to L \qquad (\alpha,-) \mapsto \begin{cases} 1 \text{ if } \alpha < 0 \\ b \text{ if } 0 \leq \alpha < 1 \\ 0 \text{ if } \alpha \geq 1 \end{cases} \qquad \chi_b^l \in \overline{\mathcal{R}}_l(L)$$

$$a \vee b = 1 \Leftrightarrow \chi_a^u \leq \chi_b^l$$