Complete normality and complete regularity in the category of locales via real-valued functions

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— joint work with J. Gutiérrez García (Bilbao)

REAL-VALUED FUNCTIONS?

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CONTINUOUS

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$h : X \to \mathbb{R}$ GENERAL ?

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point-set topology

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≪*(...) Therefore $\mathsf{C}(X)$ is a commutative ring, a subring of $\mathbb{R}^X$.*≫

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THEOREM. TFAE for a space X:

(1) X is completely normal.

(2) For every $f_1, f_2 \in \mathbb{R}^X$, if $f_1^- \leq f_2$ and $f_1 \leq f_2^\circ$, then there exists a $g \in \mathsf{LSC}(X)$ such that $f_1 \leq g \leq g^- \leq f_2$.

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  Remains to be done: complete regularity.

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(the geometric motivation reads backwards)

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THE SETTING

$\mathsf{FRM}(\mathfrak{L}(\mathbb{R}), \mathcal{S}L)$

ring $\mathsf{F}(L)$

- $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}L$ $\boxed{\text{usc}}$

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$$\boxed{\mathsf{USC}(L) \cap \mathsf{LSC}(L) = \mathsf{C}(L)}$$

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$$\boxed{f \leq g \equiv \ f(p,-) \leq g(p,-),\ \forall p \in \mathbb{Q} \ \Leftrightarrow \ g(-,q) \leq f(-,q),\ \forall q \in \mathbb{Q}}$$

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J. Gutiérrez García, T. Kubiak & J. Picado
[Localic real-valued functions: a general setting, *J. Pure Appl. Algebra* 213 (2009) 1064-1074]

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[J. Isbell, 1985]

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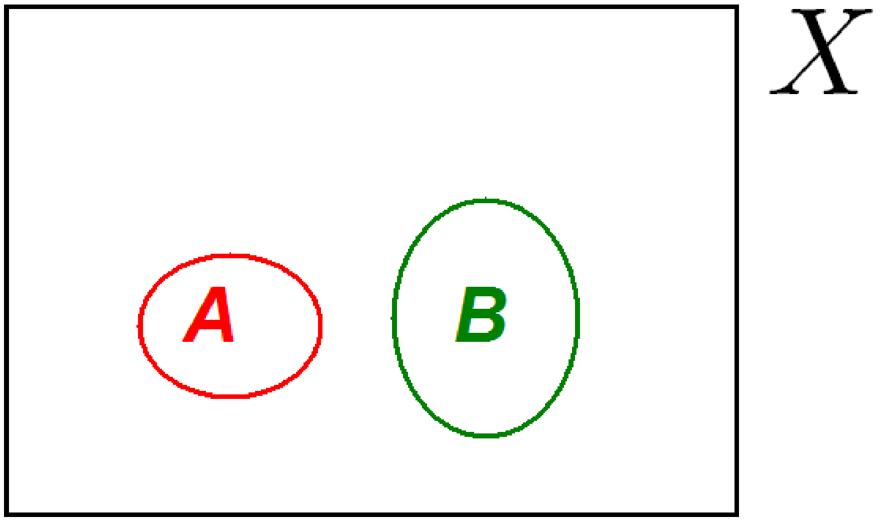

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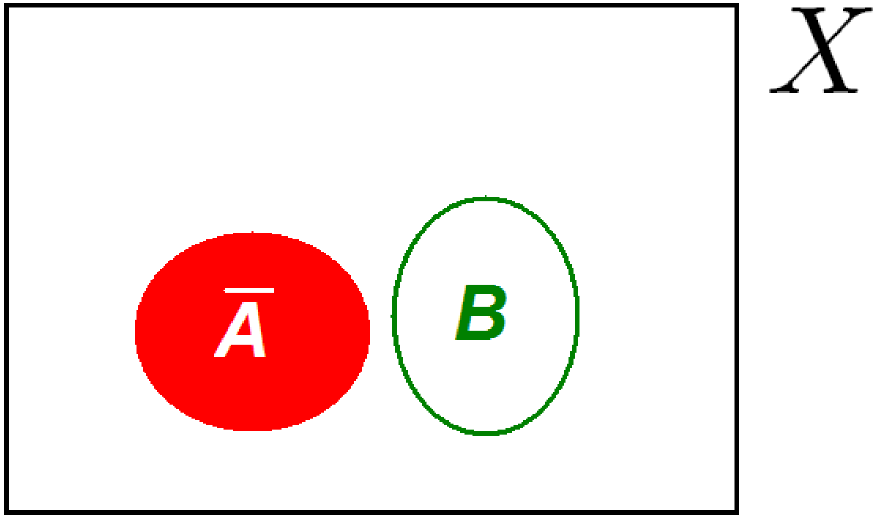

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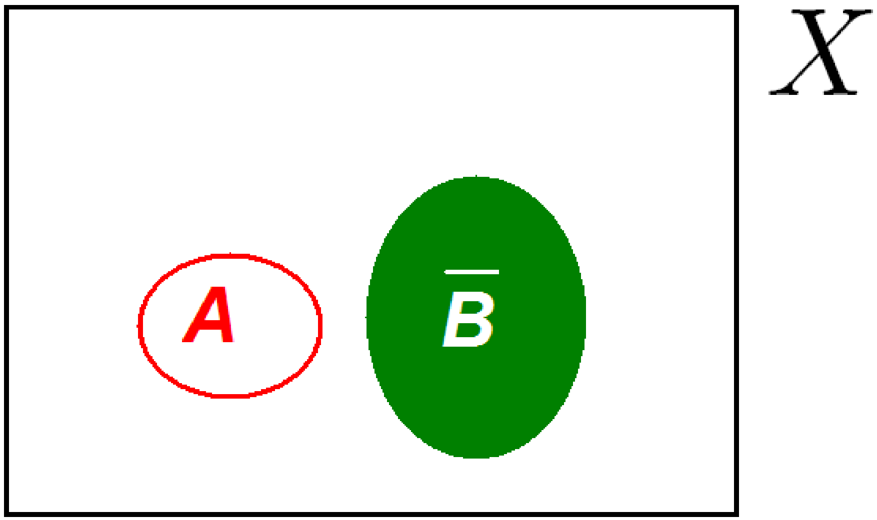

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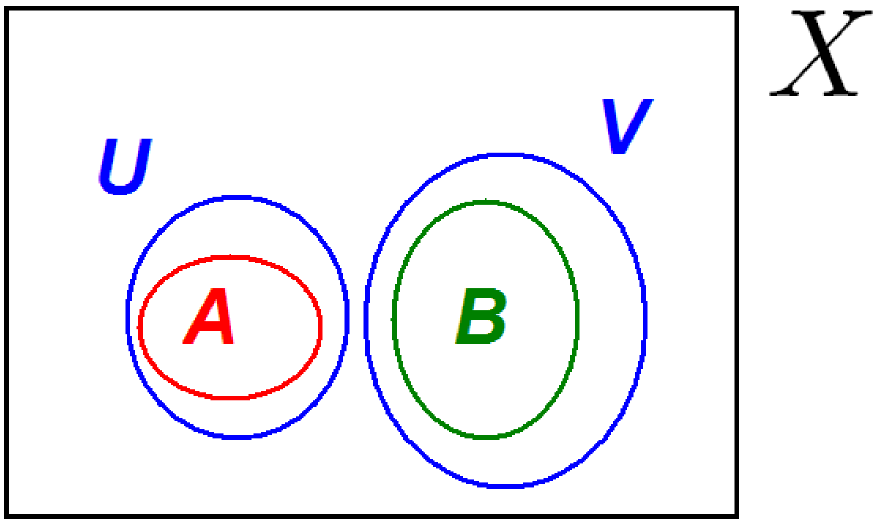

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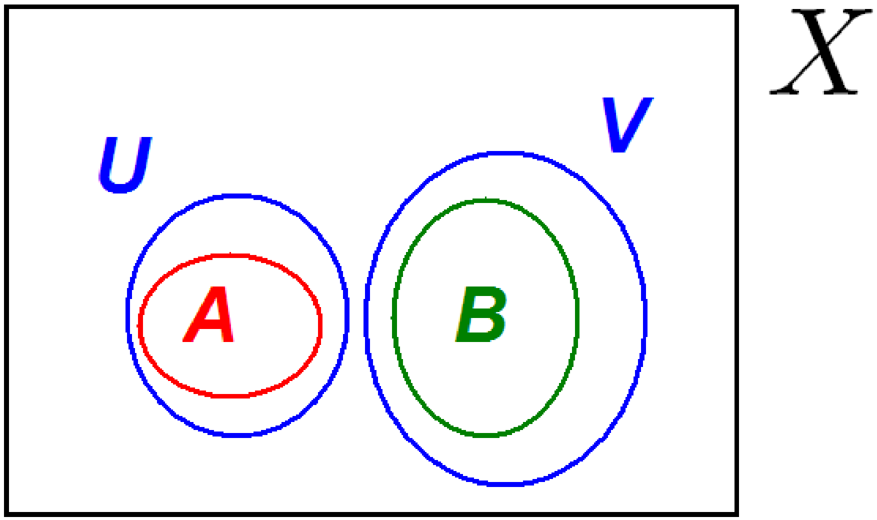

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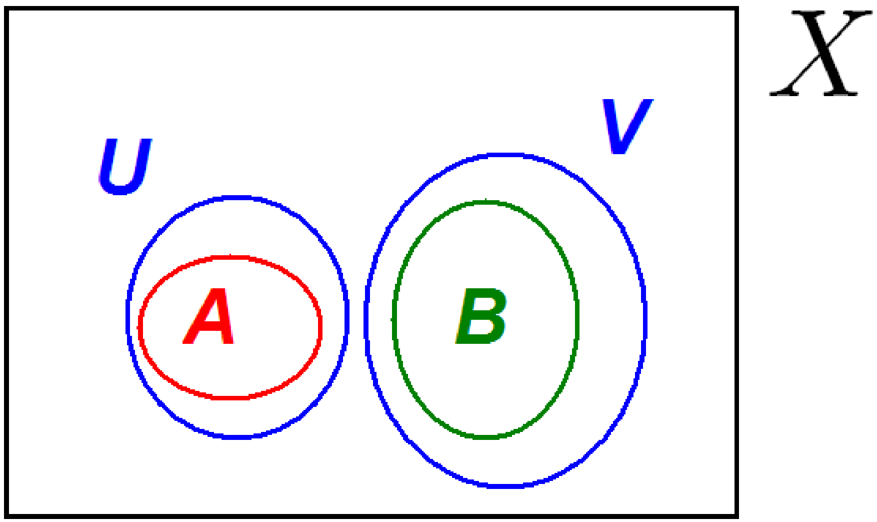

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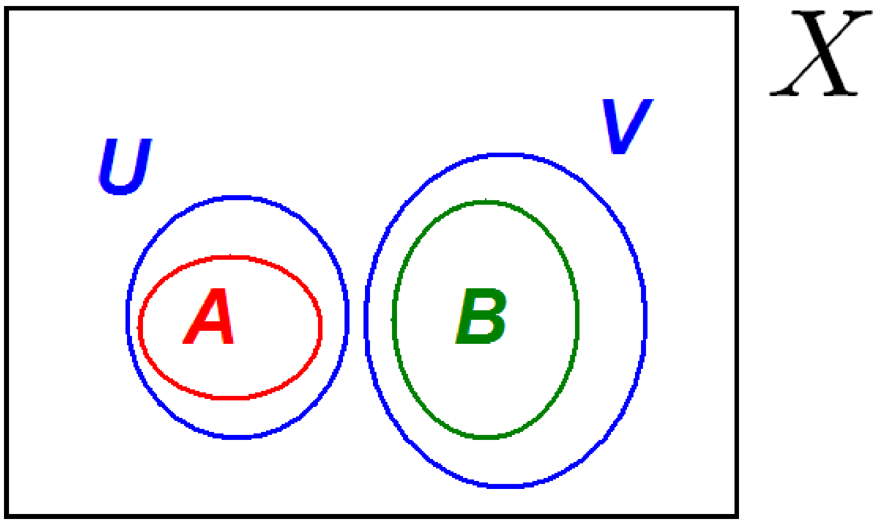

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$$\underbrace{S \vee \overline{T} = 1 = \overline{S} \vee T}_{\text{separated sublocales}} \quad \Rightarrow \quad \underbrace{\exists \text{ open } U, V : \ U \vee V = 1,\ U \leq S,\ V \leq T}_{\text{separated by open sublocales}}$$

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PROPOSITION. TFAE for a frame L:

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[H. Simmons, 1978]: a space X is CN iff $L = \mathcal{O}X$ sat. (4).

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J. Gutiérrez García & T. Kubiak J. Picado
[Lower and upper regularizations of frame semicontinuous real functions, *Algebra Universalis* 60 (2009) 169-184]

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LEMMA. Let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathsf{USC}(L)$ and $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathsf{LSC}(L)$.

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M. J. FERREIRA & J. GUTIÉRREZ GARCÍA & J. PICADO
[Completely normal frames and real-valued functions, *Topology and its Applications*, in press]

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COROLLARY. TFAE on a frame L:

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(work in progress...)