Semicontinuity in pointfree topology: monotonization and semiregularization

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Centre for Mathematics - University of Coimbra

PORTUGAL

(joint work with J. Gutiérrez Garcia and T. Kubiak)

MOTIVATION: J. Gutiérrez García and J. P.

On the algebraic representation of semicontinuity

J. Pure Appl. Algebra 210 (2007) 299-306.

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  Monotone insertion and monotone extension of frame homomorph.
  To appear in *Journal of Pure and Applied Algebra*.

SEMICONTINUITY IN POINTFREE TOPOLOGY

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  Lower and upper regularizations of frame semicont. real functions

  In preparation.

category of frames $\boxed{\text{FRM}}$

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$\boxed{\text{TOPOLOGY}}$ $\xrightarrow{\text{abstraction}}$ $\boxed{\text{POINTFREE TOPOLOGY}}$

- frames (locales)

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$$h : L \to M \qquad \left[\begin{array}{l} h(\bigvee a_i) = \bigvee h(a_i) \\[2ex] h(a \wedge b) = h(a) \wedge h(b)\ ,\ \ h(1) = 1\end{array}\right.$$

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dual adjunction

$\boxed{\text{LOC}} = \text{FRM}^{op}$

BACKGROUND: THE FRAME OF REALS $\mathfrak{L}(\mathbb{R})$

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$(-\infty, q)$

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$g : X \to (\mathbb{R}, \mathcal{T}_u)$

LSC

$g : \mathfrak{L}_u(\mathbb{R}) \to L$

$(-\infty, q)$

LSC(L) $\boxed{\bigvee_{p\in\mathbb{Q}} \Delta_{g(p,-)} = 1}$

UPPER AND LOWER REGULARIZATIONS

$$
\begin{tikzcd}
\mathsf{FRM}(\mathfrak{L}_l(\mathbb{R}), L) \\
\mathsf{USC}(L) \arrow[u, "\subset"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{FRM}(\mathfrak{L}_u(\mathbb{R}), L) \\
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\end{tikzcd}
$$

f $\qquad\qquad$ g

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\subset \big\uparrow & & \big\uparrow \subset \\
\text{USC}(L) & & \text{LSC}(L) \\
\color{blue}{f} & & \color{blue}{g}
\end{array}
$$

$$\color{blue}{f}\,\color{red}{\leq}\,\color{blue}{g} \;\equiv\; f(-,q) \vee g(p,-) = 1 \text{ for every } p < q \in Q$$

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$$g \leq f \;\equiv\; f(-, r) \wedge g(r, -) = 0 \text{ for every } r \in \mathbb{Q}$$

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$$\boxed{\text{Given } f \in \mathrm{USC}(L),\ \exists\, g \in \mathrm{LSC}(L) : g \leq f\,?}$$

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In that case, there exists the largest such g:

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$$f_\vee(p, -) = \bigvee_{p' > p} f(-, p')^*$$

$\boxed{\text{LOWER REGULARIZATION OF } f}$

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$\exists\ f_\vee$

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\mathsf{CUSC}(L) \arrow[u, "\subset"] \arrow[r, leftrightarrow, "\simeq"] & \mathsf{C}(L) \arrow[r, leftrightarrow, "\simeq"] & \mathsf{CLSC}(L) \arrow[u, "\subset"']
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REGULARIZATION: INSERTION THEOREM

THEOREM:

The following conditions on a frame L are equivalent:

(1) L is extremally disconnected (i.e. $a^* \vee a^{**} = 1$).

(2) If $f \in \mathrm{SUSC}(L)$ then $f_\vee$ is continuous.

(3) If $g \in \mathrm{SLSC}(L)$ then $g^\wedge$ is continuous.

(4) If $g \in \mathrm{LSC}(L), f \in \mathrm{USC}(L)$ and $g \leq f$, then there exists $h \in \mathrm{C}(L)$ such that $g \leq h \leq f$.

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- constructive
- extends the corresponding classical result of M. Stone [1949]

REGULARIZATION: EXTENSION THEOREM

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MONOTONE CONCEPT

usually: specialization

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$\boxed{\text{MONOTONE CONCEPT}}$

usually: specialization

concept $\rightsquigarrow$ different monotonizations

[Classically: Kubiak, Good, Stares, Borges, Gutiérrez García, de Prada Vicente, ...]

AN ILLUSTRATION

$L \in \mathsf{FRM}$

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new concept: monotonically normal frame

MONOTONIZATION

particular result

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particular result $\overset{\rightsquigarrow}{}$ ***monotonization procedure***

MONOTONIZATION

particular result $\xrightarrow[\textit{monotonization procedure}]{}$ monotone variant

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pointfree insertion theorems

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Pointfree Katětov-Tong Insertion Theorem:

L is normal iff for every usc real function $f : \mathfrak{L}_l(\mathbb{R}) \to L$ and every lsc real function $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $f \leq g$ there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.

MONOTONIZATION: INSERTION THEOREM

UL(L)$= \{(f,g) \in \mathsf{USC}(L) \times \mathsf{LSC}(L) : f \leq g\}$

ordered by $(f_1, g_1) \leq (f_2, g_2) \; \equiv \; f_2 \leq f_1, g_1 \leq g_2$.

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The following conditions on a frame L are equivalent:

(1) L is monotonically normal.

(2) There exists a monotone $\Theta : \text{UL}(L) \to \text{C}(L)$ such that $f \leq \Theta(f,g) \leq g$ for every $(f,g) \in \text{UL}(L)$.

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EXTENSION THEOREM ...