

Semicontinuity in pointfree topology: monotonization and semiregularization

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(joint work with J. Gutiérrez Garcia and T. Kubiak)

SEMICONTINUITY IN POINTFREE TOPOLOGY

MOTIVATION: J. Gutiérrez García and J. P.
On the algebraic representation of semicontinuity
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Lower and upper regularizations of frame semicont. real functions
In preparation.

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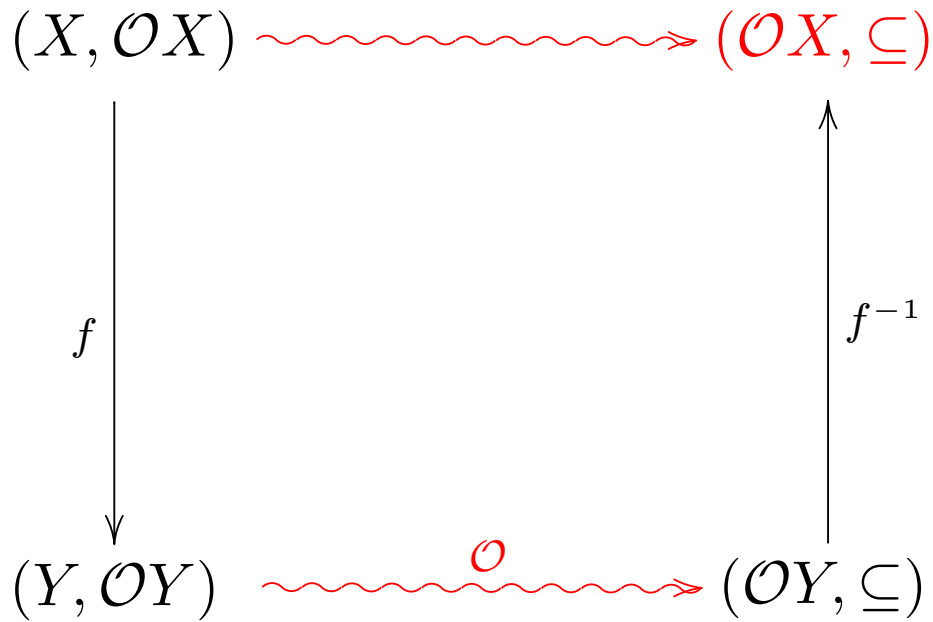
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$$(Y, \mathcal{O}Y)$$

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$$\begin{array}{ccc} (X, \mathcal{O}X) & \rightsquigarrow & (\mathcal{O}X, \subseteq) \\ \downarrow f & & \\ (Y, \mathcal{O}Y) & \rightsquigarrow & (\mathcal{O}Y, \subseteq) \end{array}$$

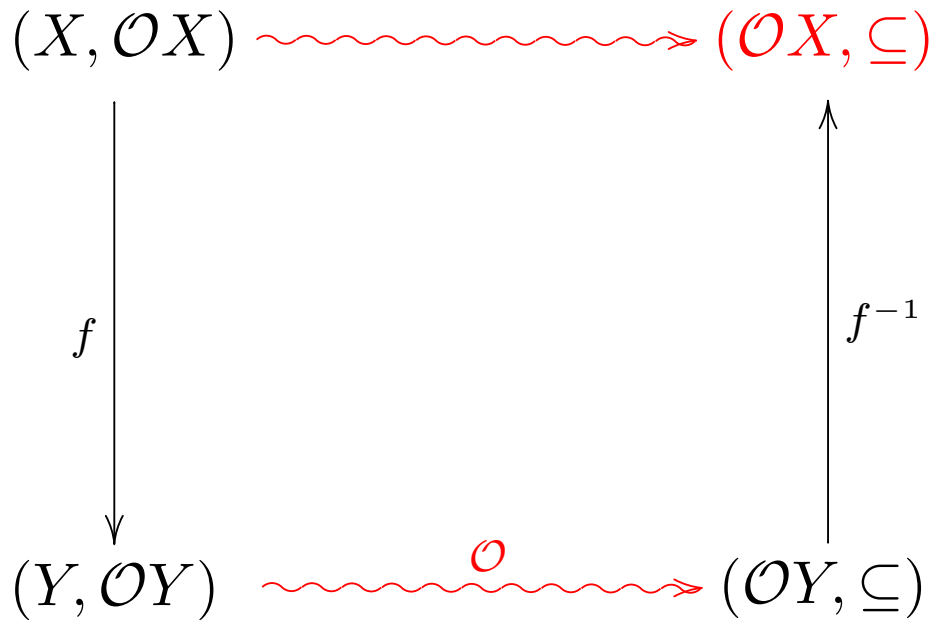
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POINTFREE TOPOLOGY

category of frames FRM

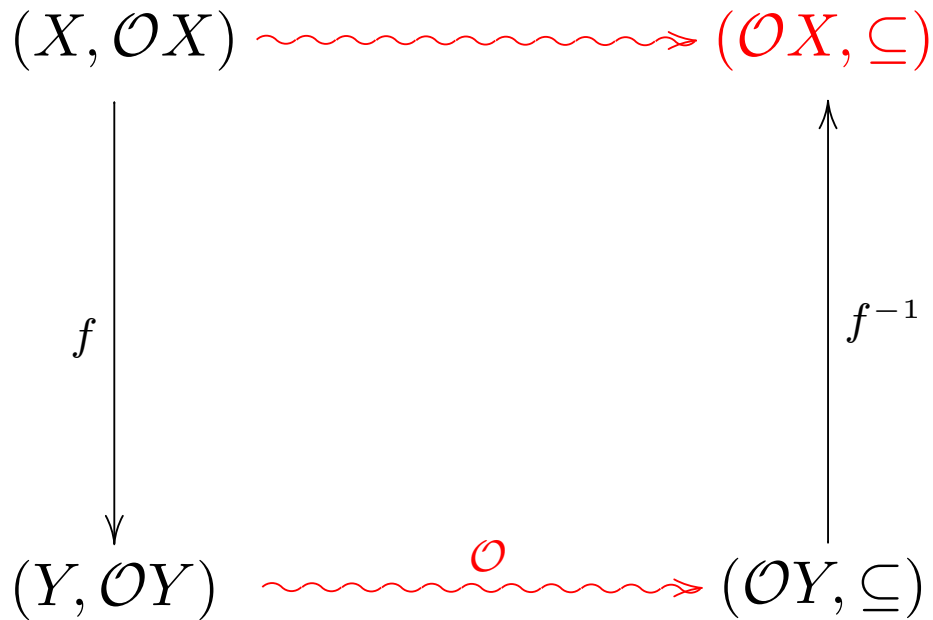


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TOPOLOGY

abstraction \rightsquigarrow

POINTFREE TOPOLOGY

- frames (locales)

L

complete lattice

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- homomorphisms

 $h : L \rightarrow M$

$$h\left(\bigvee a_i\right) = \bigvee h(a_i)$$

$$h(a \wedge b) = h(a) \wedge h(b) , h(1) = 1$$

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$$\text{TOP} \xrightarrow{\mathcal{O}} \text{FRM}$$

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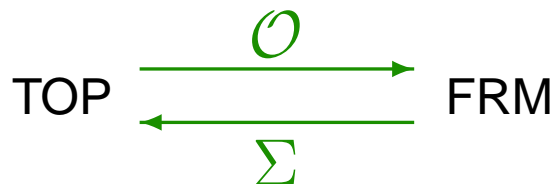
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dual adjunction

$$\boxed{\text{LOC}} = \text{FRM}^{op}$$

BACKGROUND: THE FRAME OF REALS $\mathfrak{L}(\mathbb{R})$

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(SEMI)CONTINUOUS REAL FUNCTIONS

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CONTINUOUS

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USC

$$(-\infty, q)$$

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USC(L)

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$$\text{USC}(L) \quad \bigvee_{q \in \mathbb{Q}} \Delta_{f(-,q)} = 1$$

$$g : X \rightarrow (\mathbb{R}, \mathcal{T}_u)$$

LSC

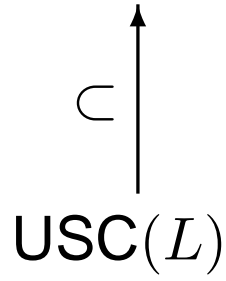
$$g : \mathfrak{L}_u(\mathbb{R}) \rightarrow L$$

$$(-\infty, q)$$

$$\text{LSC}(L) \quad \bigvee_{p \in \mathbb{Q}} \Delta_{g(p,-)} = 1$$

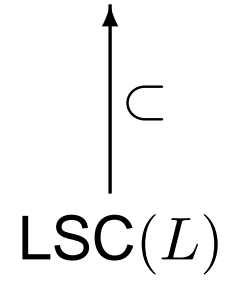
UPPER AND LOWER REGULARIZATIONS

$\text{FRM}(\mathcal{L}_l(\mathbb{R}), L)$



f

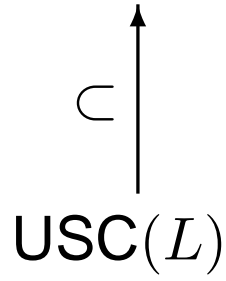
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g

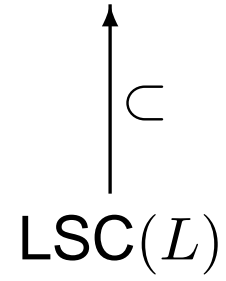
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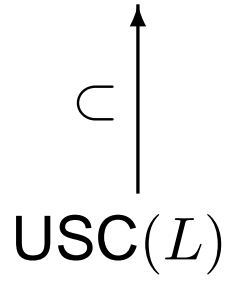


g

$$f \leq g \equiv f(-, q) \vee g(p, -) = 1 \text{ for every } p < q \in Q$$

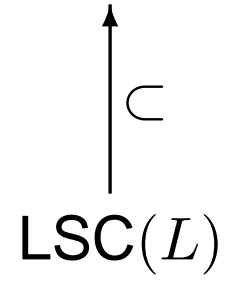
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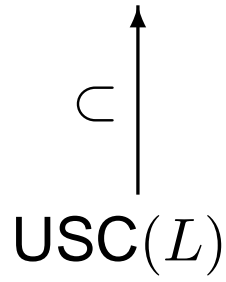


g

$$g \leq f \equiv f(-, r) \wedge g(r, -) = 0 \text{ for every } r \in \mathbb{Q}$$

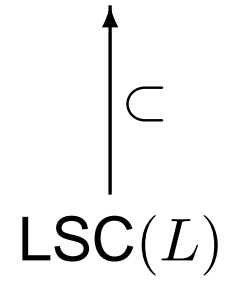
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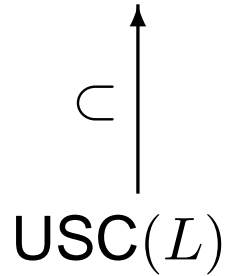


g

Given $f \in \text{USC}(L)$, $\exists g \in \text{LSC}(L) : g \leq f$?

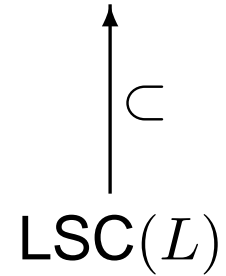
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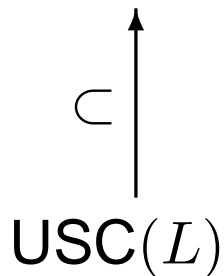
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strong usc $\text{SUSC}(L)$

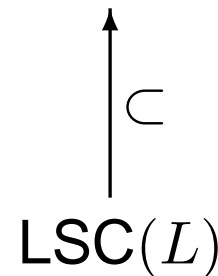
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In that case, there exists the largest such g :

UPPER AND LOWER REGULARIZATIONS

$$\text{FRM}(\mathcal{L}_l(\mathbb{R}), L)$$

$$\begin{array}{c} \subset \uparrow \\ \text{USC}(L) \end{array}$$

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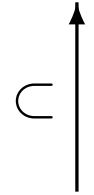
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$$f_{\vee}(p, -) = \bigvee_{p' > p} f(-, p')^*$$

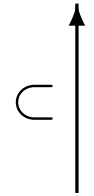
LOWER REGULARIZATION OF f

UPPER AND LOWER REGULARIZATIONS

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$\text{USC}(L)$



$\text{SUSC}(L)$

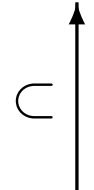
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UPPER AND LOWER REGULARIZATIONS

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$f \in \text{SUSC}(L)$

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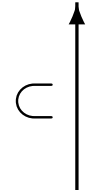
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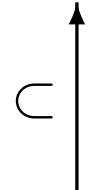


$\text{SLSC}(L) \ni g$

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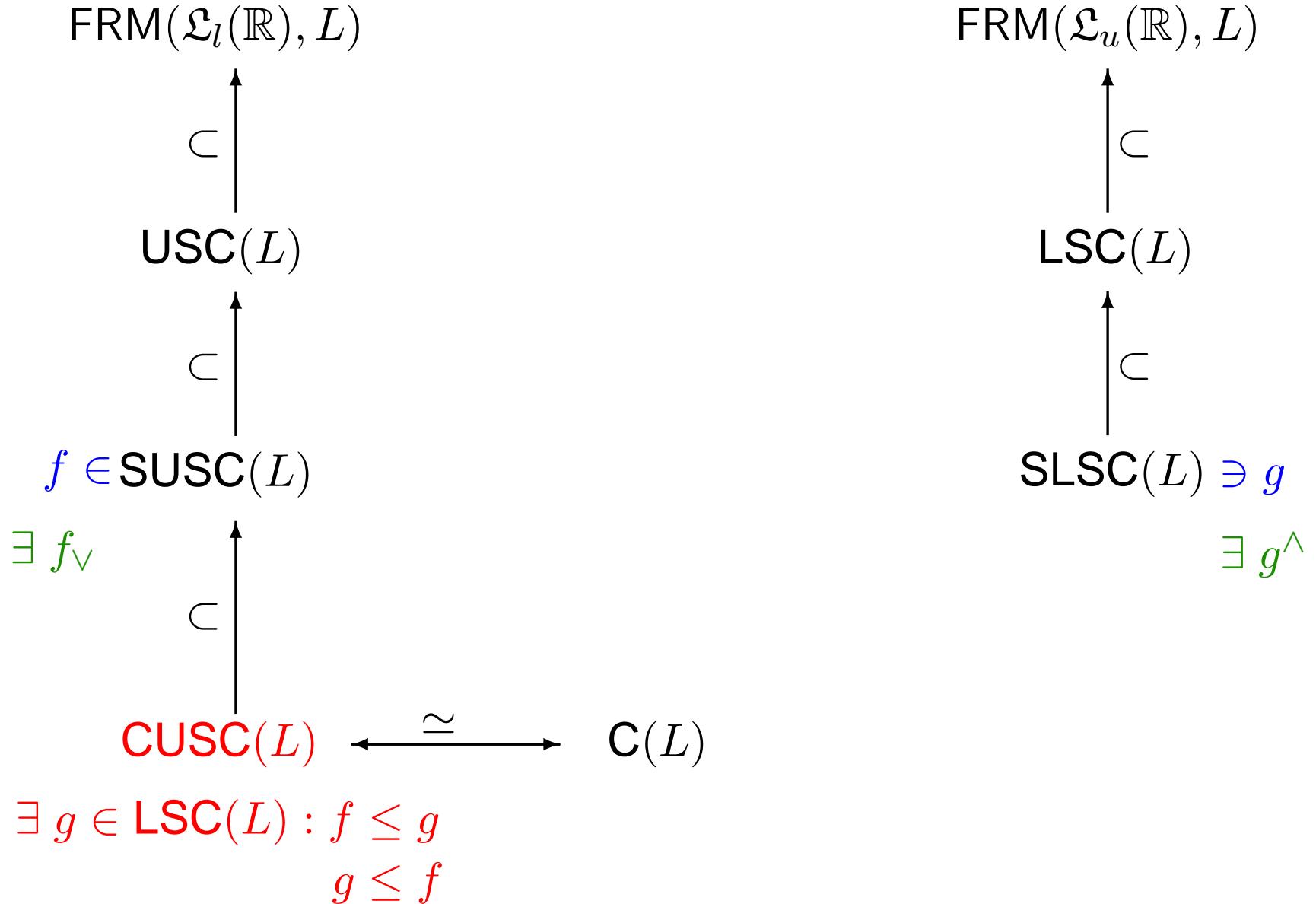


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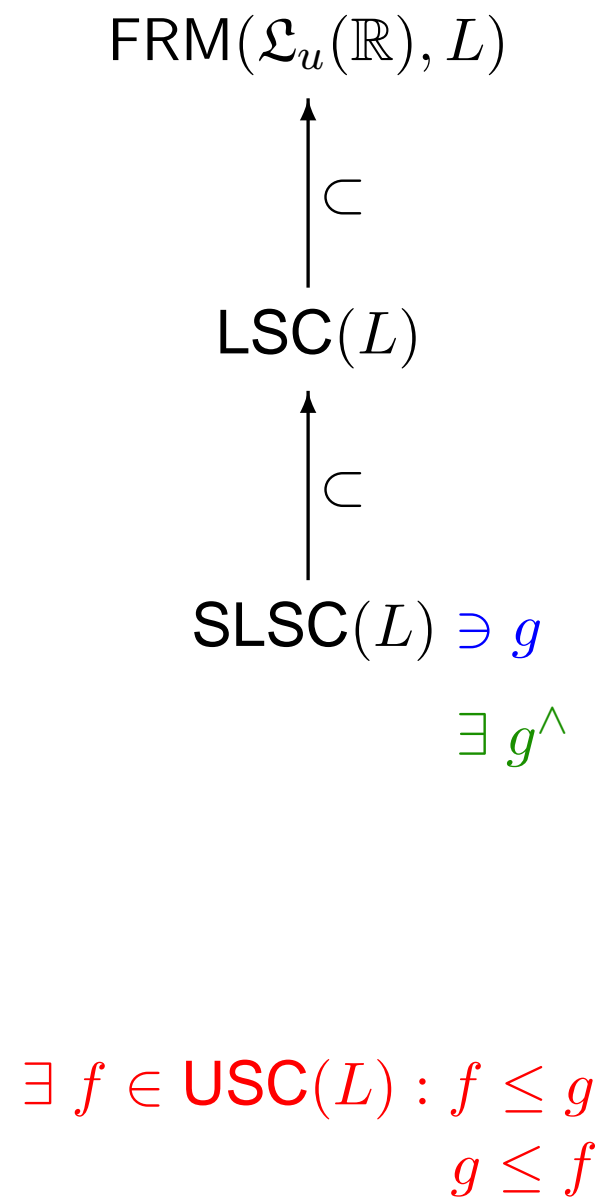
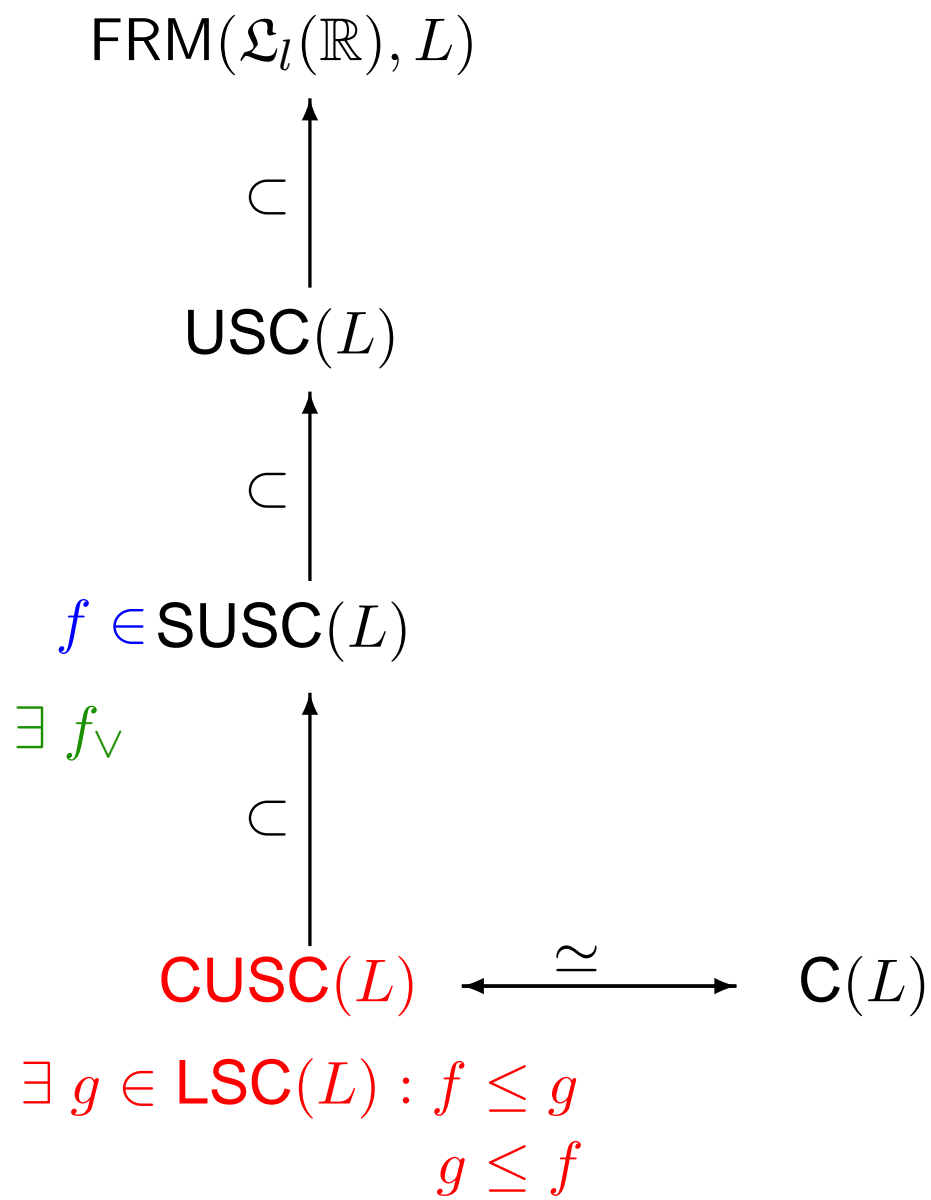
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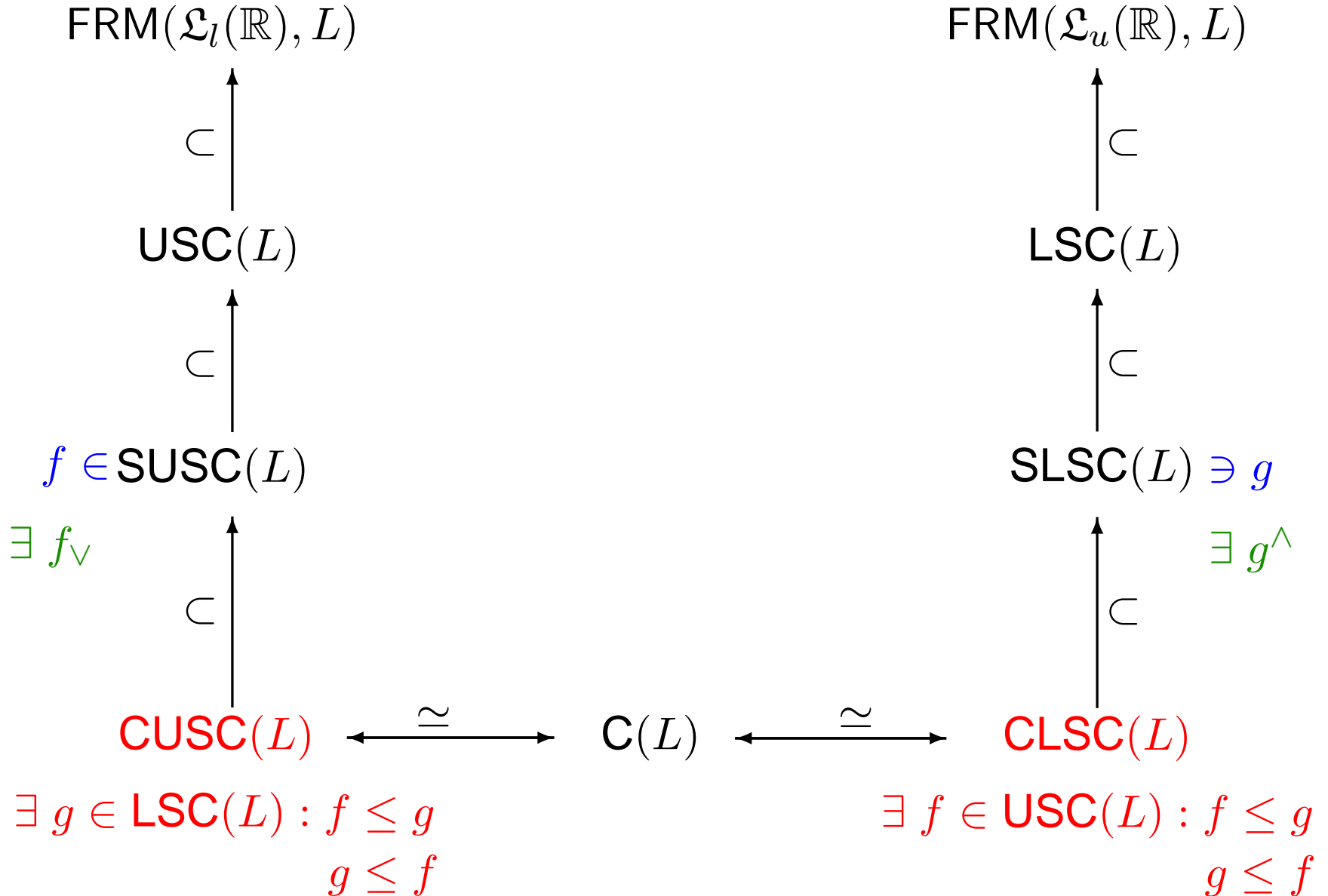
UPPER AND LOWER REGULARIZATIONS



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UPPER AND LOWER REGULARIZATIONS



THEOREM:

The following conditions on a frame L are equivalent:

- (1) L is extremally disconnected (i.e. $a^* \vee a^{**} = 1$).
- (2) If $f \in \text{SUSC}(L)$ then f_{\vee} is continuous.
- (3) If $g \in \text{SLSC}(L)$ then g^{\wedge} is continuous.
- (4) If $g \in \text{LSC}(L)$, $f \in \text{USC}(L)$ and $g \leq f$, then there exists $h \in \mathbf{C}(L)$ such that $g \leq h \leq f$.
- ⋮

REGULARIZATION: INSERTION THEOREM

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- **constructive**
- **extends the corresponding classical result of M. Stone [1949]**

REGULARIZATION: EXTENSION THEOREM

THEOREM:

A frame L is extremally disconnected iff for each open sublocale S of L and each $h \in \mathbf{C}(S)$, there exists a continuous extension $\tilde{h} \in \mathbf{C}(L)$ such that

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$$\begin{array}{ccc} & & L \\ & \nearrow \tilde{h} & \downarrow \\ \mathfrak{L}(\mathbb{R}) & \xrightarrow{h} & S \end{array}$$

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CONCEPT: $\left\{ \begin{array}{l} \text{sets } P, Q \\ \theta : P \rightarrow Q \end{array} \right.$

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usually: specialization

WHAT IS A MONOTONIZATION ?

CONCEPT: $\left\{ \begin{array}{l} \text{sets } P, Q \\ \theta : P \rightarrow Q \end{array} \right.$ posets $(P, \leq_P), (Q, \leq_Q)$
 $\theta : (P, \leq_P) \rightarrow (Q, \leq_Q)$ is **monotone**

MONOTONE CONCEPT

usually: specialization

concept  different monotonicizations

[Classically: Kubiak, Good, Stares, Borges, Gutiérrez García, de Prada Vicente, ...]

AN ILLUSTRATION

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NORMALITY OPERATOR

AN ILLUSTRATION: monotonically normal frames

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MONOTONE NORMALITY OPERATOR

new concept: monotonically normal frame

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monotonization procedure

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Pointfree Katětov-Tong Insertion Theorem:

L is normal iff for every usc real function $f : \mathfrak{L}_l(\mathbb{R}) \rightarrow L$ and every lsc real function $g : \mathfrak{L}_u(\mathbb{R}) \rightarrow L$ with $f \leq g$ there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \rightarrow L$ such that $f \leq h \leq g$.

MONOTONIZATION: INSERTION THEOREM

$$\mathbf{UL}(L) = \{(f, g) \in \mathbf{USC}(L) \times \mathbf{LSC}(L) : f \leq g\}$$

ordered by $(f_1, g_1) \leq (f_2, g_2) \equiv f_2 \leq f_1, g_1 \leq g_2$.

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The following conditions on a frame L are equivalent:

- (1) L is monotonically normal.
- (2) There exists a monotone $\Theta : \mathbf{UL}(L) \rightarrow \mathbf{C}(L)$ such that $f \leq \Theta(f, g) \leq g$ for every $(f, g) \in \mathbf{UL}(L)$.

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EXTENSION THEOREM ...