Semicontinuous real functions pointfreely

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Center for Mathematics - University of Coimbra

PORTUGAL

MOTIVATION

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[M.J.F. and J.P., *The semicontinuous quasi-uniformity of a frame*, Kyungpook Math. J. 46 (2006) 189-200]

SEMICONTINUITY CLASSICALLY

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(α, β)	$(-\infty, \alpha)$	$(\alpha, +\infty)$

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$A \in \mathcal{O}X$, $\chi_A : X \to \mathbb{R}$ is lsc

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Question: $\left.\begin{array}{c} f \text{ usc} \\ \\ g \text{ lsc} \end{array}\right\} \overset{?}{\Rightarrow} \exists$ continuous $h : f \leq h \leq g$

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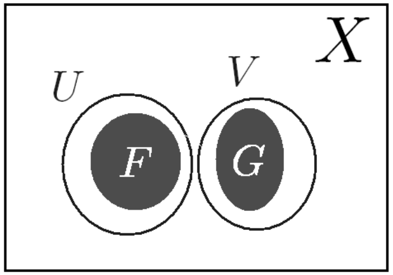

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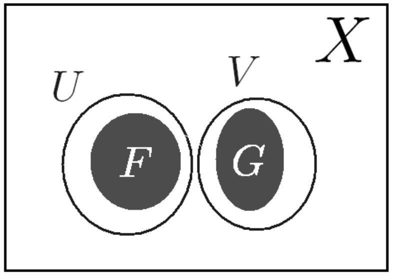

RELEVANCE of the question: it extends Urysohn's Lemma

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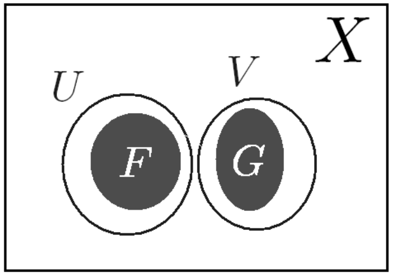

Urysohn's Lemma (for normal spaces X)

$$\forall\, F, G \in \mathcal{F}X, F \cap G = \emptyset,\ \exists\, h : X \to [0,1] \ :\ h(F) = \{1\}, h(G) = \{0\}$$

$$\Updownarrow$$

$$\forall\, F \in \mathcal{F}X,\ \forall\, A \in \mathcal{O}X,\ F \subseteq A,\ \exists\, h : X \to \mathbb{R} \ :\ \chi_F \leq h \leq \chi_A$$

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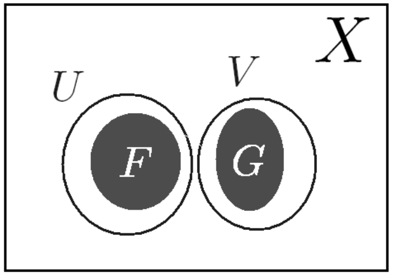

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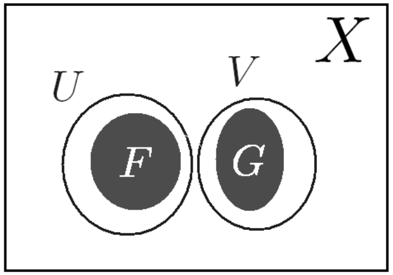

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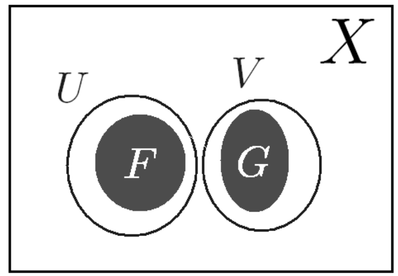

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Yes, if X is NORMAL [Katetoǒ-Tong, 1948]

POINTFREE TOPOLOGY

category of frames **FRM**

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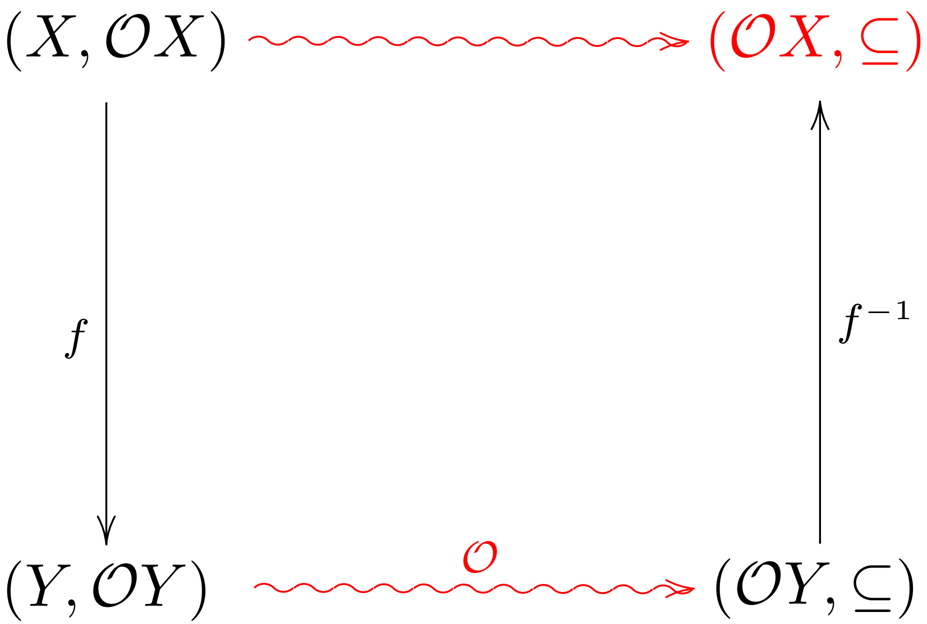

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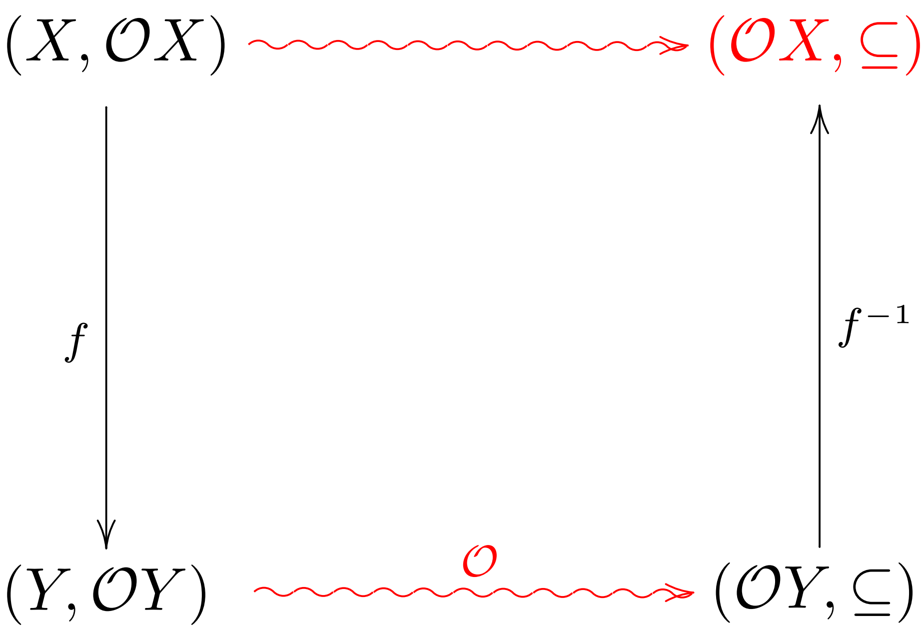

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TOPOLOGY $\overset{\text{abstraction}}{\rightsquigarrow}$ POINTFREE TOPOLOGY

- frames (locales)

L

complete lattice

$$a \wedge \bigvee b_i = \bigvee (a \wedge b_i)$$

category of frames FRM

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$$\left[\begin{array}{l} \text{complete lattice} \\ a \wedge \bigvee b_i = \bigvee (a \wedge b_i) \end{array}\right.$$

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$h : L \to M$

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[A. Joyal, B. Banaschewski]

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CONTINUOUS REAL FUNCTIONS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree top.*, Coimbra, 1997]

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$$\Sigma\mathfrak{L}(\mathbb{R}) \cong (\mathbb{R}, \mathcal{T}_e)$$

general frame L: $\qquad \text{FRM}(\mathfrak{L}(\mathbb{R}), L)$

continuous real functions on L

[B. Banaschewski, *The real numbers in pointfree top.*, Coimbra, 1997]

SEMICONTINUOUS REAL FUNCTIONS POINTFREELY

[Li and Wang, *Localic Katětov-Tong insertion theorem*, 1997]

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upper semicontinuous real function on L: $f : \mathfrak{L}_l(\mathbb{R}) \to L$

where $\mathfrak{L}_l(\mathbb{R})$ is the subframe of $\mathfrak{L}(\mathbb{R})$ generated by elements

$$(-, \alpha) := \bigvee_{\beta \in \mathbb{Q}} (\beta, \alpha) \qquad (\alpha \in \mathbb{Q})$$

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lower semicontinuous real function on L: $f : \mathfrak{L}_u(\mathbb{R}) \to L$

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Localic Katětov-Tong Insertion Theorem:

L is normal iff for every usc real function $f : \mathfrak{L}_l(\mathbb{R}) \to L$ and every lsc real function $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $f \leq g$ there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.

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but NOT TRUE!!!

[J.P., *A new look at localic interpolation theorems*, Topology Appl. 153 (2006) 3203-3218]

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$(\mathbb{R} \cup \{-\infty\}, \mathcal{T}_l)$

$(\mathbb{R}, \mathcal{T}_l)$ is not sober: $\Sigma L \not\cong (\mathbb{R}, \mathcal{T}_l)$

EXAMPLES

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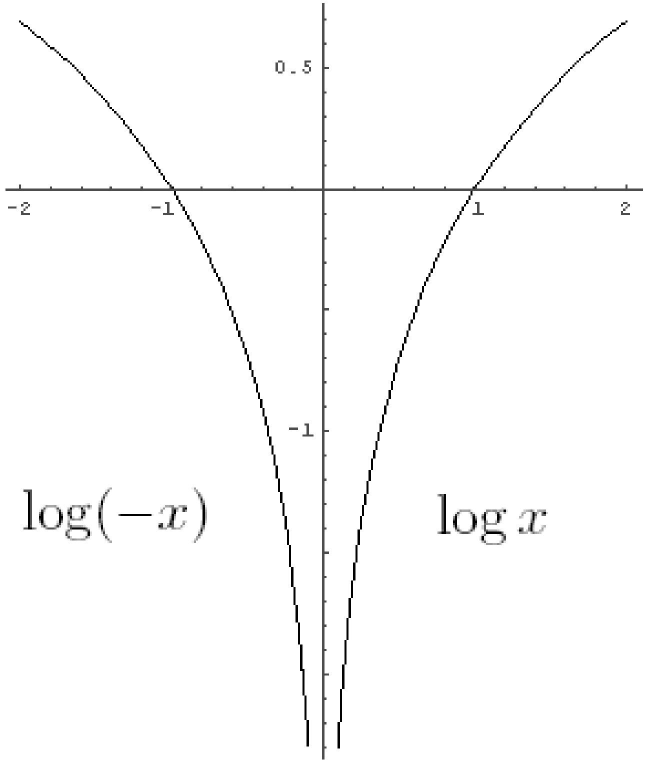

$f(0) = -\infty$

ALGEBRAIC DESCRIPTION FOR SPATIAL FRAMES

$$\mathsf{FRM}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X) \quad \stackrel{\cong}{\longleftrightarrow} \quad \mathsf{TOP}(X, (\mathbb{R} \cup \{-\infty\}, \mathcal{T}_l))$$

ALGEBRAIC DESCRIPTION FOR SPATIAL FRAMES

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\Big\uparrow & & \subset \Big\uparrow \\
\color{red}{?} & \xleftrightarrow{\;\cong\;} & \mathsf{TOP}(X, (\mathbb{R}, \mathcal{T}_l))
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$$

ALGEBRAIC DESCRIPTION FOR SPATIAL FRAMES

$$\begin{array}{ccc} \mathsf{FRM}(\mathfrak{L}_l(\mathbb{R}),\mathcal{O}X) & \xleftrightarrow{\;\cong\;} & \mathsf{TOP}(X,(\mathbb{R}\cup\{-\infty\},\mathcal{T}_l)) \\ \Big\uparrow & & \subset \Big\uparrow \\ ? & \xleftrightarrow{\;\cong\;} & \mathsf{TOP}(X,(\mathbb{R},\mathcal{T}_l)) \end{array}$$

$h : \mathfrak{L}_l(\mathbb{R}) \xrightarrow{h} \mathcal{O}X$

$\{\alpha \in \mathbb{Q} \mid x \in h(-,\alpha)\}$ is bounded below for every $x \in X$

HOW TO DESCRIBE UPPER SEMICONTINUITY ALGEBRAICALLY?

$$f : X \to \mathbb{R} \text{ is u.s.c} \Leftrightarrow f : \underbrace{(X, \mathcal{O}X, \mathcal{C}X)}_{Sk(X)} \to (\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u) \in \text{BiTOP}$$

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$\Updownarrow$ sober bispace

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locale X

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$(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe

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$$\mathsf{BiFRM}(\mathfrak{L}(\mathbb{R}), \mathfrak{C}L) \cong \left\{ f : \mathfrak{L}_l(\mathbb{R}) \to L \in \mathsf{FRM} \mid \bigvee_{\alpha \in \mathbb{Q}} \Delta_{f(-,\alpha)} = \mathbf{1} \right\}$$

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$$\begin{aligned} h_f : (-,\alpha) &\mapsto \nabla_{f(-,\alpha)} \\ (\alpha,-) &\mapsto \bigvee_{\beta>\alpha} \Delta_{f(-,\beta)} \end{aligned} \quad \leftsquigarrow \quad f$$

CONCLUSION

usc real function on L:

$$f : \mathfrak{L}_l(\mathbb{R}) \to L \in \text{FRM s.t.} \bigvee_{\alpha \in \mathbb{Q}} \Delta_{f(-,\alpha)} = \mathbf{1}$$

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Pointfree Katětov-Tong Insertion Theorem:

L is normal iff for every usc real function $f : \mathfrak{L}_l(\mathbb{R}) \to L$ and every lsc real function $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $f \leq g$ there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.

J.P., J. Gutierrez García
[*On the algebraic representation of semicontinuity*, JPAA, to appear]

WORK IN PROGRESS (with J. Gutierrez García)

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- Semicontinuity and extremally disconnected locales.

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- Perfect normality and stratified frames. Strict insertion theorems.