Insertion in spaces, bispaces, ordered spaces and point-free spaces

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PORTUGAL

– joint work with M. J. Ferreira and J. Gutiérrez García –

CATEGORY BiFRM OF BIFRAMES [Banaschewski-Brümmer-Hardie 1983]

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biframe maps

- $h : L_0 \to M_0$ frame homomorphism
- $h(L_i) \subseteq M_i \quad (i = 1, 2)$.

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[Gutiérrez García-Kubiak-Picado 2008]

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$\therefore$ L_1-usc + L_2-lsc $\Rightarrow$ continuous on L_0.

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$$\Leftrightarrow \;\forall a \in L_1,\, a^{\bullet\bullet} \vee a^{\bullet} = 1 \qquad \Leftrightarrow \;\forall b \in L_2,\, b^{\bullet} \vee b^{\bullet\bullet} = 1.$$

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$$\left.\begin{array}{l} L \text{ is normal} \\ G \text{ is } L_1\text{-usc} \\ F \text{ is } L_2\text{-lsc} \\ G \leq F \end{array}\right\} \Rightarrow \exists (u_{\alpha_k})_k \in L_2 : \left\{\begin{array}{l} q > \alpha_k \Rightarrow G(-,q) \vee \mathfrak{c}(u_{\alpha_k}) = 1 \\ p < \alpha_k \Rightarrow F(p,-) \vee \mathfrak{c}(u^\bullet_{\alpha_k}) = 1 \\ \alpha_{k_1} < \alpha_{k_2} \Rightarrow u_{\alpha_{k_1}} \vee u^\bullet_{\alpha_{k_2}} = 1. \end{array}\right.$$

Katětov-Tong-type insertion theorem:

TFAE for a biframe (L_0, L_1, L_2):

(i) (L_0, L_1, L_2) is normal.

(ii) $$\left.\begin{array}{l}\forall G : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L_0)\ \ L_1\text{-usc} \\ \forall F : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L_0)\ \ L_2\text{-lsc}\end{array}\right\} G \leq F \ \Rightarrow \ \boxed{G \leq H \leq F}$$

for some L_1-usc and L_2-lsc $H : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L_0)$

INSERTION THEOREMS FOR BIFRAMES

Stone-type insertion theorem:

TFAE for a biframe (L_0, L_1, L_2):

(i) (L_0, L_1, L_2) is extremally disconnected.

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CONSEQUENCES: biframes (L_0, L_1, L_2)

$a \in L_1, b \in L_2 \quad \rightsquigarrow \quad \chi_{\mathfrak{c}(a)} \qquad \chi_{\mathfrak{o}(b)}$

L_1-usc $\qquad$ L_2-lsc

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K.-T. Theorem

- Urysohn-type lemma for biframes [A. Schauerte, *PhD Thesis*, 1992]

A biframe is normal iff whenever $a \vee b = 1$ $(a \in L_1, b \in L_2)$ *there exists* $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ *such that*

$$h(-,1) \leq a, \; h(0,-) \leq b.$$

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$$\boxed{a \wedge b = 0}$$

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S. Theorem

- Urysohn-type lemma for ext. disc. biframes [new]:

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> *A biframe is ext. disc. iff whenever* $a \wedge b = 0$ $(a \in L_1, b \in L_2)$ *there exists* $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ *such that*
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> $$a \wedge h(0, -) = 0,\ b \vee h(-, 1) = 1.$$

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- Tietze-type extension theorems for biframes... [both new]

CONSEQUENCES: frames (and spaces)

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Katětov-Tong-type insertion theorem:

TFAE for a frame L:

(i) L is normal.

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[Y.M. Li-Z.H. Li, *Alg. Univ.*, 2000]
[Gutiérrez García-Kubiak-Picado, *Alg. Univ.*, 2008]

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[new]

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