

# ***Insertion in spaces, bispaces, ordered spaces and point-free spaces***

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*– joint work with M. J. Ferreira and J. Gutiérrez García –*

# CATEGORY BiFRM OF BIFRAMES [Banaschewski-Brümmer-Hardie 1983]

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biframe maps

$$\left[ \begin{array}{l} h : L_0 \rightarrow M_0 \text{ frame homomorphism} \\ h(L_i) \subseteq M_i \quad (i = 1, 2). \end{array} \right]$$

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[Gutiérrez García-Kubiak-Picado 2008]

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$\therefore L_1\text{-usc} + L_2\text{-lsc} \Rightarrow \text{continuous on } L_0.$



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$$\Leftrightarrow \forall a \in L_1, a^{\bullet\bullet} \vee a^\bullet = 1 \Leftrightarrow \forall b \in L_2, b^\bullet \vee b^{\bullet\bullet} = 1.$$

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$$\left. \begin{array}{l} L \text{ is normal} \\ G \text{ is } L_1\text{-usc} \\ F \text{ is } L_2\text{-lsc} \\ G \leq F \end{array} \right\} \Rightarrow \exists (u_{\alpha_k})_k \in L_2 : \left\{ \begin{array}{l} q > \alpha_k \Rightarrow G(-, q) \vee \mathfrak{c}(u_{\alpha_k}) = 1 \\ p < \alpha_k \Rightarrow F(p, -) \vee \mathfrak{c}(u_{\alpha_k}^\bullet) = 1 \\ \alpha_{k_1} < \alpha_{k_2} \Rightarrow u_{\alpha_{k_1}} \vee u_{\alpha_{k_2}}^\bullet = 1. \end{array} \right.$$

## Katětov-Tong-type insertion theorem:

TFAE for a biframe  $(L_0, L_1, L_2)$ :

(i)  $(L_0, L_1, L_2)$  is **normal**.

(ii)  $\left. \begin{array}{l} \forall G : \mathcal{L}(\mathbb{R}) \rightarrow \mathfrak{C}(L_0) \text{ } L_1\text{-usc} \\ \forall F : \mathcal{L}(\mathbb{R}) \rightarrow \mathfrak{C}(L_0) \text{ } L_2\text{-lsc} \end{array} \right\} G \leq F \Rightarrow \boxed{G \leq H \leq F}$

for some  **$L_1$ -usc and  $L_2$ -lsc**  $H : \mathcal{L}(\mathbb{R}) \rightarrow \mathfrak{C}(L_0)$

## Stone-type insertion theorem:

TFAE for a biframe  $(L_0, L_1, L_2)$ :

(i)  $(L_0, L_1, L_2)$  is **extremally disconnected**.

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## CONSEQUENCES: biframes $(L_0, L_1, L_2)$

$$a \in L_1, b \in L_2 \quad \rightsquigarrow \quad \begin{array}{ll} \chi_{\mathfrak{c}(a)} & \chi_{\mathfrak{o}(b)} \\ L_1\text{-usc} & L_2\text{-lsc} \end{array}$$

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$$\begin{array}{ccc} L_1\text{-usc} & \swarrow & L_2\text{-lsc} \\ & K\text{-T. Theorem} & \end{array}$$

- Urysohn-type lemma for biframes [A. Schauerte, *PhD Thesis, 1992*]

A *biframe* is normal iff whenever  $a \vee b = 1$  ( $a \in L_1, b \in L_2$ ) there exists  $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \rightarrow (L_0, L_1, L_2)$  such that

$$h(-, 1) \leq a, \quad h(0, -) \leq b.$$

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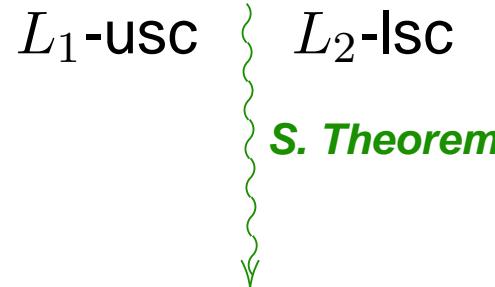
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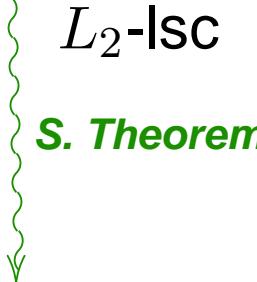
- Urysohn-type lemma for ext. disc. biframes [new]:

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A *biframe* is ext. disc. iff whenever  $a \wedge b = 0$  ( $a \in L_1, b \in L_2$ ) there exists  $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \rightarrow (L_0, L_1, L_2)$  such that

$$a \wedge h(0, -) = 0, \quad b \vee h(-, 1) = 1.$$

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- Tietze-type extension theorems for biframes... [both new]

## CONSEQUENCES: frames (and spaces)

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### Katětov-Tong-type insertion theorem:

TFAE for a frame  $L$ :

(i)  $L$  is normal.

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[Gutiérrez García-Picado, J. Pure Appl. Alg., 2007]

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### Stone-type insertion theorem:

TFAE for a frame  $L$ :

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[Y.M. Li-Z.H. Li, *Alg. Univ.*, 2000]

[Gutiérrez García-Kubiak-Picado, *Alg. Univ.*, 2008]

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**[H. Priestley, J. London Math. Soc., 1971]**

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