Rings of (extended) real functions in frames

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PORTUGAL

— joint work with J. Gutiérrez García (UPV-EHU, Bilbao, Spain)

THE REALS: $\mathfrak{L}(\mathbb{R})$

$\mathfrak{L}(\mathbb{R}) =$ Frm$\langle\ (-,q),(p,-)(p,q\in\mathbb{Q})\ |$ (1) $(-,q)\wedge(p,-)=0$ for $q\leq p$,

(2) $(-,q)\vee(p,-)=1$ for $q>p$,

(3) $(-,q)=\bigvee_{s<q}(-,s)$,

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$(—,q)$ $\quad\mathfrak{L}_l(\mathbb{R})\left\{\begin{array}{l}\textbf{(3)}\ (—,q)=\bigvee_{s<q}(—,s),\\ \textbf{(4)}\ \bigvee_{q\in\mathbb{Q}}(—,q)=1,\end{array}\right.$

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REAL FUNCTIONS

J. GUTIÉRREZ GARCÍA& T. KUBIAK & J. PICADO

[Localic real-valued functions: a general setting, *J. Pure Appl. Algebra* 213 (2009) 1064-1074]

REAL FUNCTIONS

$$f \leq g \equiv f(p,—) \leq g(p,—),\ \forall p \in \mathbb{Q} \Leftrightarrow g(—,q) \leq f(—,q),\ \forall q \in \mathbb{Q}$$

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KATĚTOV-TONG INSERTION THEOREM. TFAE on a frame L:

(1) L is normal.

(2) For every $f \in \mathsf{USC}(L)$ and every $g \in \mathsf{LSC}(L)$ satisfying $f \leq g$, there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.

J. GUTIÉRREZ GARCÍA & J. PICADO
[On the algebraic representation of semicontinuity,
Journal of Pure and Applied Algebra **210 (2007) 299–306]**

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$$g \in \mathsf{LSC}(L),\ g \leq f \Rightarrow g \leq f^\circ.$$

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[Lower and upper regularizations of frame semicontinuous real functions, *Algebra Universalis* 60 (2009) 169–184]

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MORE: bounded insertion (Michael), strict insertion (Dowker), monotone insertion (Kubiak), ...

GENERAL INSERTION THEOREM. TFAE on a frame L:

(1) L is completely normal (= hereditarily normal).

(2) For every $h_1, h_2 \in \mathsf{F}(L)$, if $h_1^- \leq h_2$ and $h_1 \leq h_2^\circ$, then there exists $g \in \mathsf{LSC}(L)$ such that $h_1 \leq g \leq g^- \leq h_2$.

M. J. FERREIRA & J. GUTIÉRREZ GARCÍA & J. PICADO
[Completely normal frames and real-valued functions,
***Topology and its Applications* 156 (2009) 2932–2941]**

BACKGROUND: the commutative f-ring $\mathrm{C}(L)$

P. T. Johnstone,

Stone Spaces, CUP, 1982.

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B. Banaschewski,

The real numbers in pointfree topology,

Textos de Matemática, vol. 12, Universidade de Coimbra, 1997.

$(\mathsf{C}(L), +, \cdot, \leq)$ is a commutative archimedean and strong f-ring with unit

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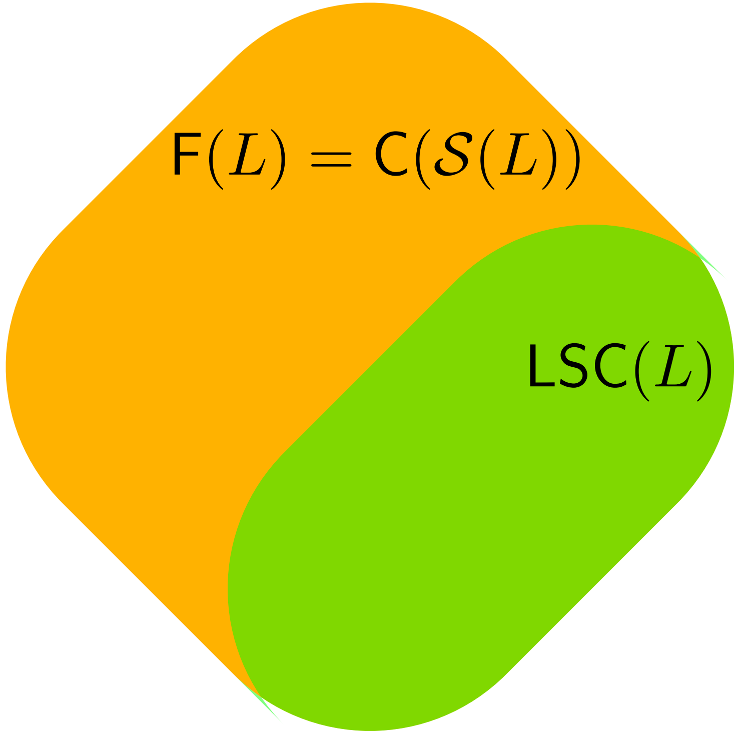

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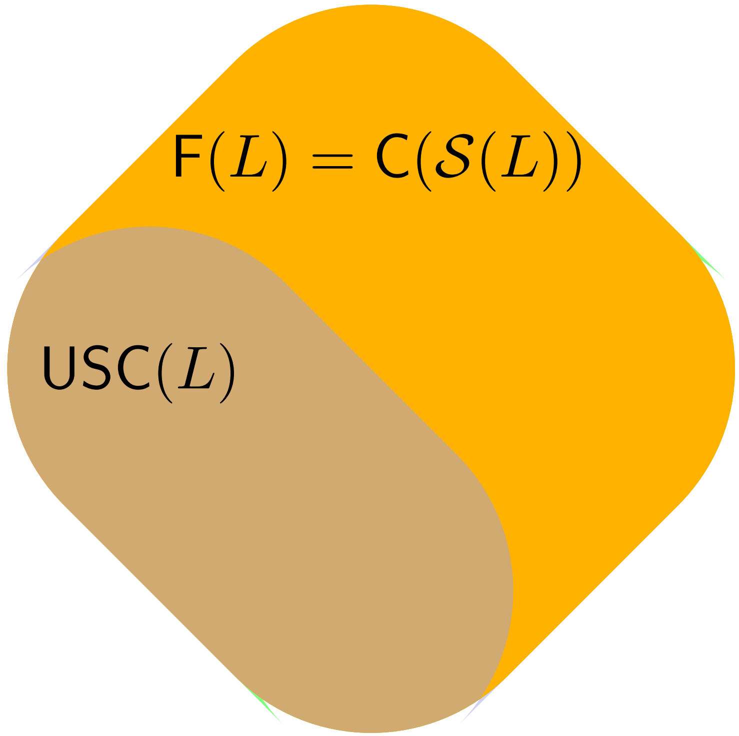

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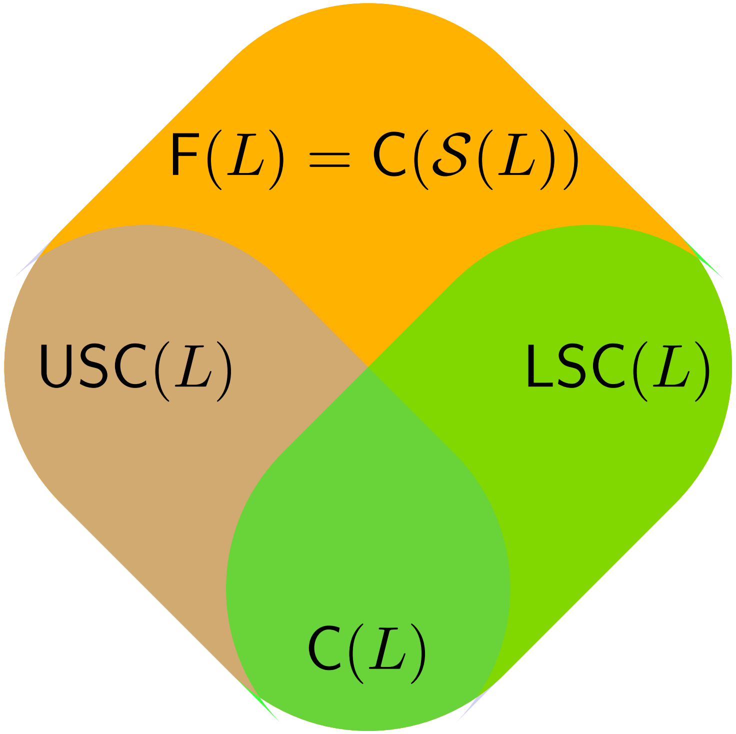

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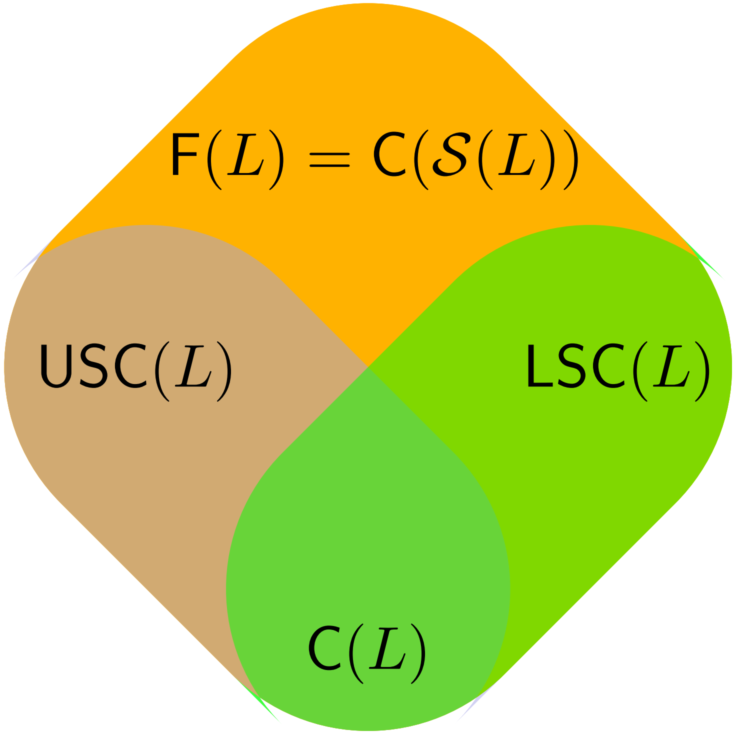

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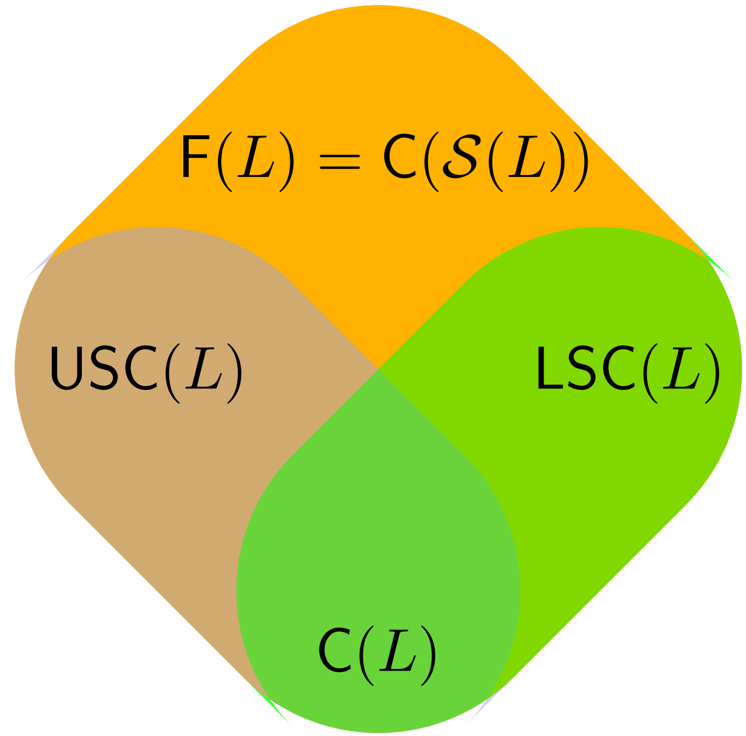

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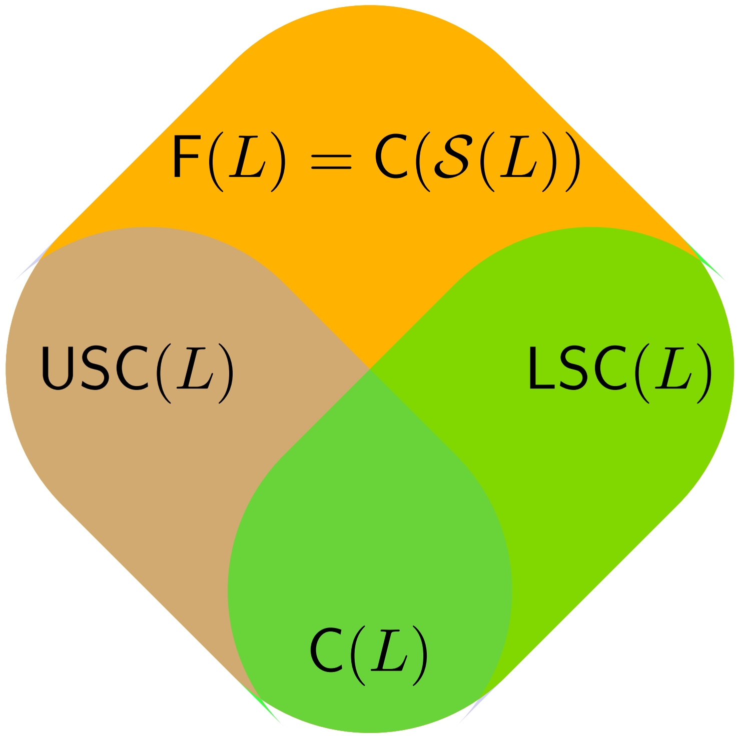

$\ni -f$

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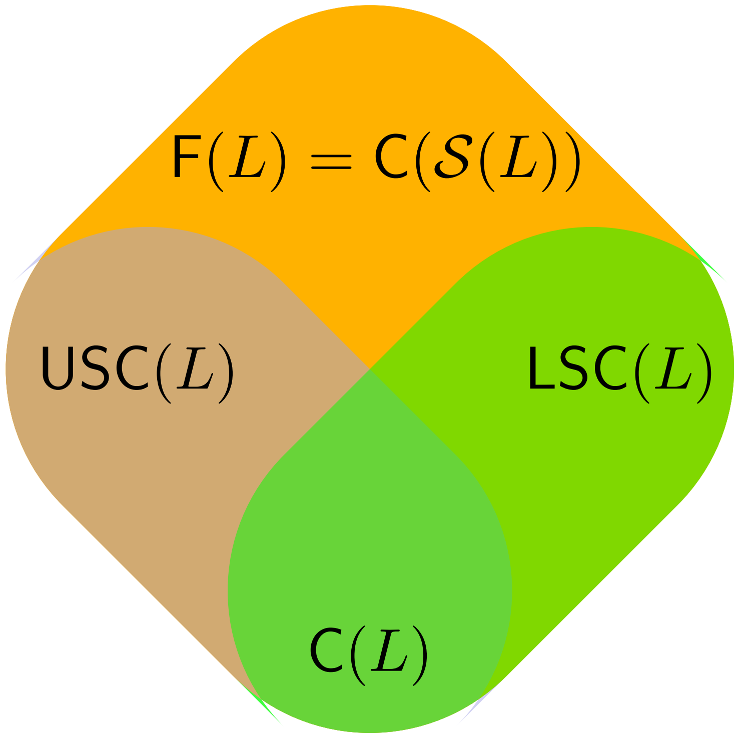

$f \in \mathsf{USC}(L)$ ... $\mathsf{LSC}(L) \ni -f$

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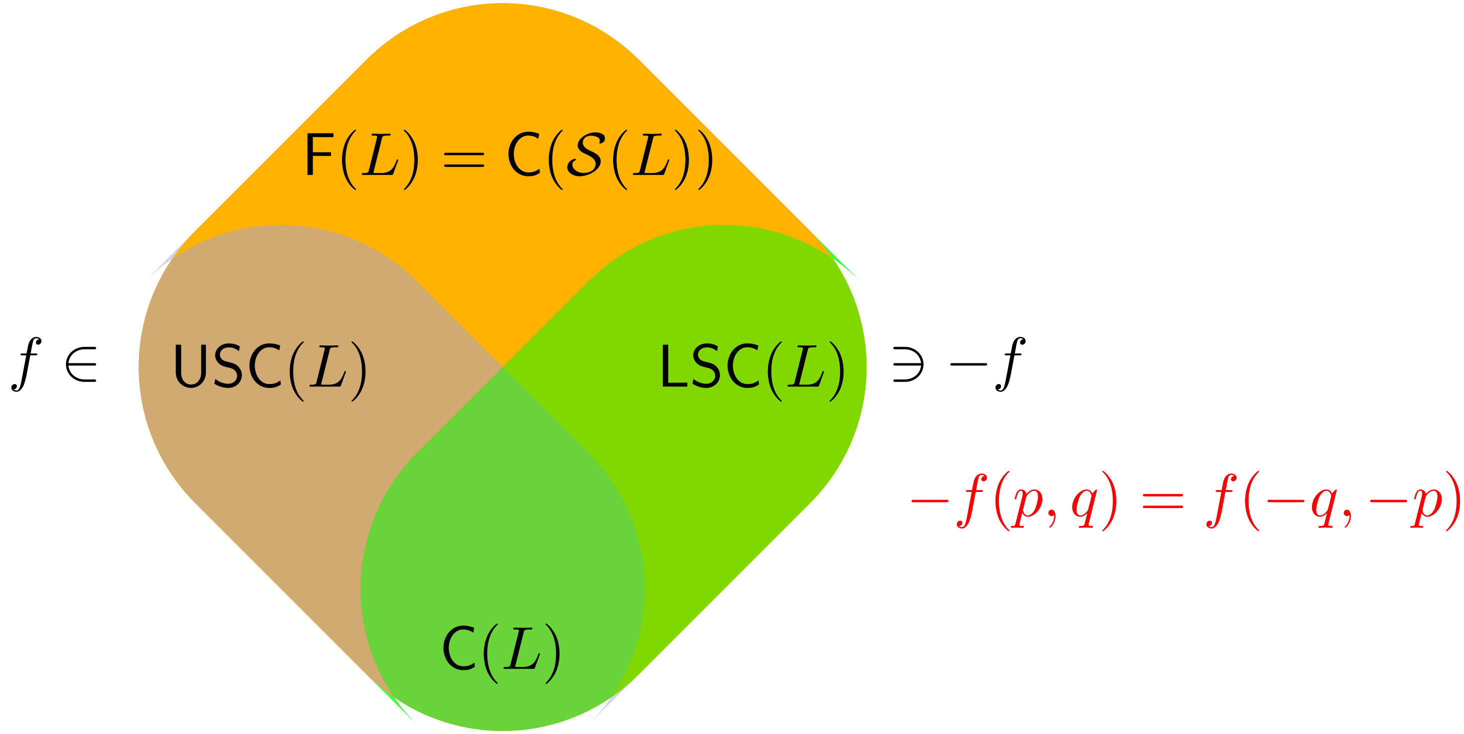

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- What else can be said about $(\mathsf{USC}(L), +, \cdot, \leq)$ and $(\mathsf{LSC}(L), +, \cdot, \leq)$?

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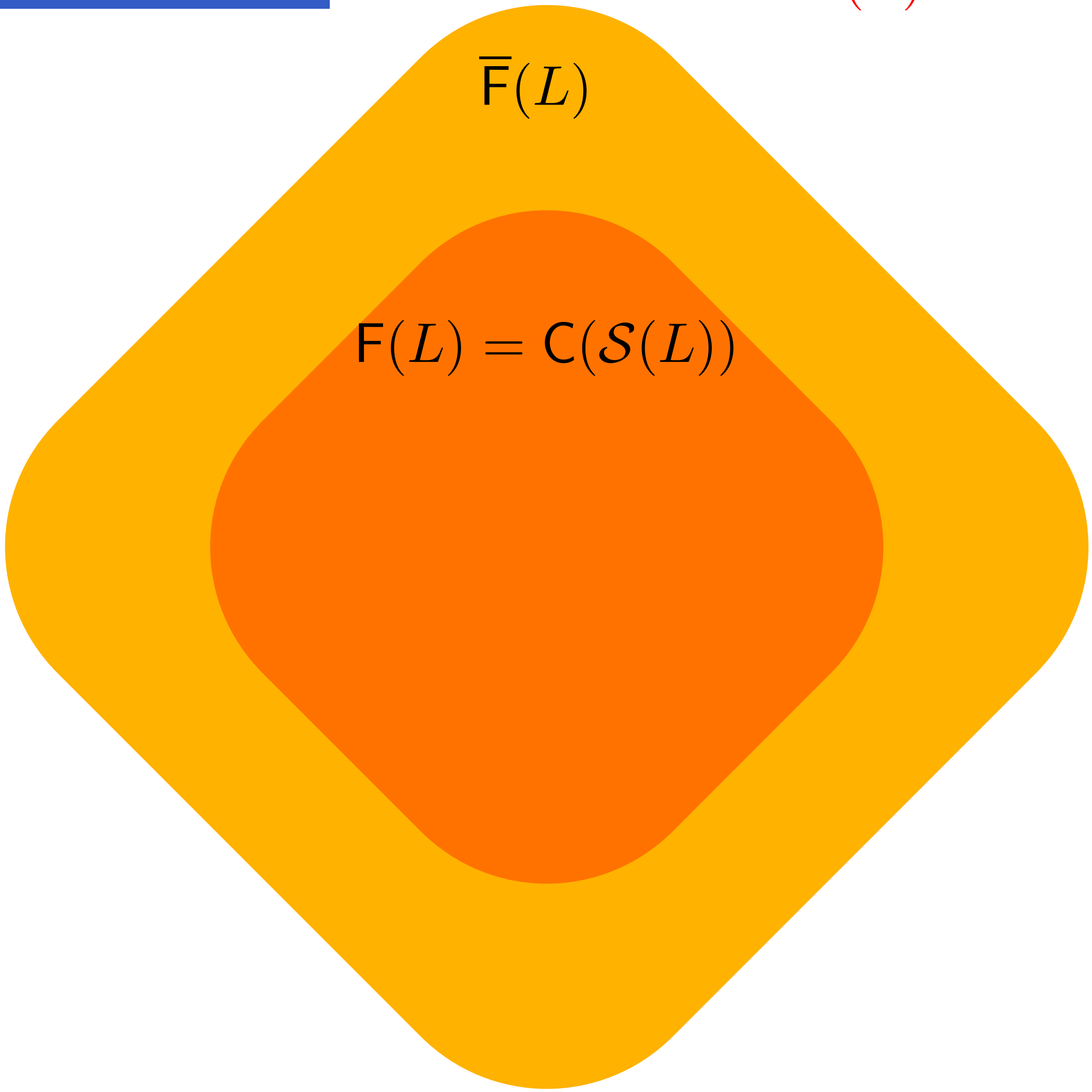

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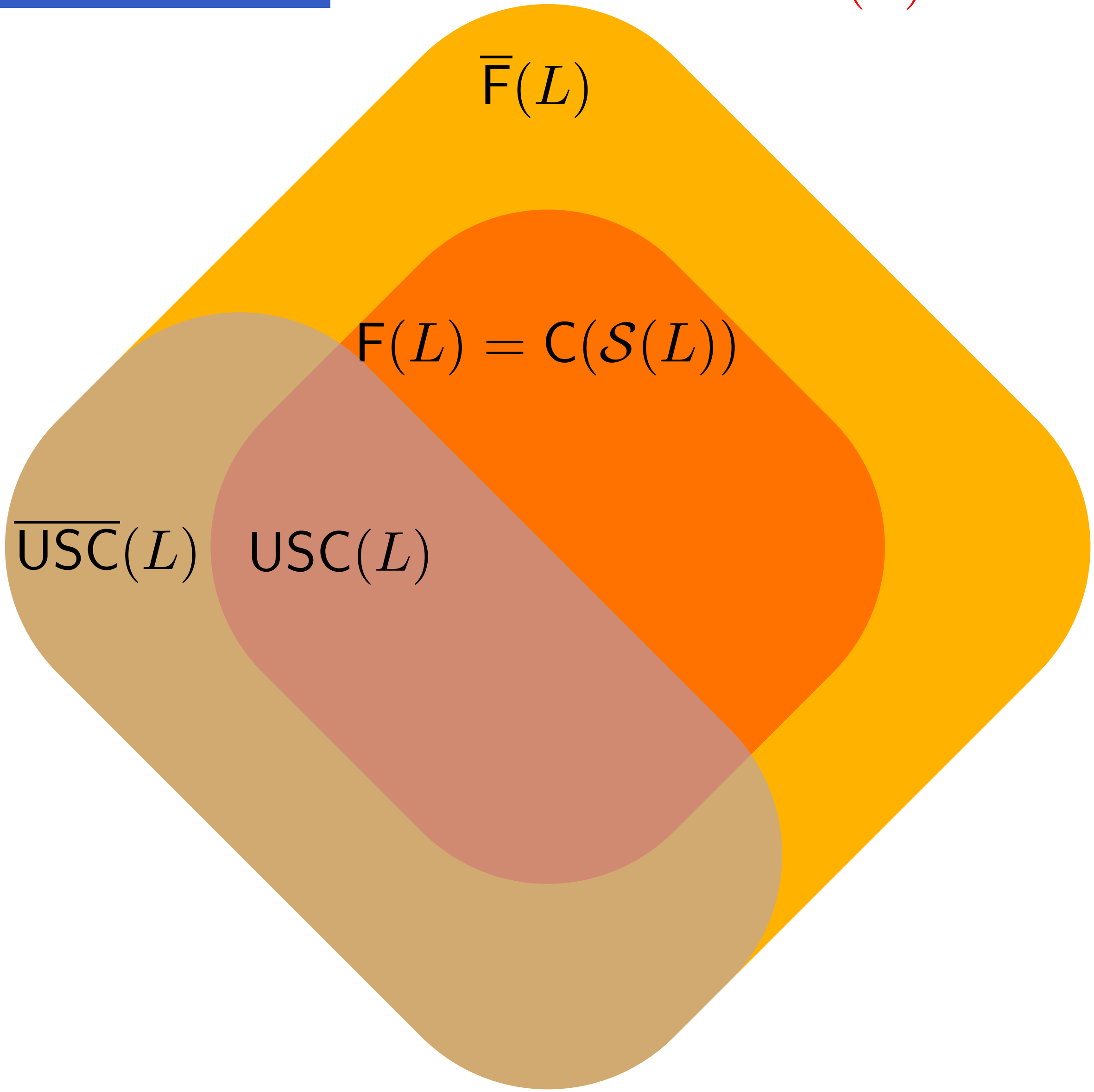

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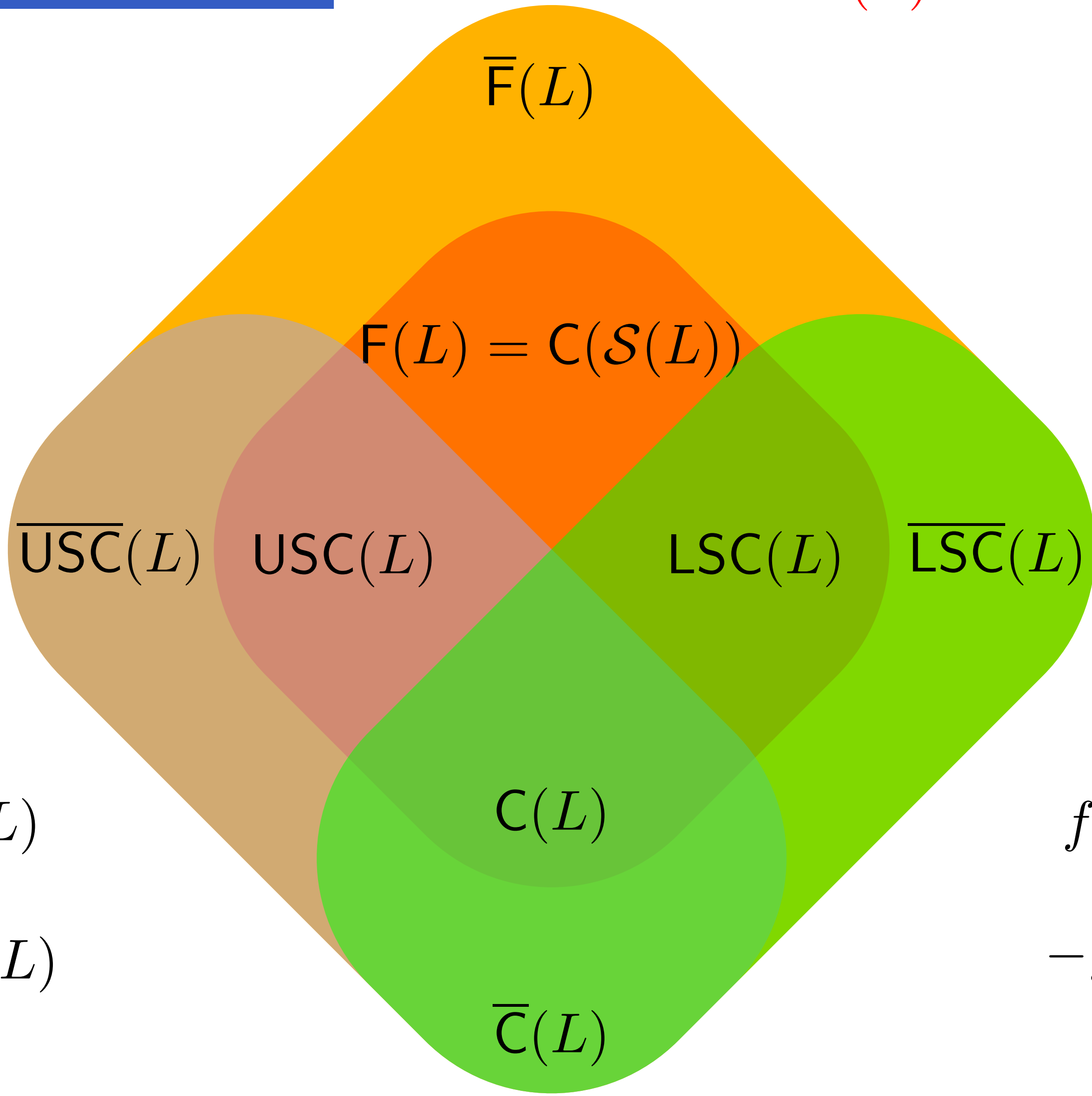

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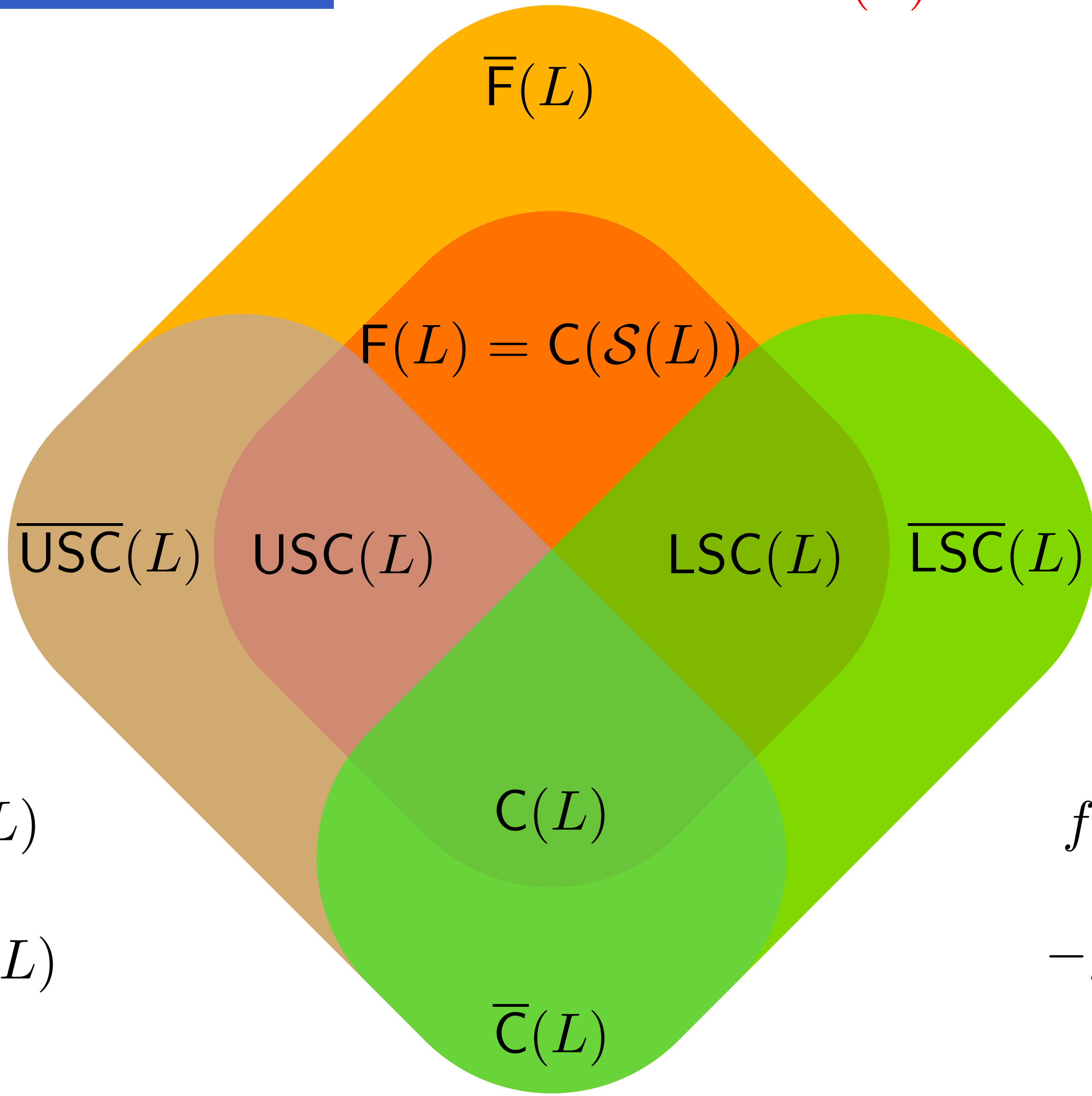

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- QUESTION 2: How to deal with extended real functions ?

SCALES: constructing real functions

$\mathcal{C} = \{c_p \mid p \in \mathbb{Q}\} \subseteq L$

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$\boxed{\mathbf{r}}$ $\qquad \mathcal{C}_r = \{c_p^r \mid p \in \mathbb{Q}\} = \{\dots, 1, 1, \dots, 1, \dots, 0, 0, \dots, 0, \dots\}$

$\uparrow$

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$r \in \mathbb{Q}, \quad \mathbf{r} : \mathfrak{L}(\mathbb{R}) \to L$

$\boxed{\mathbf{r}}$ $\quad \mathcal{C}_r = \{c_p^r \mid p \in \mathbb{Q}\} = \{\ldots, 1, 1, \ldots, 1, \ldots, 0, 0, \ldots, 0, \ldots\}$

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ALGEBRAIC OPERATIONS in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$

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$$= f^- \cdot g^- = \underbrace{(-f)}_{\mathsf{USC}(L)} \cdot \underbrace{(-g)}_{\mathsf{USC}(L)} \in \mathsf{USC}(L).$$

ALGEBRAIC OPERATIONS in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$

PROPOSITION. Let $f, g \in \mathsf{F}(L)$.

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⋮

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$f, g \in \overline{\mathsf{F}}(L)$ are sum compatible if

$$\left[\begin{array}{l} (a_f^+ \vee a_g^-) \wedge (a_g^+ \vee a_f^-) = 1, \\ a_f^+, a_f^-, a_g^+ \text{ and } a_g^- \text{ are complemented.} \end{array} \right.$$

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$\therefore$ $f \vee g \in \mathrm{D}(L)$. By inversion $f \wedge g = -((-f) \vee (-g)) \in \mathrm{D}(L)$.

$\therefore$ $\mathrm{D}(L)$ is a lattice with inversion

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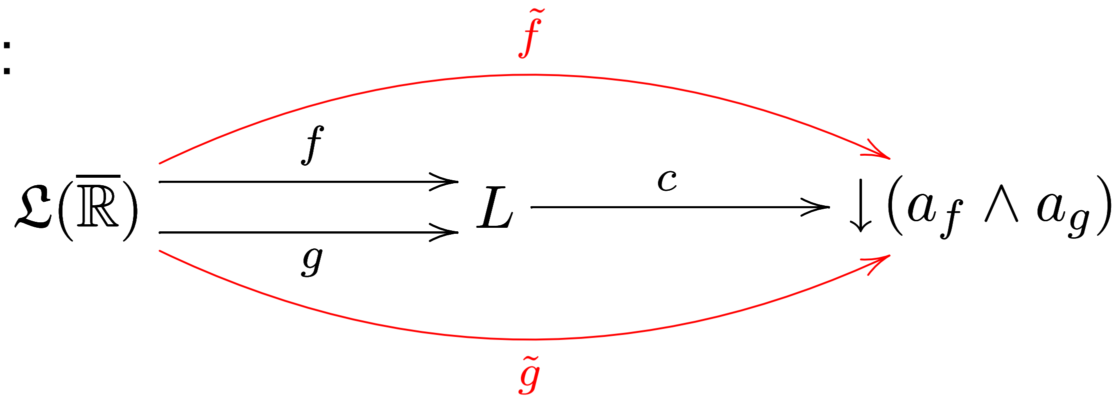

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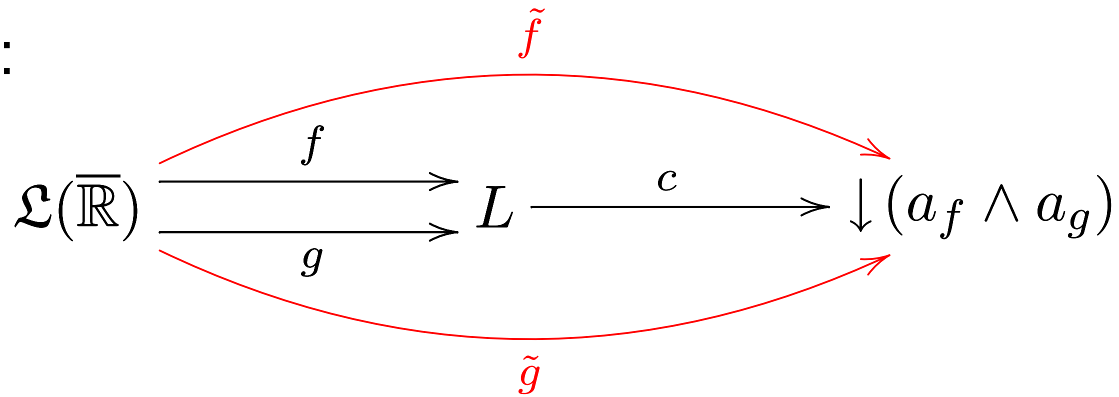

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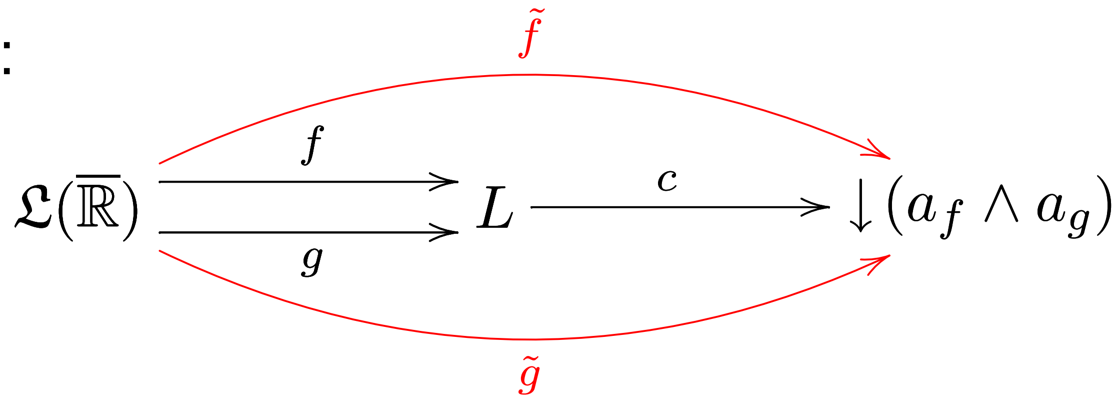

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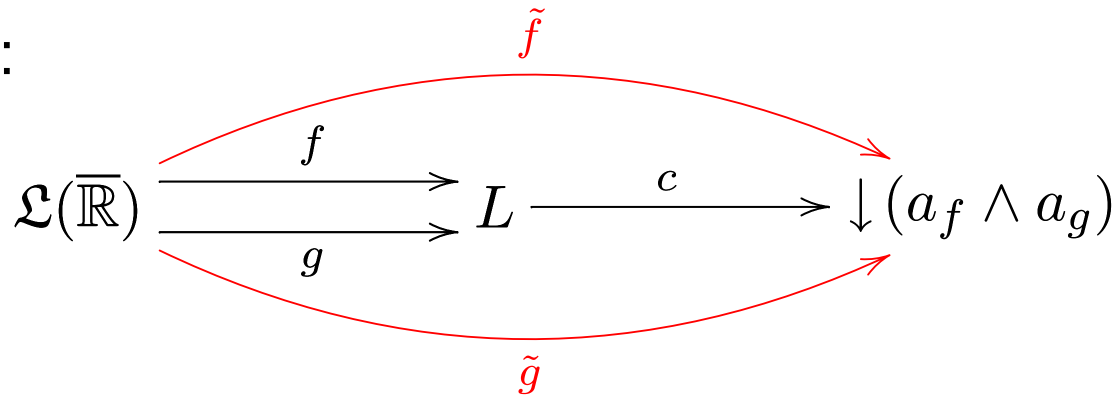

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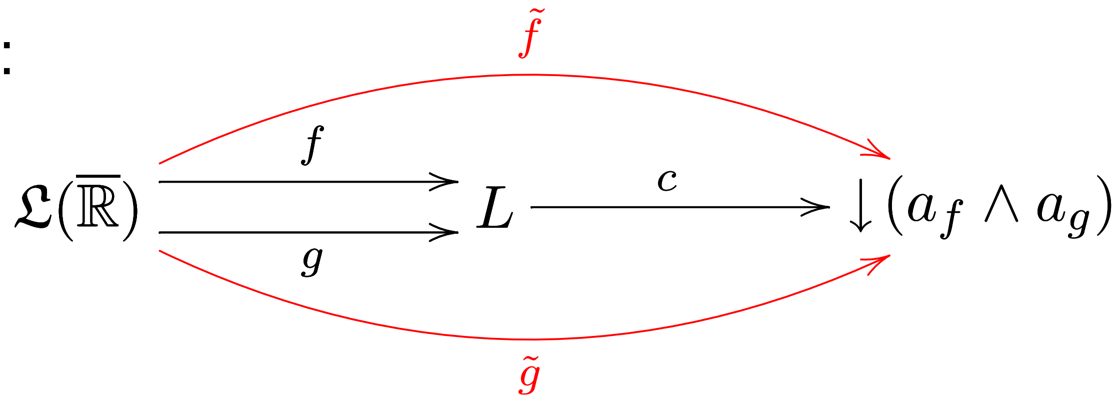

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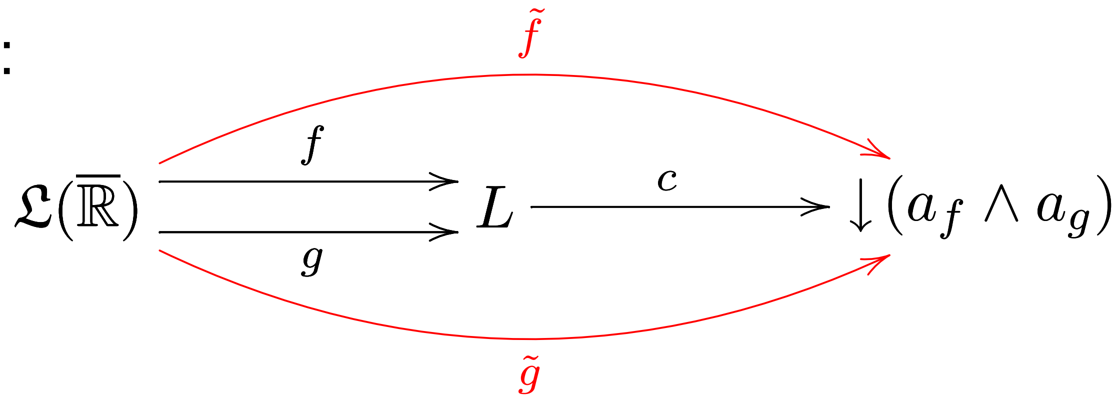

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