On the duality between normality and extremal disconnectedness

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— joint work with J. Gutiérrez García (UPV-EHU, Bilbao, Spain)

THEOREM. TFAE for a locale L:

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5. The image of L under any surjective closed localic map is normal.

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ext. disc.

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open

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5. The image of L under any surjective ~~closed~~ open localic map is ~~normal~~ ext. disc..

This shapes the idea that the two notions are somehow dual to each other and therefore can be studied in parallel.

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- there is a variety of classical insertion type results
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  Can one unify them under a single general result ?

NORMALITY vs EXTREMAL DISCONNECTEDNESS

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[De Morgan frames]

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as lattices: L is normal iff L^{op} is extremally disconnected.

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need only for $a, b \in L^*$ (<u>regular</u>) $(a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0)$

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(FRAME) (CO-FRAME)

VARIANTS OF NORMALITY

Normal

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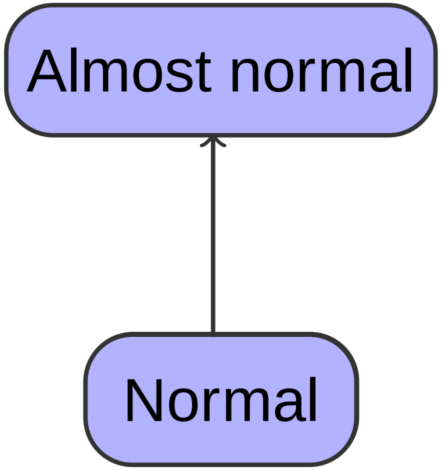
$(a, b \in L)$

VARIANTS OF NORMALITY



a: <u>regular</u>

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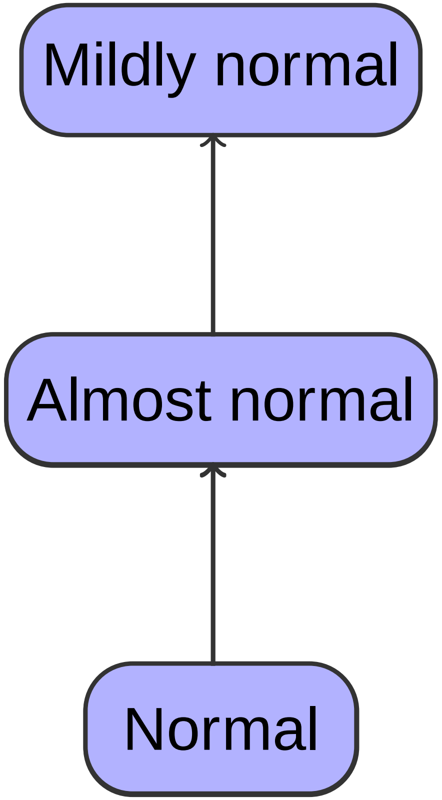

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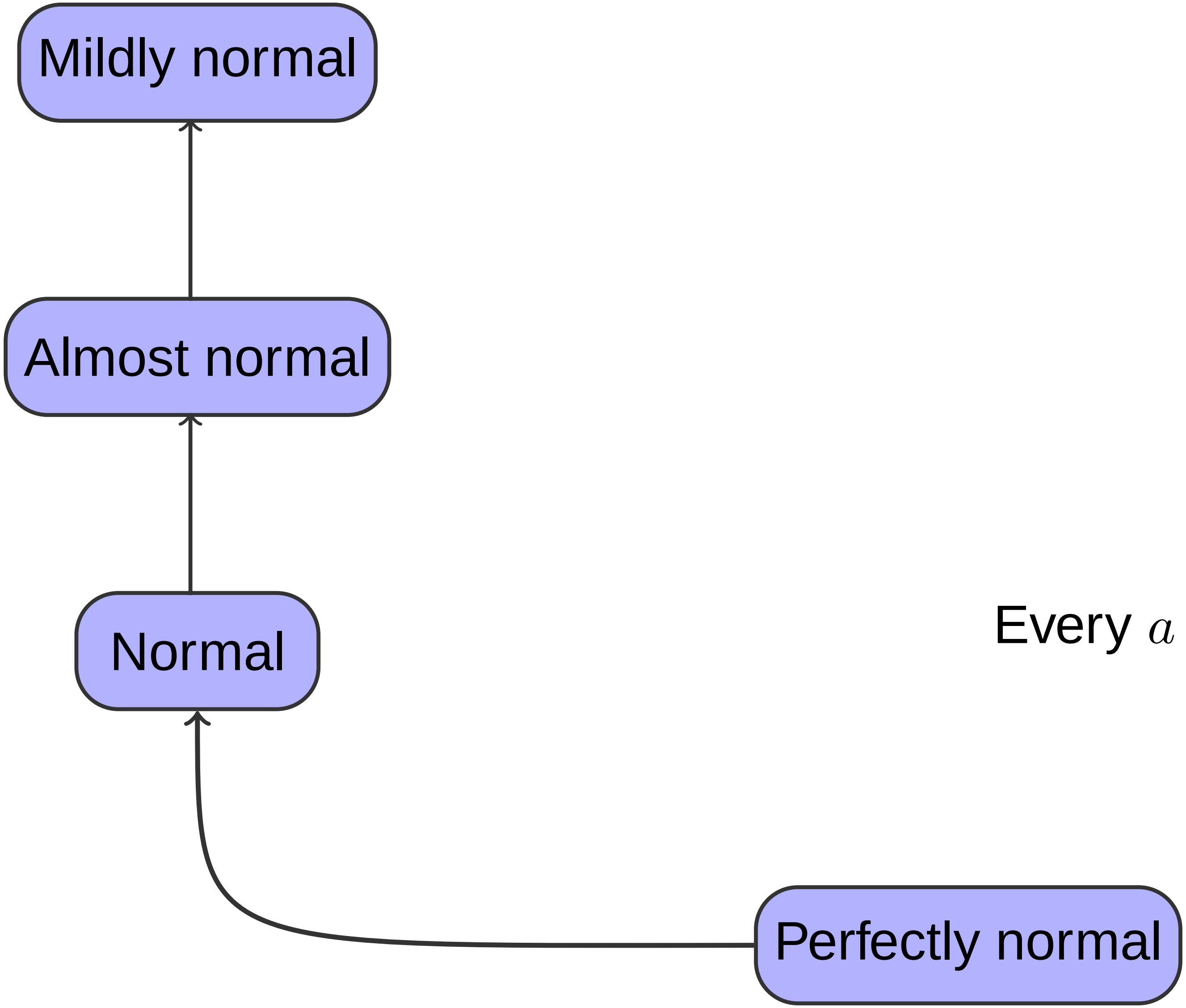


Every $a \in L$ is regular G_δ

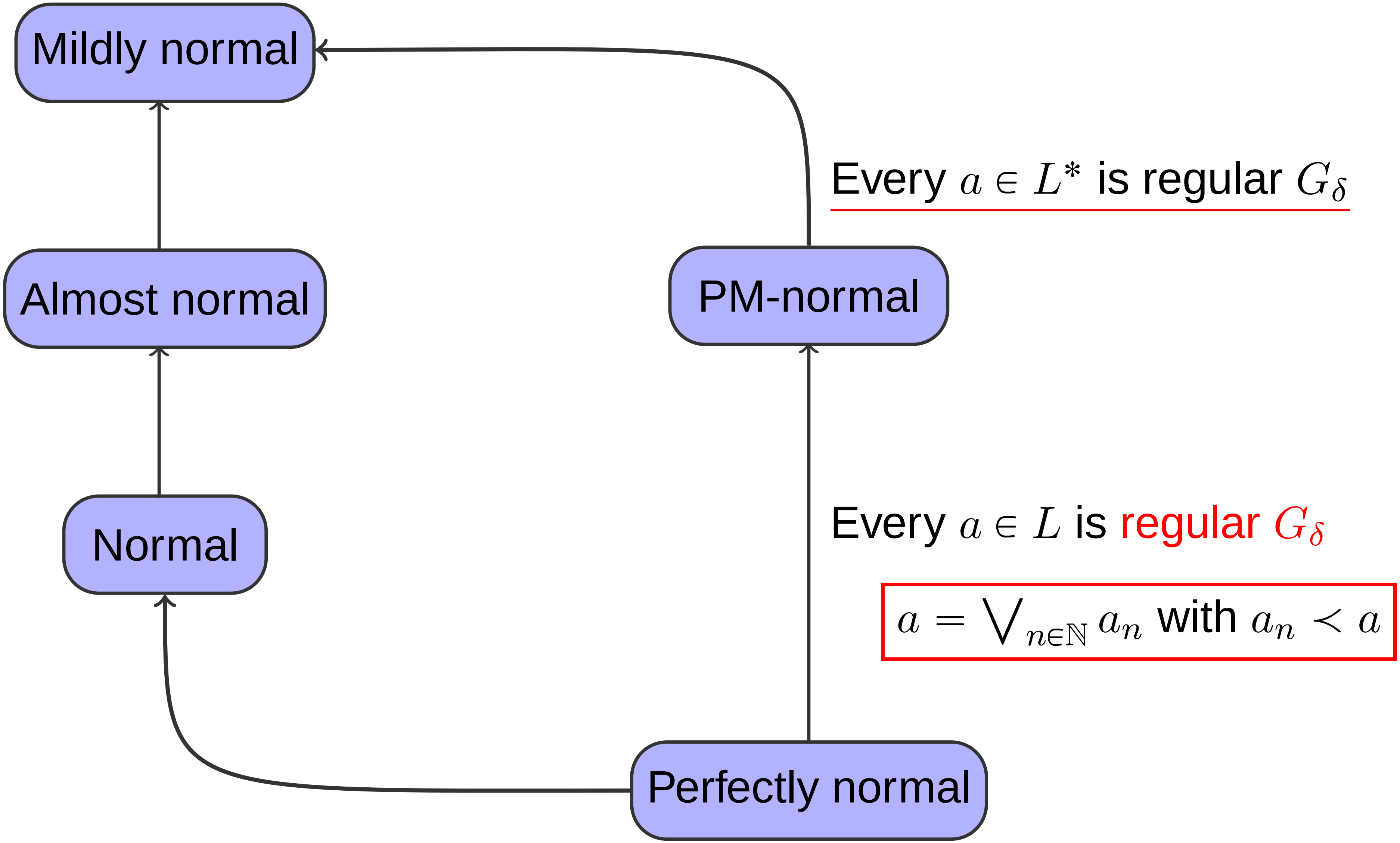
VARIANTS OF NORMALITY

$$
\begin{tikzcd}
\text{Mildly normal} & \\
\text{Almost normal} \arrow[u] & \\
\text{Normal} \arrow[u] & \\
& \text{Perfectly normal} \arrow[ul]
\end{tikzcd}
$$

Every $a \in L$ is regular G_δ

$$\boxed{a = \bigvee_{n\in\mathbb{N}} a_n \text{ with } a_n \prec a}$$

VARIANTS OF NORMALITY



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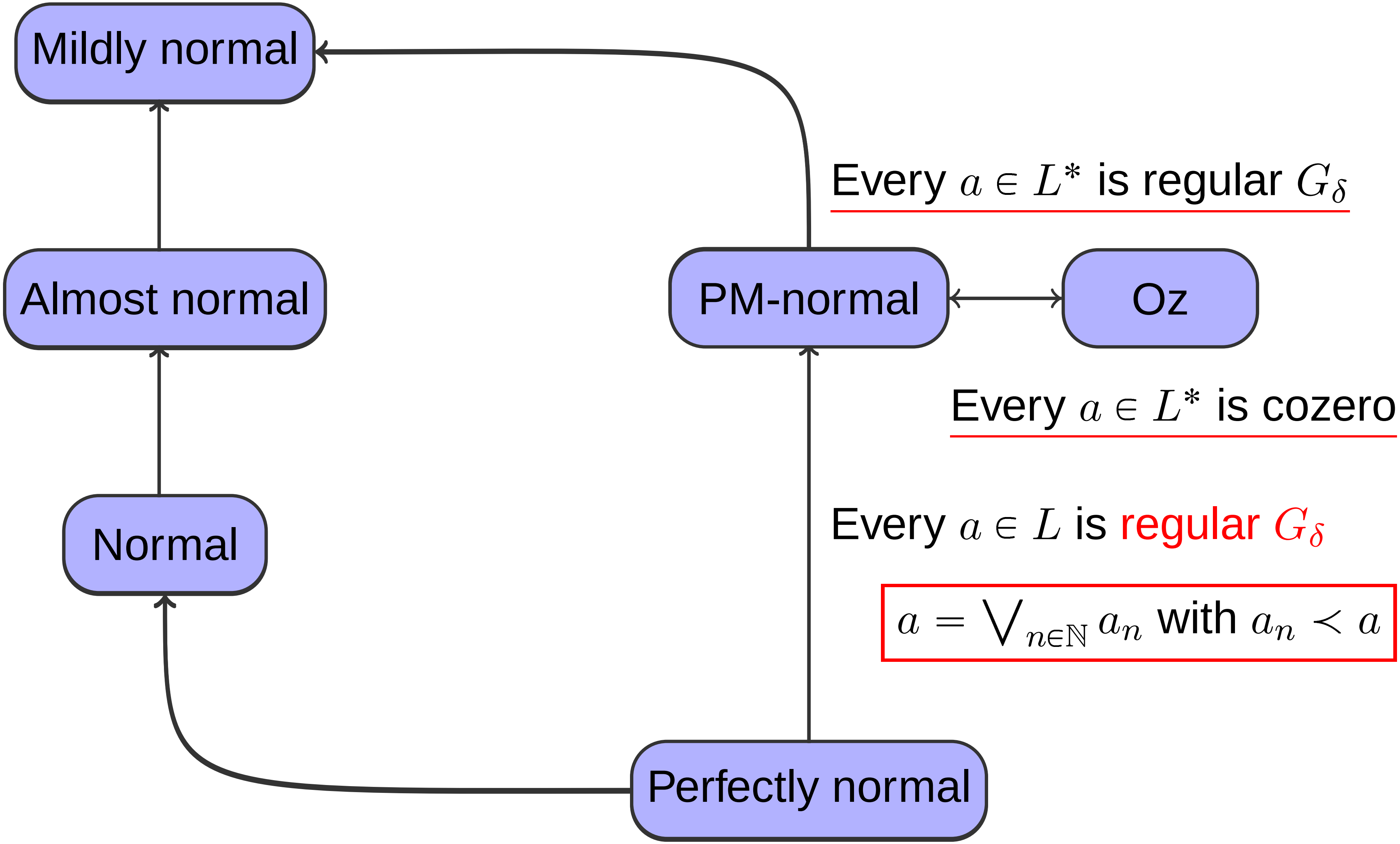

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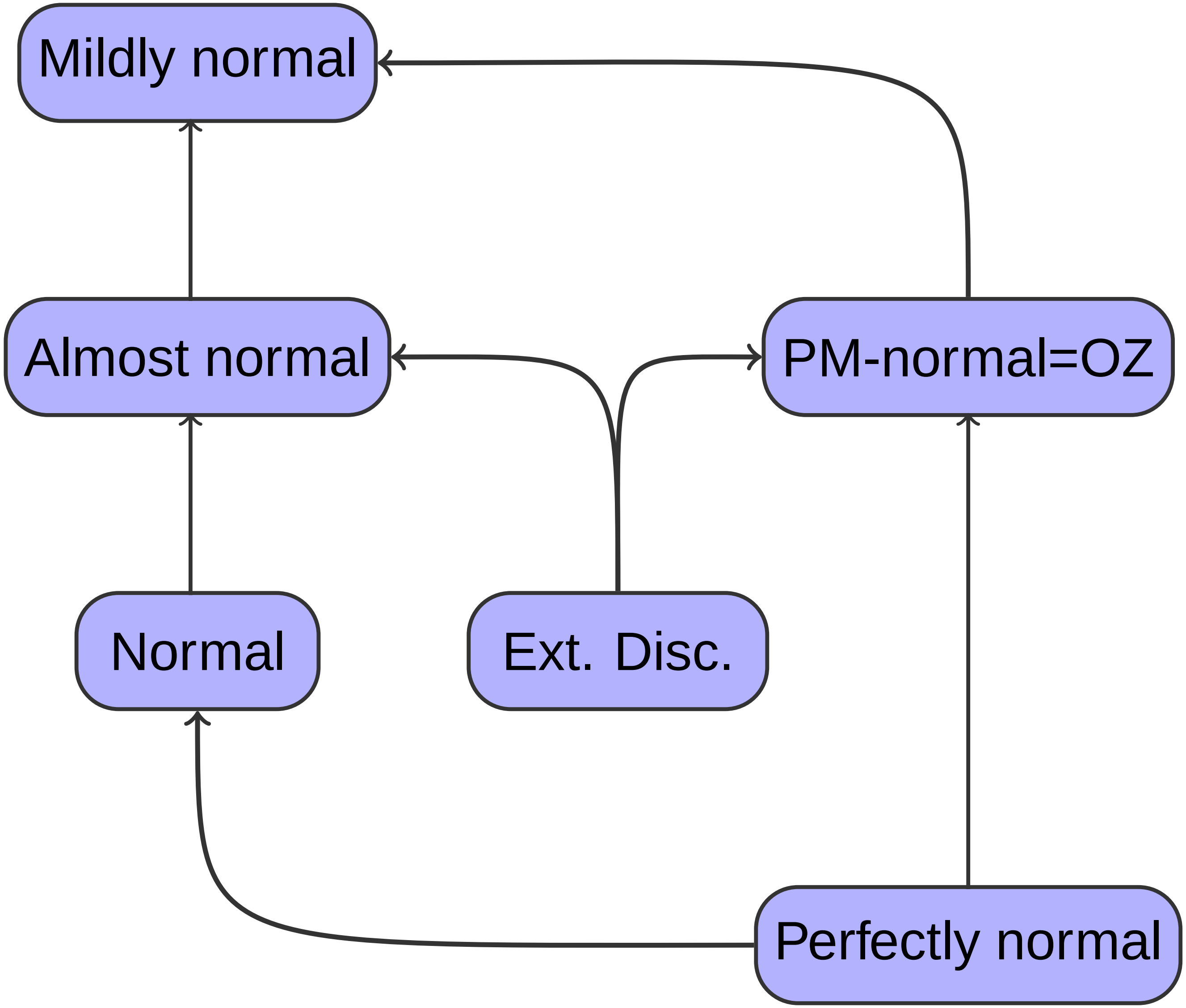

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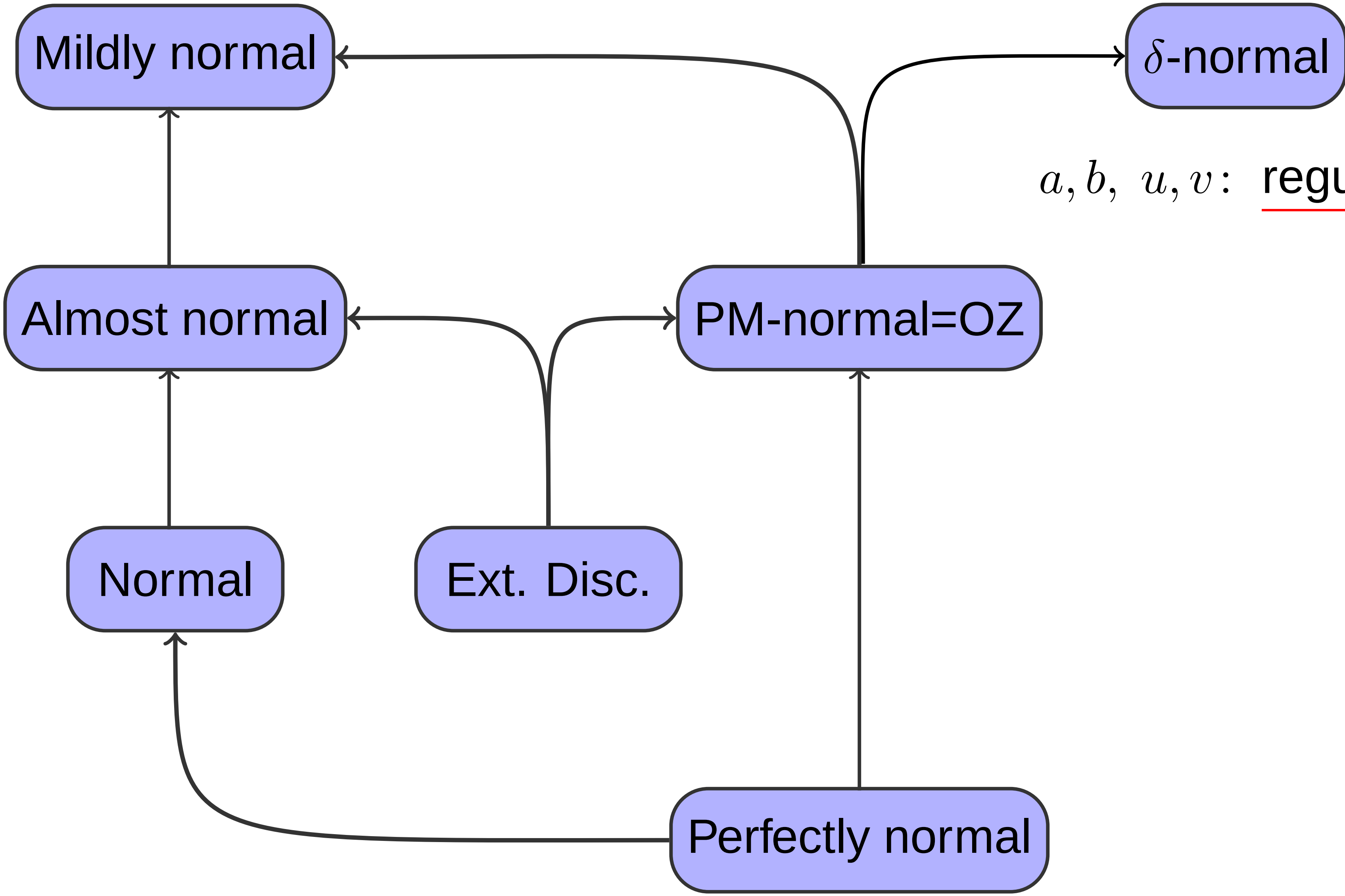
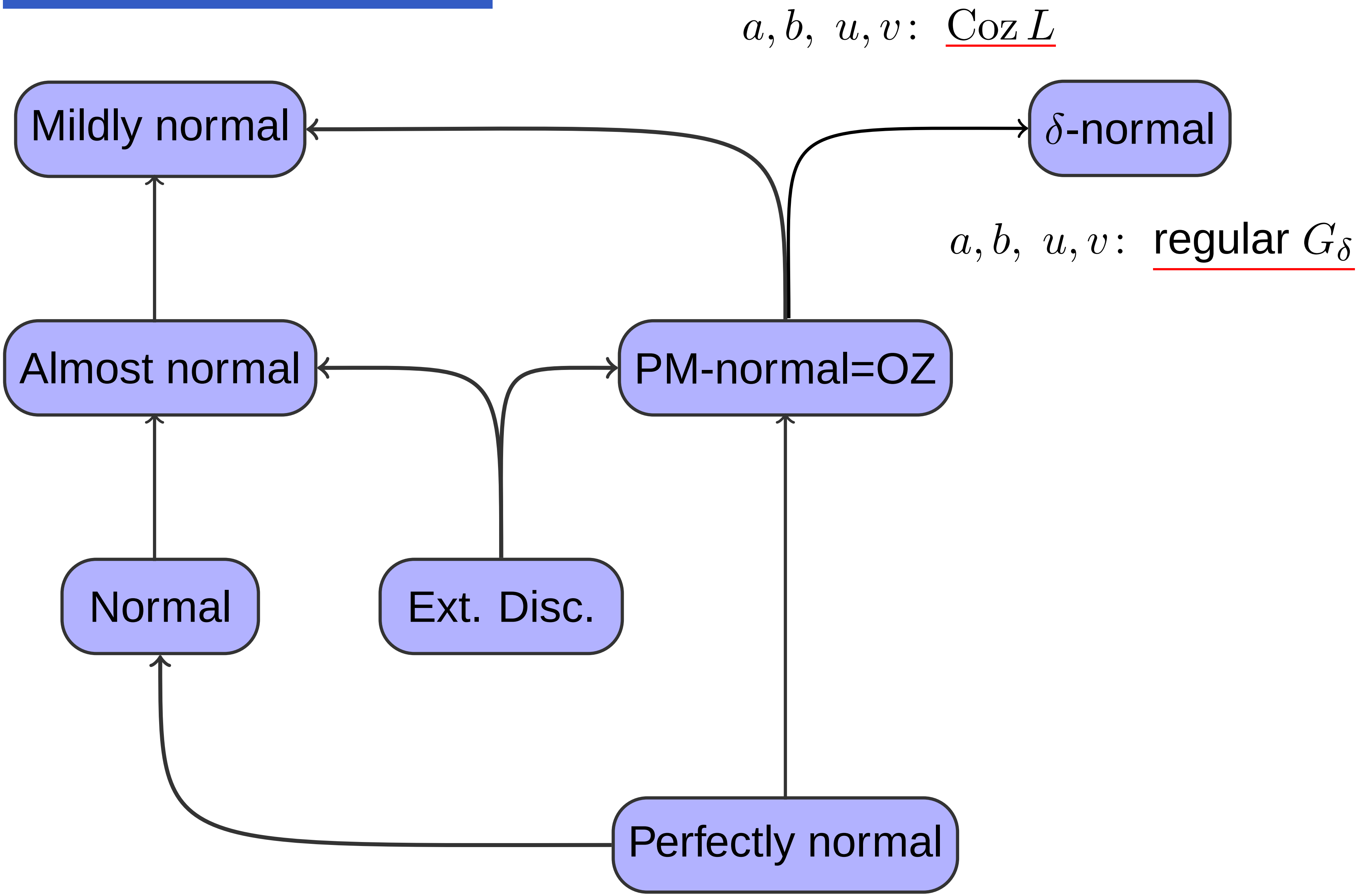


$a, b,\ u, v:$ regular G_δ

VARIANTS OF NORMALITY



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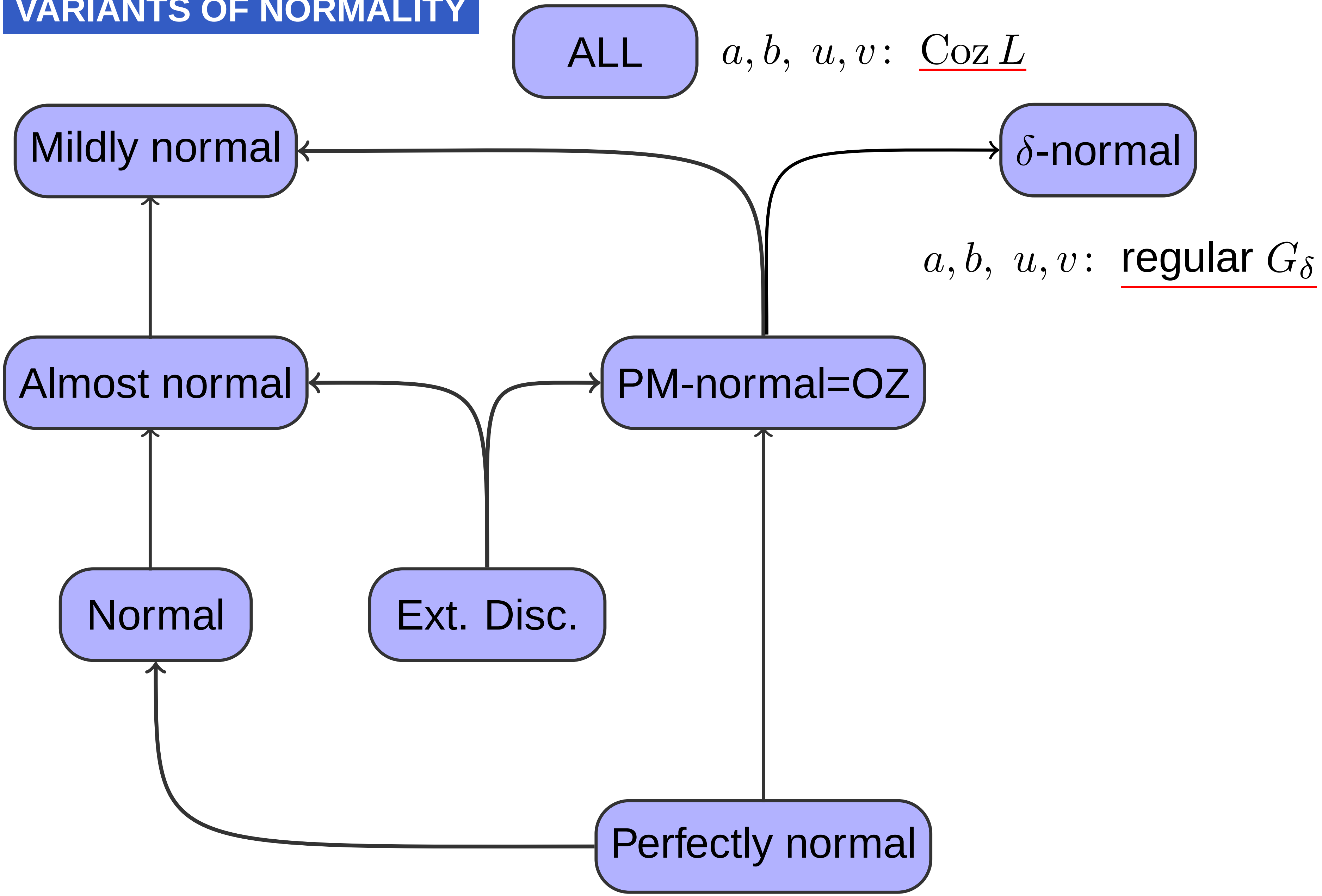
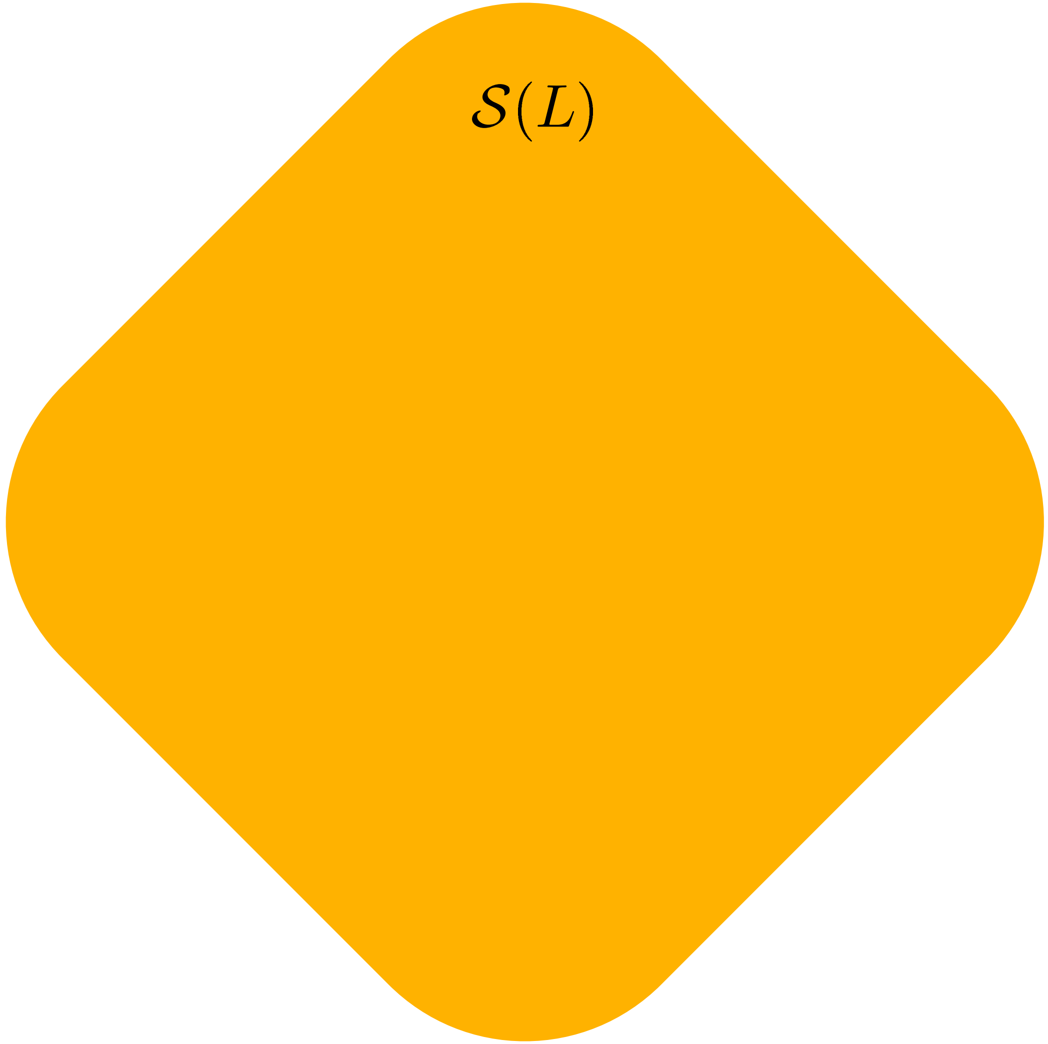


IDEA: go to $\mathcal{S}(L)$, take complements

the frame of sublocales



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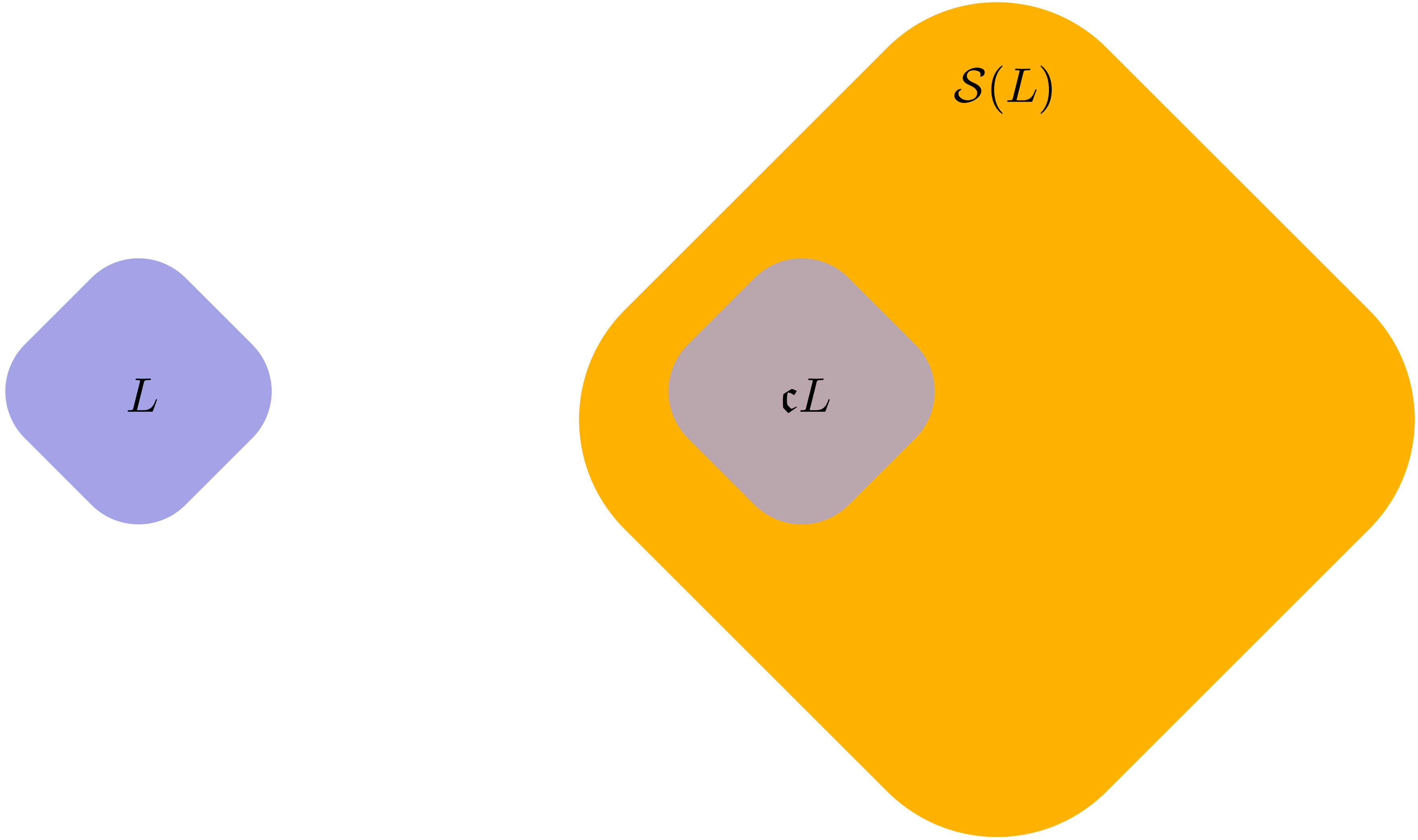

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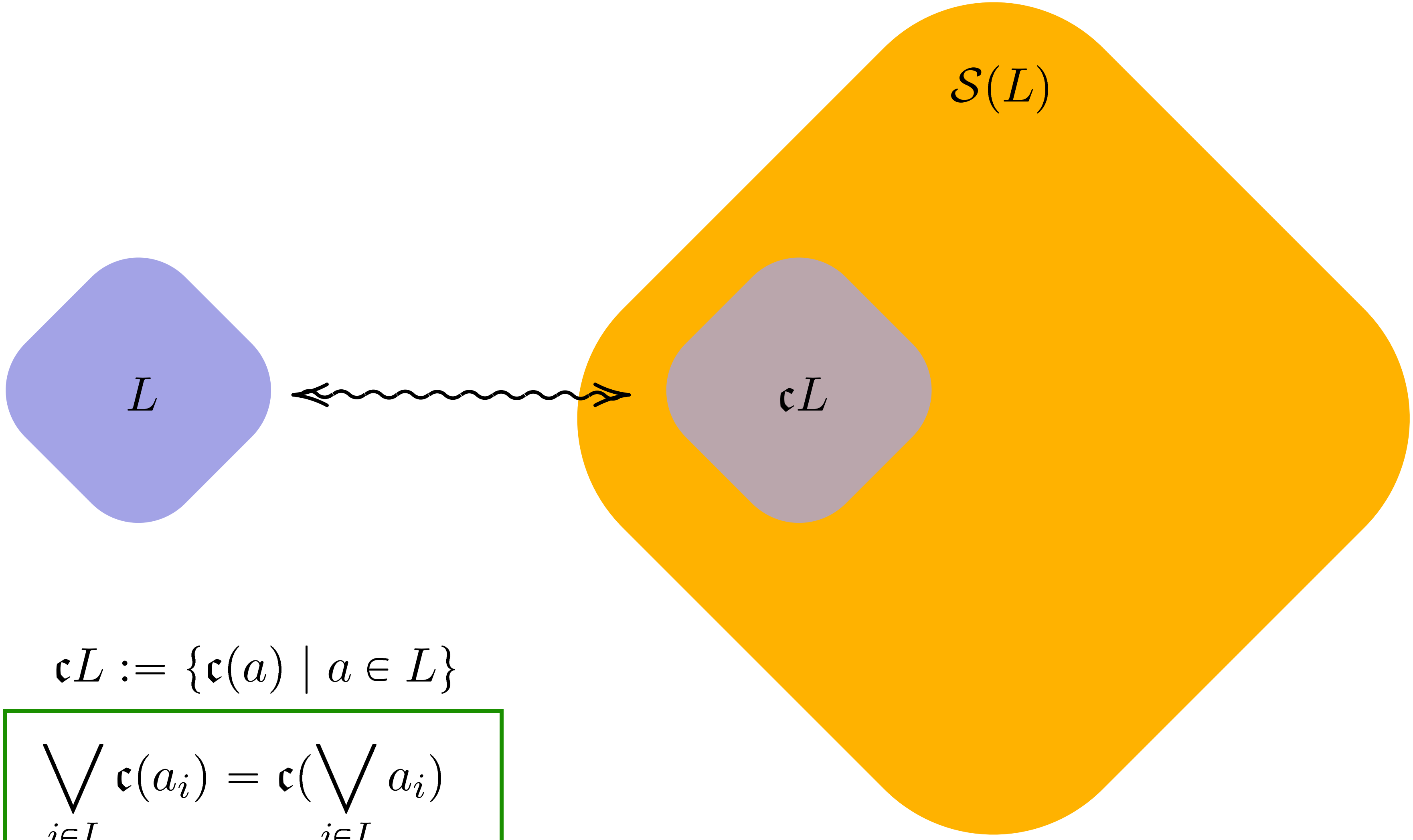

$$\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\boxed{\begin{aligned} \bigvee_{i\in I} \mathfrak{c}(a_i) &= \mathfrak{c}\Big(\bigvee_{i\in I} a_i\Big) \\ \mathfrak{c}(a) \wedge \mathfrak{c}(b) &= \mathfrak{c}(a \wedge b) \end{aligned}}$$

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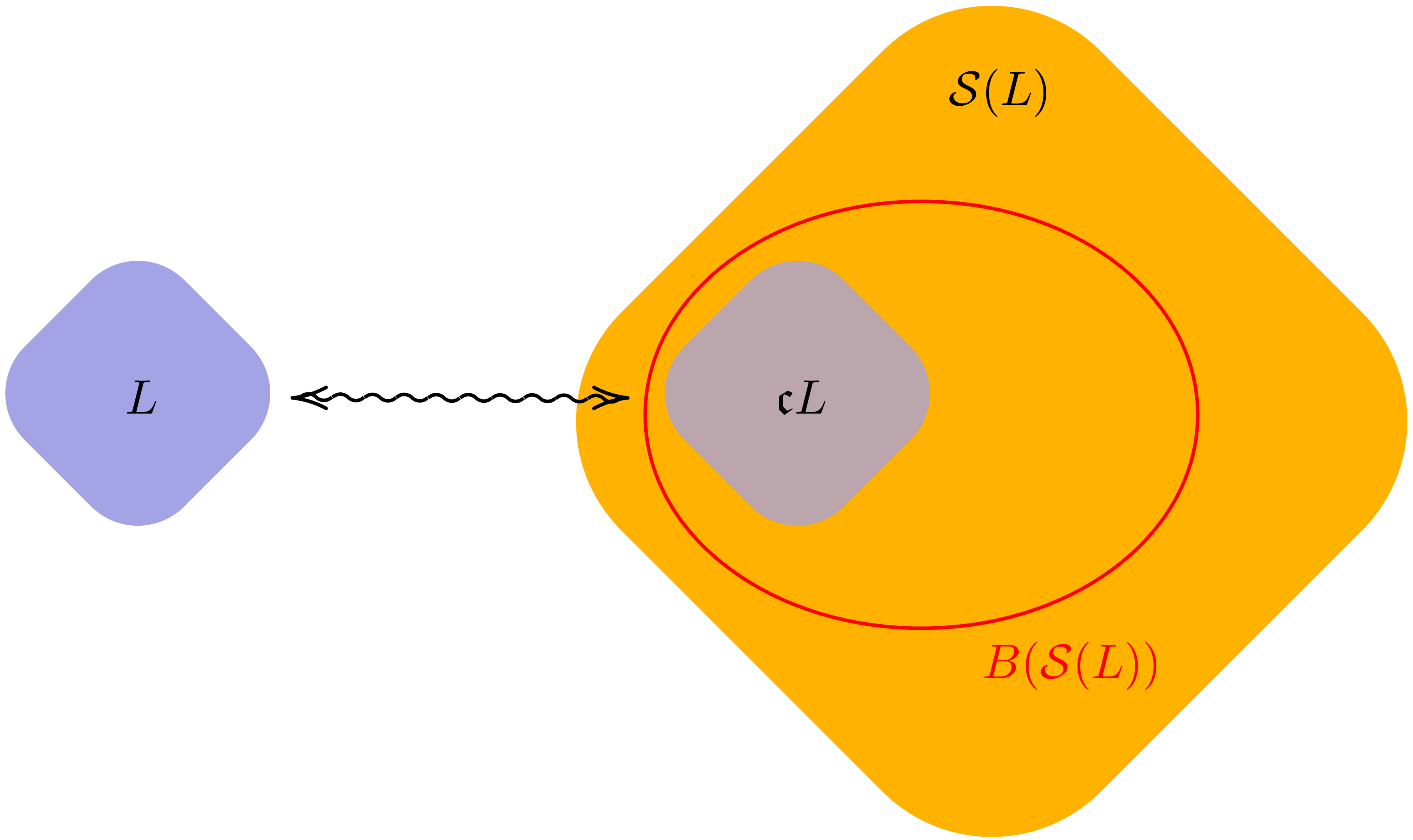

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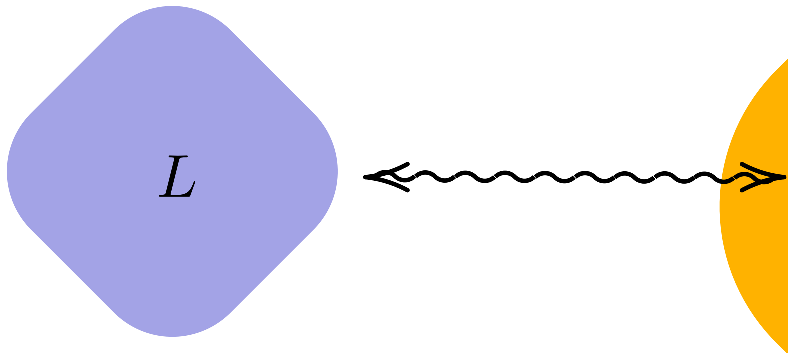

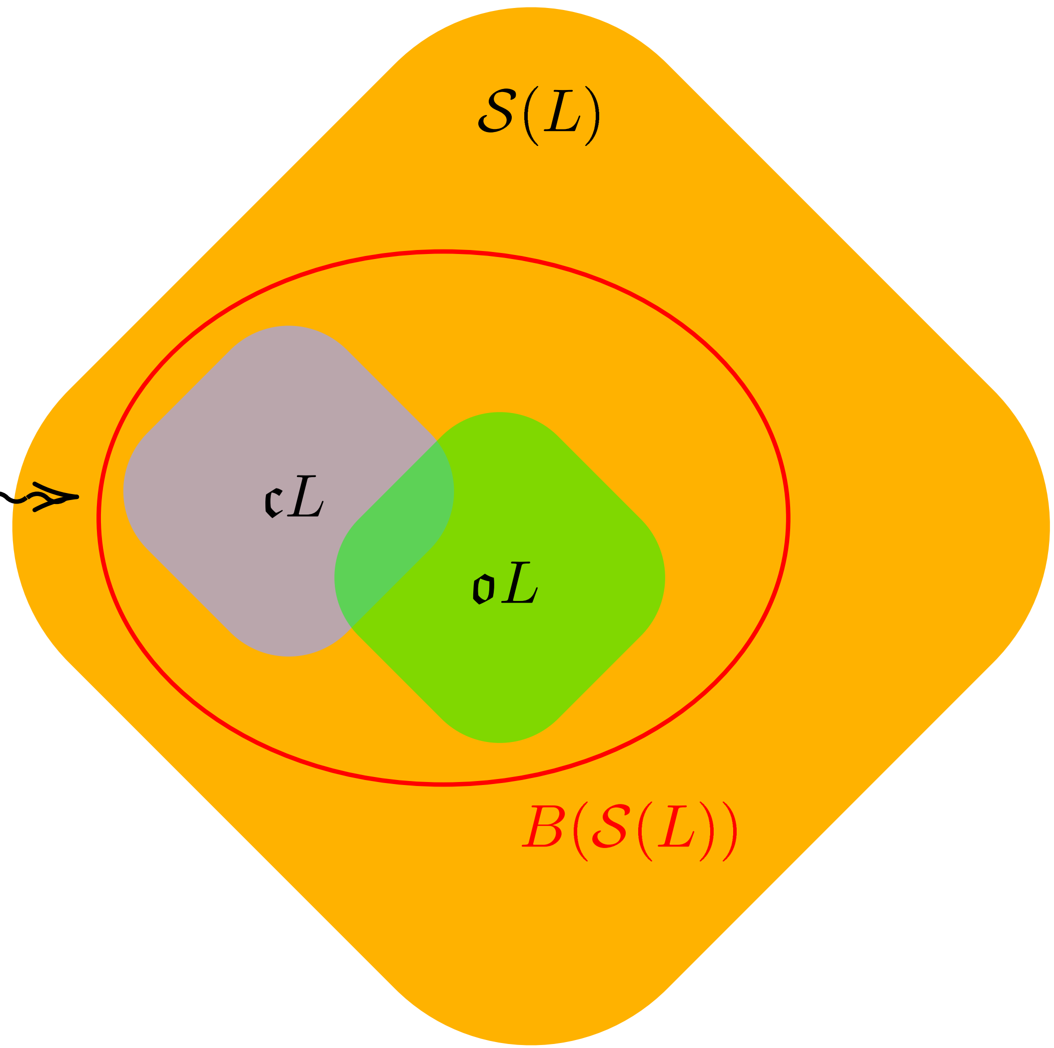

$$\mathfrak{o}L := \{\mathfrak{o}(a) \mid a \in L\}$$

$$\boxed{\begin{array}{l} \bigwedge_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \\ \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \end{array}}$$

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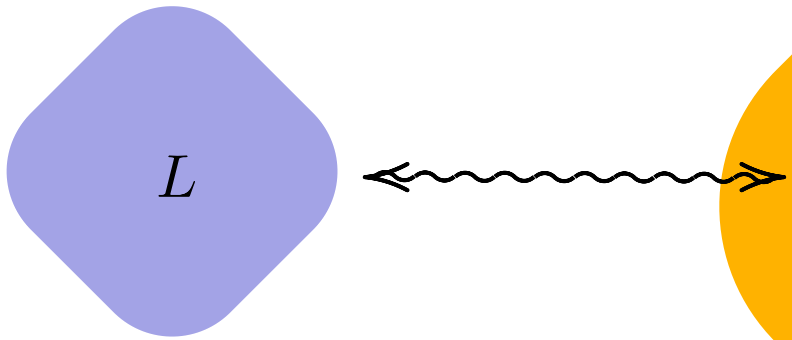

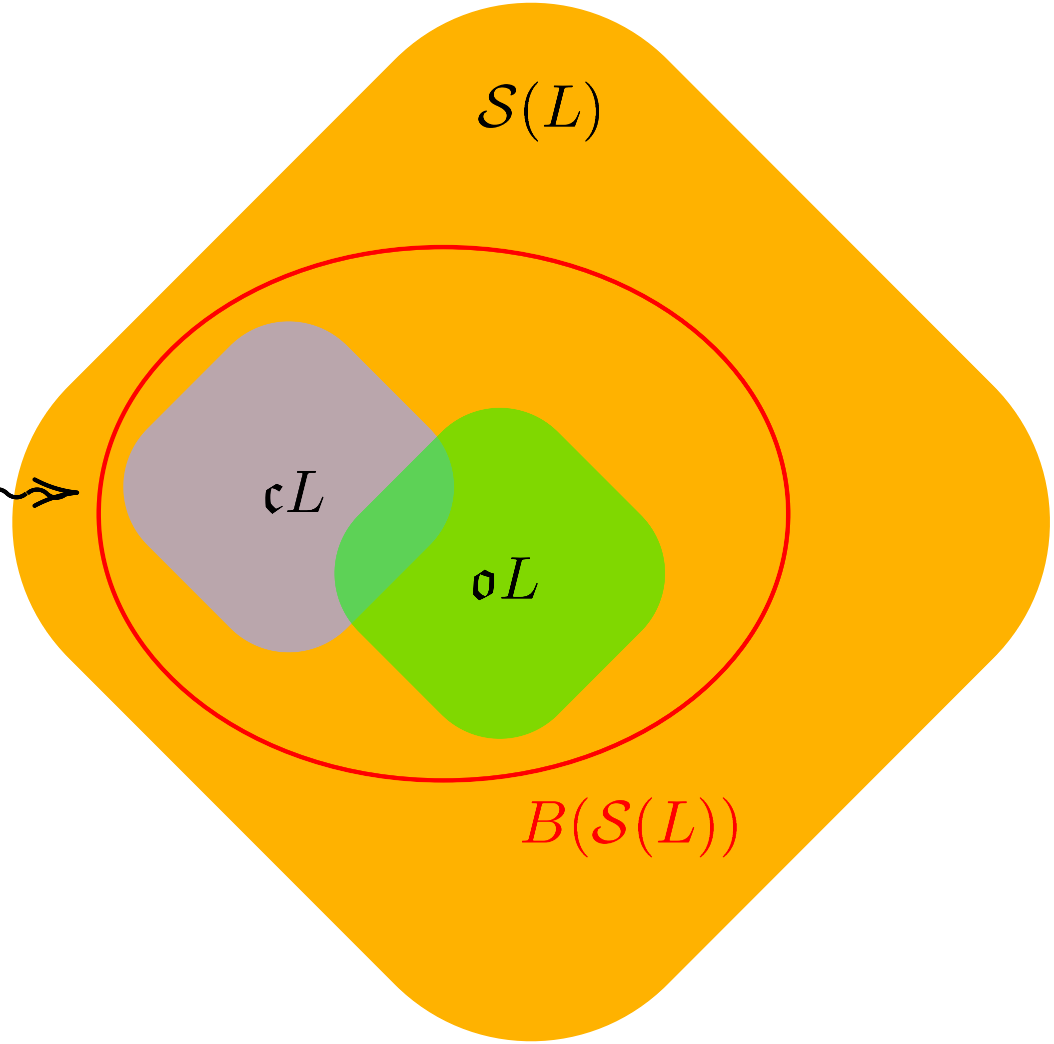

L is normal $\Leftrightarrow$ $\mathfrak{c}L$ is normal

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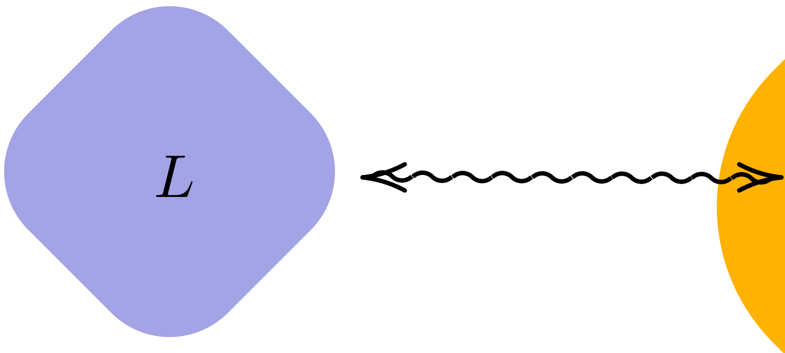
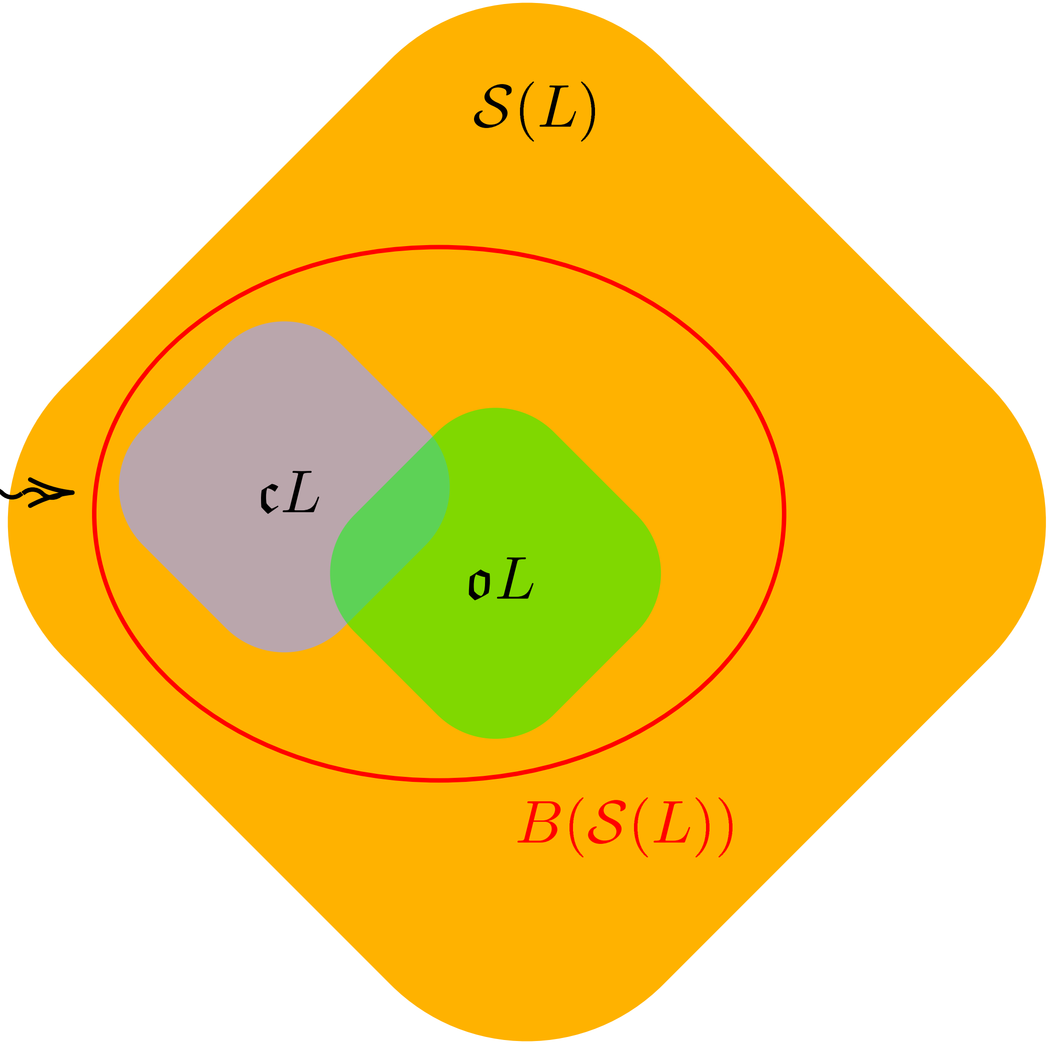


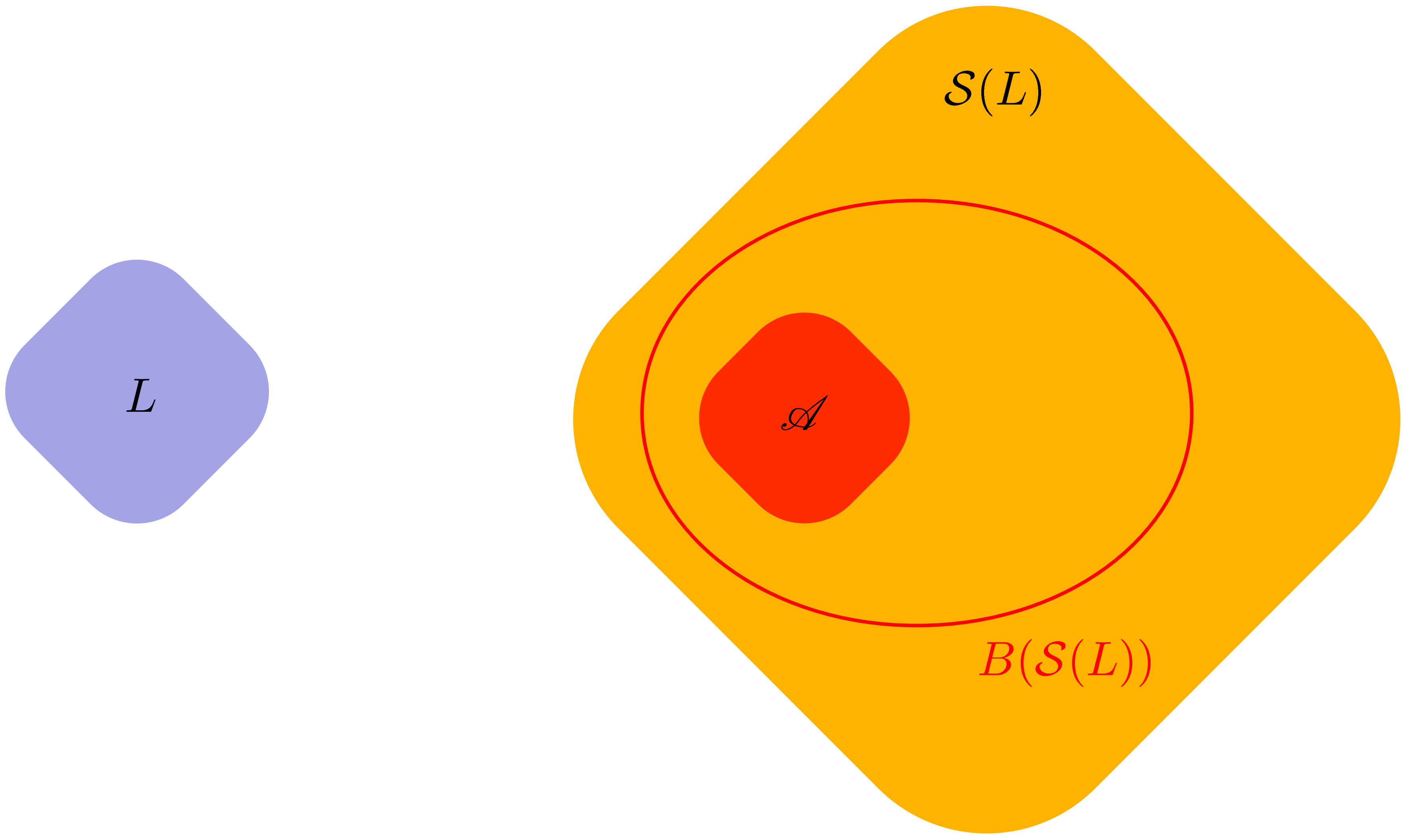


L is normal $\Leftrightarrow$ $\mathfrak{c}L$ is normal

$\Leftrightarrow$ $\mathfrak{o}L$ is extremally disconnected

$\mathscr{A}$-normality vs $\mathscr{A}$-extremal disconnectedness

the frame of sublocales



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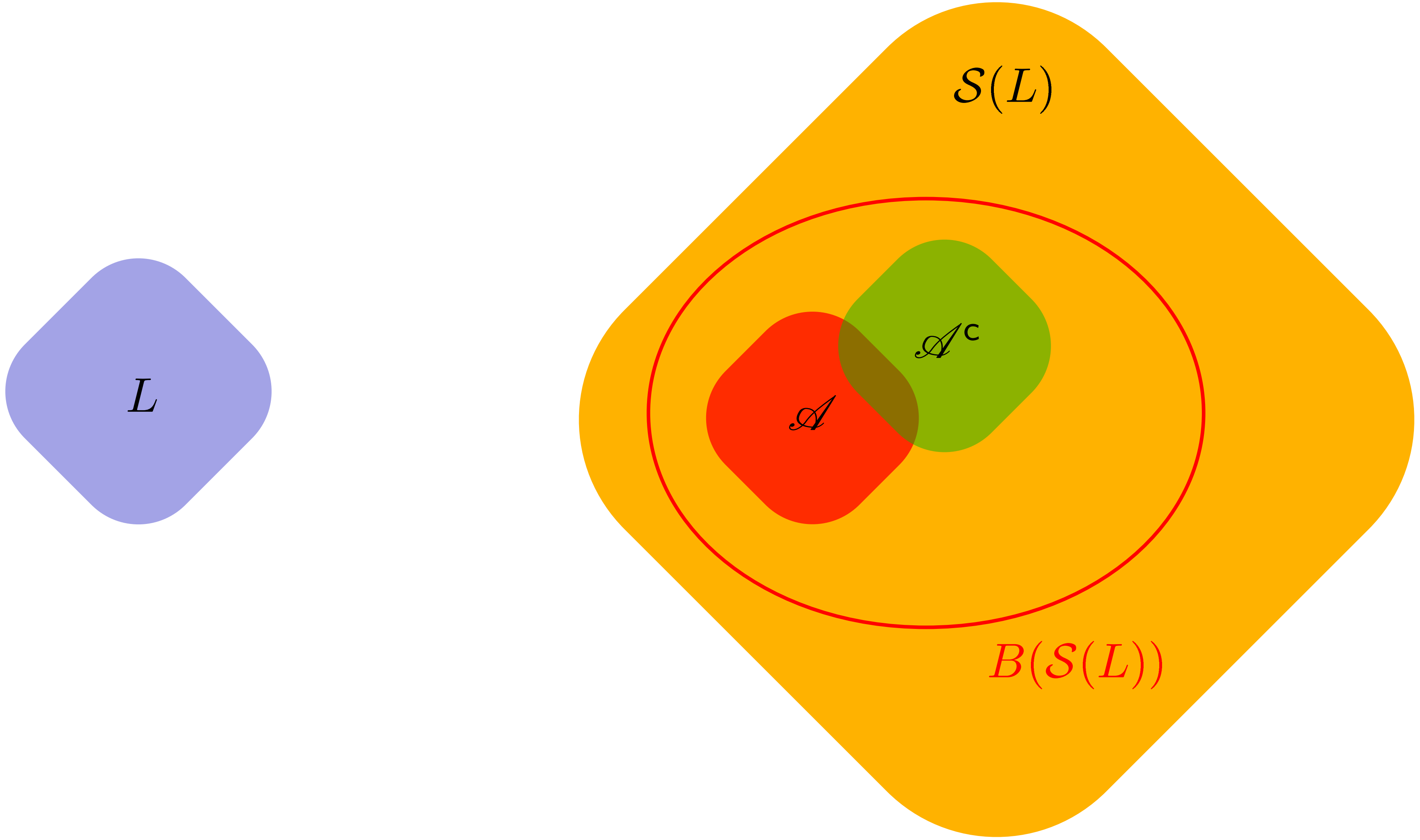

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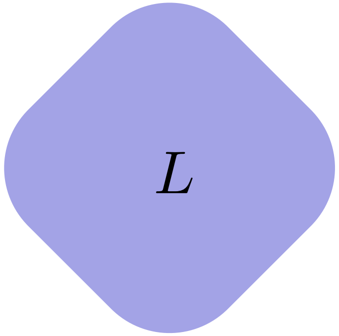

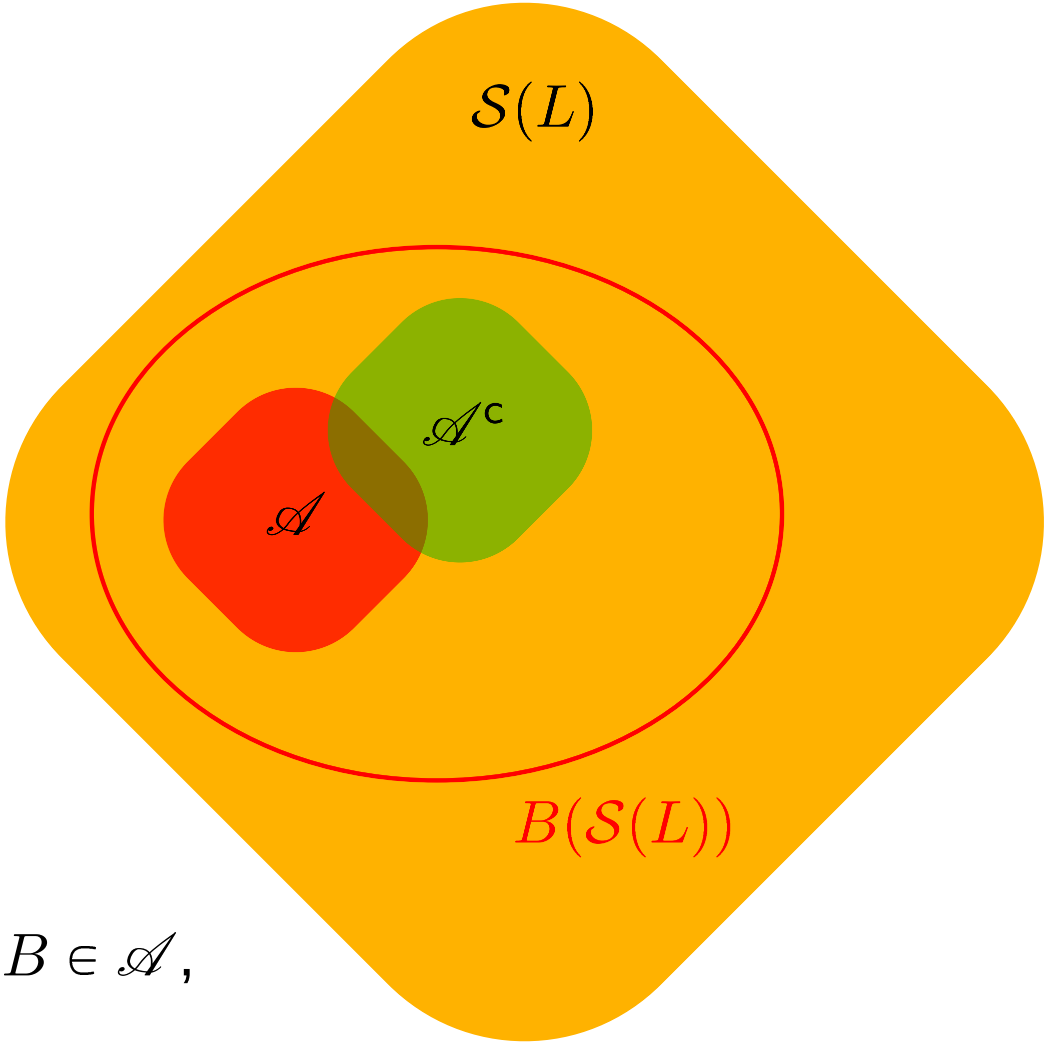

L is $\mathscr{A}$-normal $\equiv$ For any $A, B \in \mathscr{A}$,

$$A \vee B = 1 \;\Rightarrow\; \exists\, U, V \in \mathscr{A} : U \wedge V = 0,\; A \vee U = 1 = B \vee V.$$

$\mathscr{A}$-normality vs $\mathscr{A}$-extremal disconnectedness

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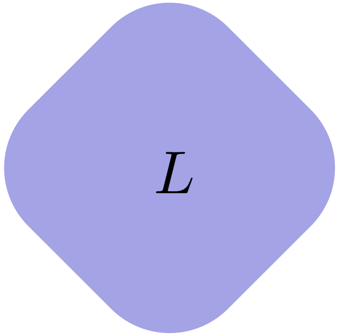

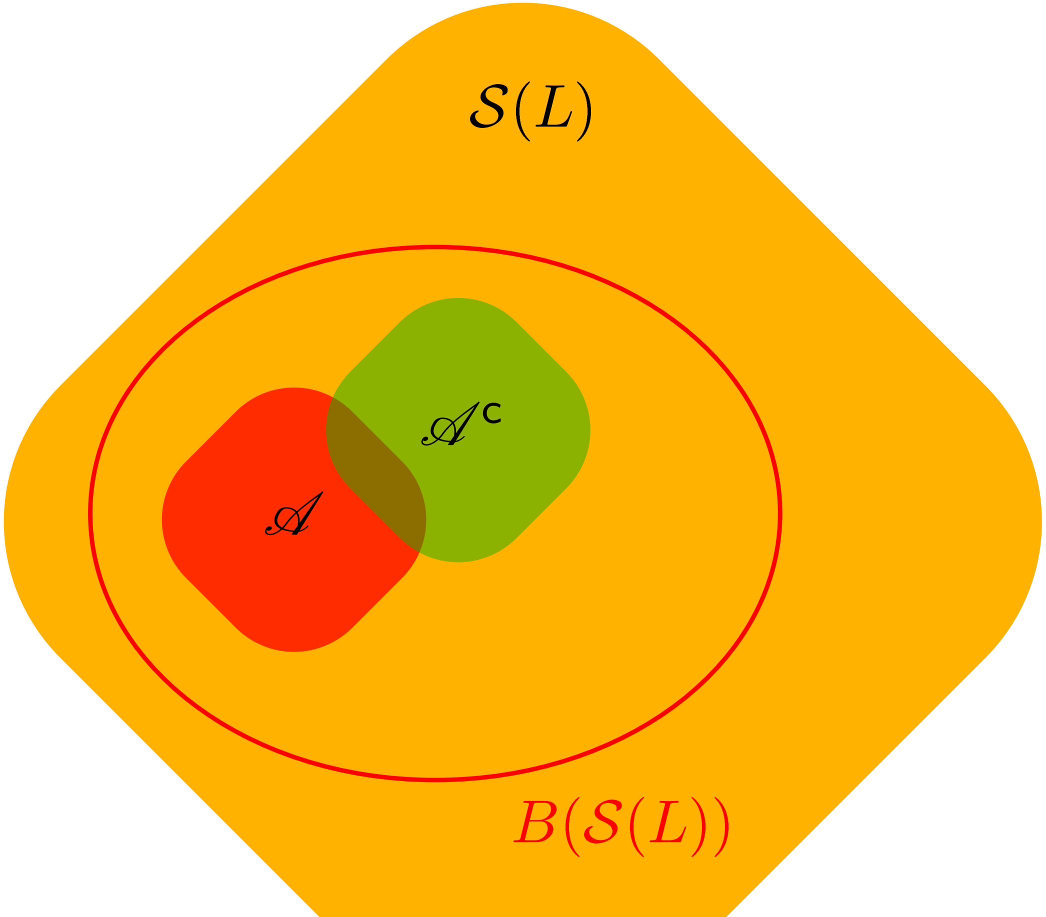

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L is $\mathscr{A}$-extremally disconnected $\equiv$ L is $\mathscr{A}^{\mathsf{c}}$-normal.

$\mathscr{A}$-normality: EXAMPLES

$$\boxed{\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}}$$

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- $\mathscr{A}_1$-normal frames: normal
- $\mathscr{A}_1$-extremally disconnected frames: extremally disconnected

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- $\mathscr{A}_1$-extremally disconnected frames: extremally disconnected

$\boxed{\mathscr{A}_2 = \{\mathfrak{c}(a^*) : a \in L\}}$

$\mathscr{A}$-normality: EXAMPLES

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- $\mathscr{A}_1$-normal frames: normal
- $\mathscr{A}_1$-extremally disconnected frames: extremally disconnected

$$\boxed{\mathscr{A}_2 = \{\mathfrak{c}(a^*) : a \in L\}}$$

- $\mathscr{A}_2$-normal frames: mildly normal
- $\mathscr{A}_2$-extremally disconnected frames: extremally disconnected

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$\mathscr{A}$-normality: EXAMPLES

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- $\mathscr{A}_3$-extremally disconnected: δ-extremally disconnected

$\boxed{\mathscr{A}_4 = \{\mathfrak{c}(\mathrm{coz} f) \colon f \in \mathsf{C}(L)\}}$

- $\mathscr{A}_4$-normal: all frames
- $\mathscr{A}_4$-extremally disconnected: F-frames

REAL FUNCTIONS ON L

$$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$$

BACKGROUND: the frame of reals

$$\mathfrak{L}(\mathbb{R}) \;:=\; \textsf{Frm}\big\langle\, (-,q),(p,-)\,(p,q \in \mathbb{Q}) \;\big|$$

BACKGROUND: the frame of reals

$\mathfrak{L}(\mathbb{R}) \;:=\;$ Frm$\langle\, (—, q), (p, —)(p, q \in \mathbb{Q}) \mid$ (1) $(—, q) \wedge (p, —) = 0$ for $q \leqslant p$,

(2) $(—, q) \vee (p, —) = 1$ for $q > p$,

(3) $(—, q) = \bigvee_{s<q} (—, s)$,

(4) $\bigvee_{q \in \mathbb{Q}} (—, q) = 1$,

(5) $(p, —) = \bigvee_{r>p} (r, —)$,

(6) $\bigvee_{p \in \mathbb{Q}} (p, —) = 1 \,\rangle$.

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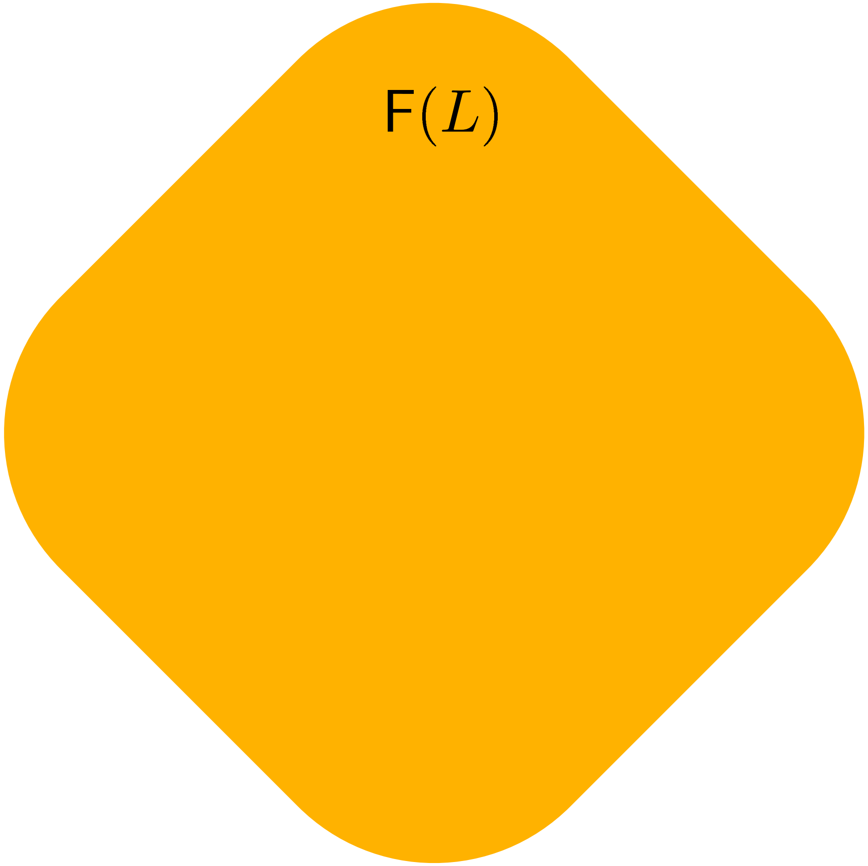

SEMICONTINUITY AND CONTINUITY

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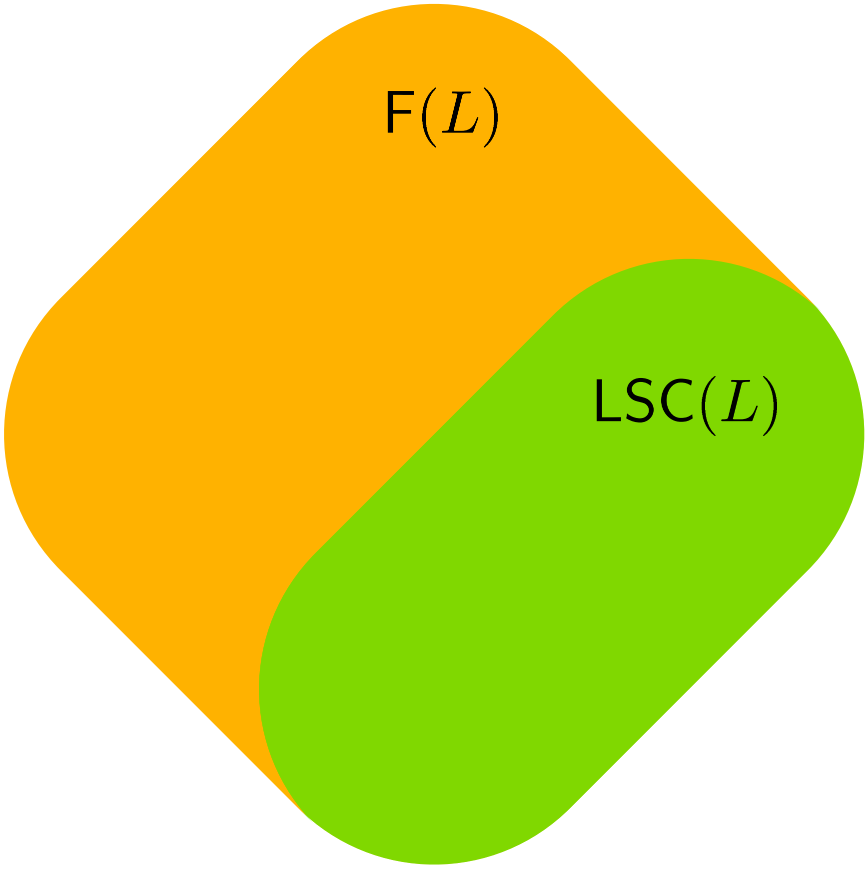

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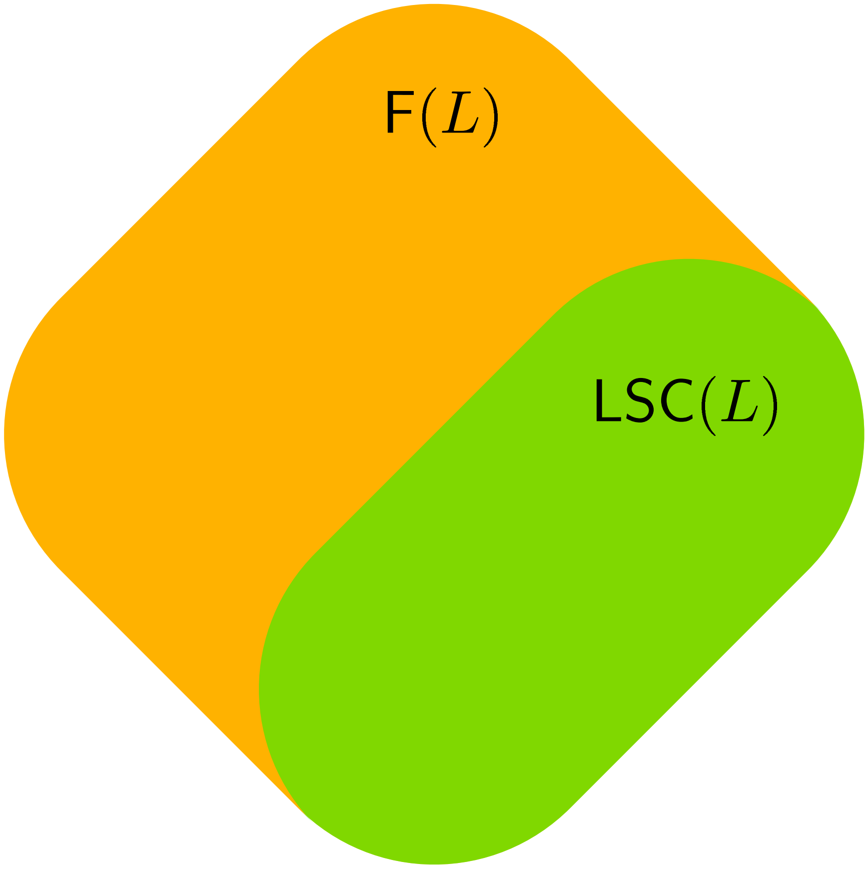

$f(p, \text{—}) \in \mathfrak{c}L$

SEMICONTINUITY AND CONTINUITY

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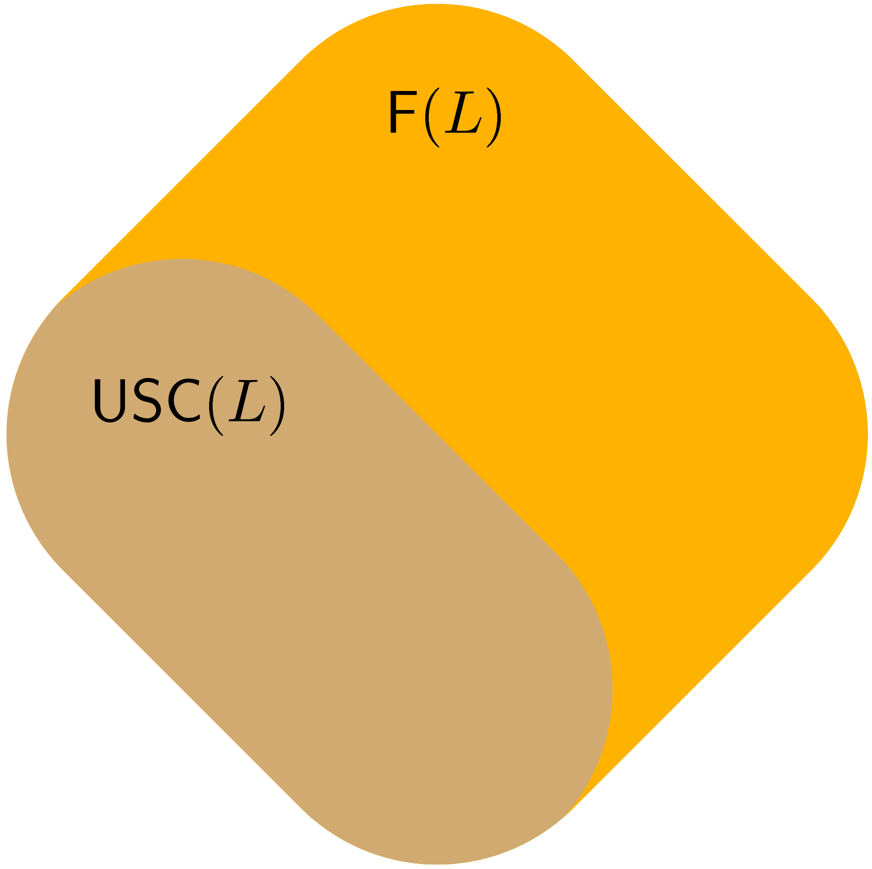
$f(\text{—}, q) \in \mathfrak{c}L$

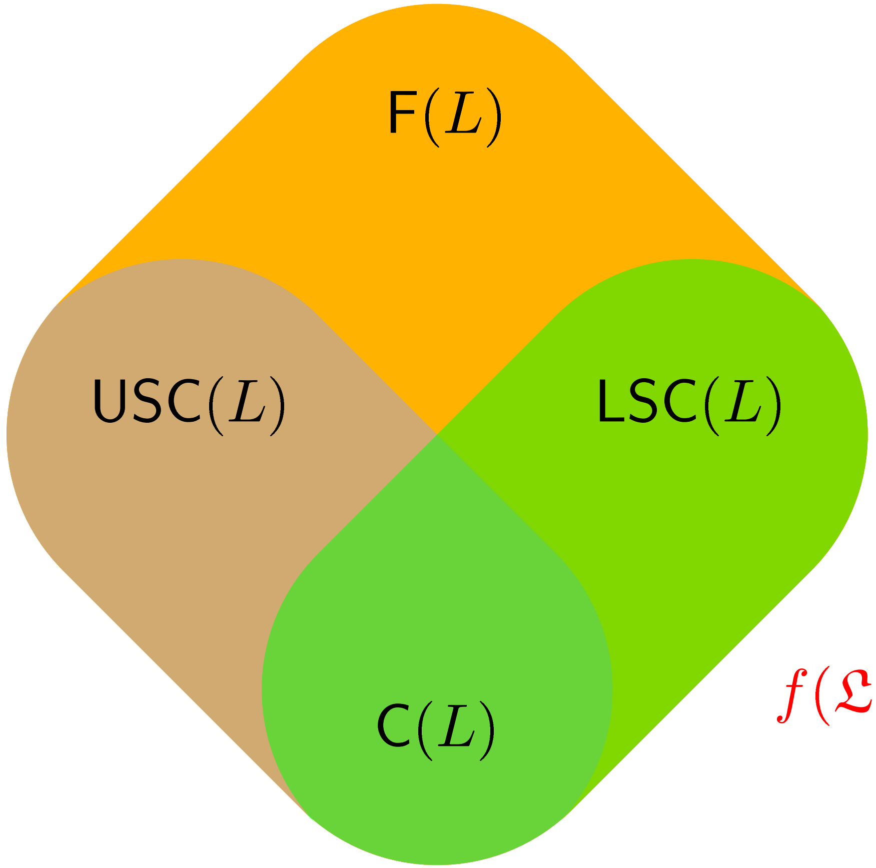


$f(p, \text{—}) \in \mathfrak{c}L$

SEMICONTINUITY AND CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$f(-, q) \in \mathfrak{c}L$



$f(p, -) \in \mathfrak{c}L$

$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

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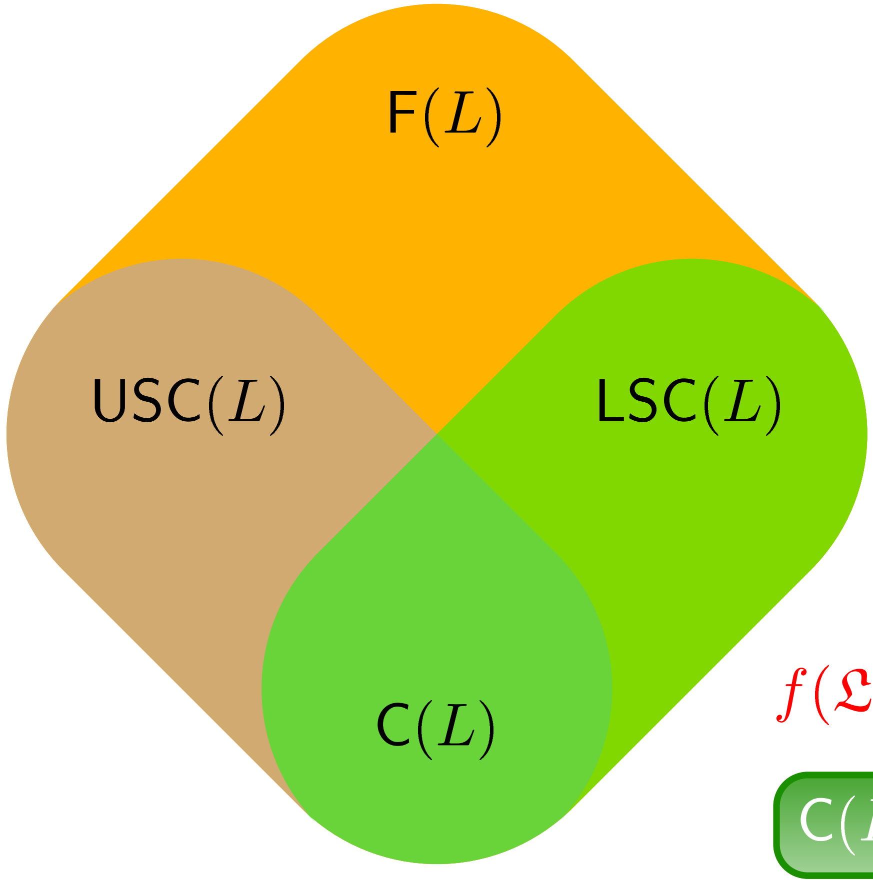


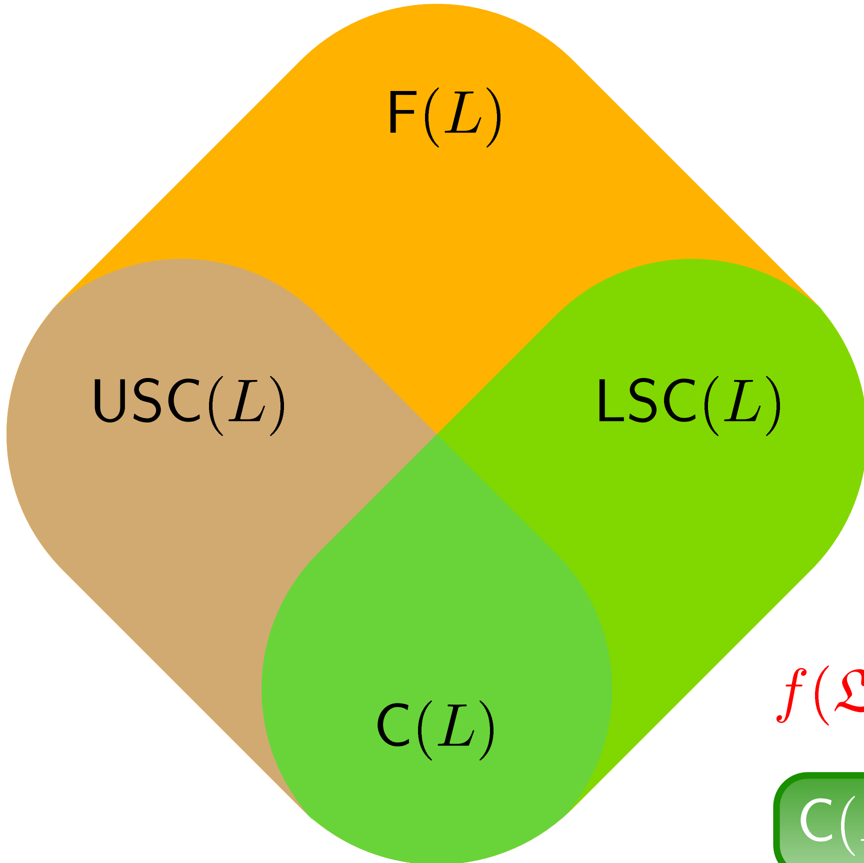
$f(p, -) \in \mathfrak{c}L$

$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

$\boxed{\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$

SEMICONTINUITY AND CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$



$f(-, q) \in \mathfrak{c}L$

$f(p, -) \in \mathfrak{c}L$

$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

$\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$

$$\boxed{f \leqslant g \;\equiv\; f(p, -) \leqslant g(p, -),\ \forall p \in \mathbb{Q}}$$

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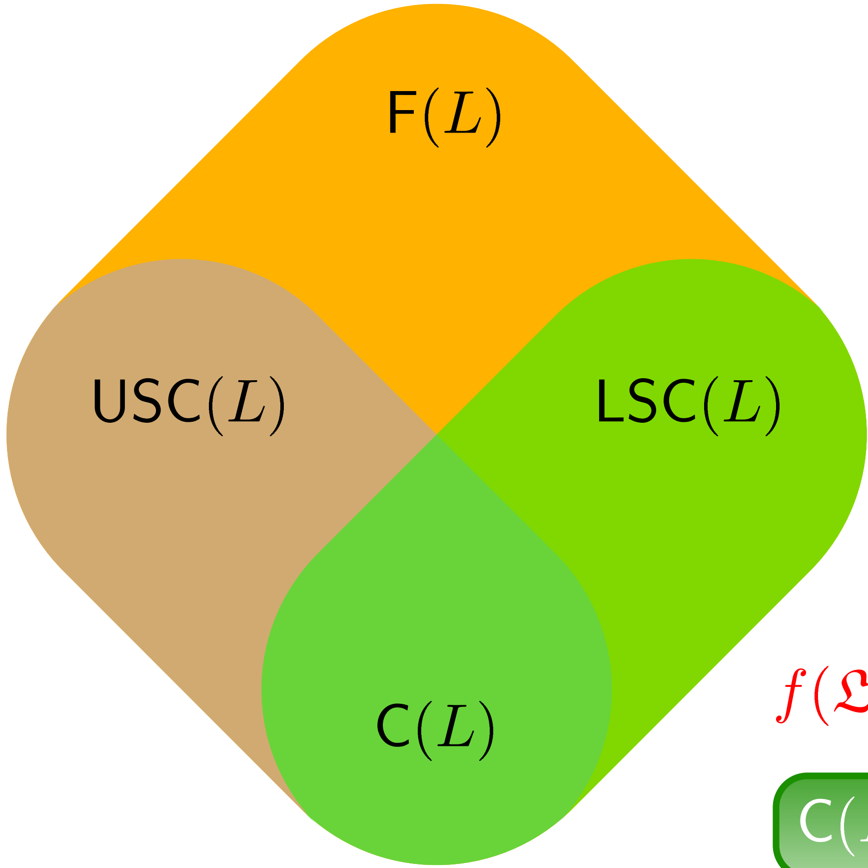

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J. Gutiérrez García, T. Kubiak & J. P.
Localic real functions: a general setting, J. Pure Appl. Algebra *213 (2009)* .

$$f \in \mathsf{USC}(L) \iff \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L : \;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \;:\;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

["$\Leftarrow$": $f(\text{—}, q) = \bigvee_{r<q} f(\text{—}, r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(\text{—}, q)$.]

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

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$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\;\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

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$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} : \;\; f(q, -) \leqslant F_{p,q} \leqslant f(p, -).$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \;:\;\; f(—, p) \leqslant F_{p,q} \leqslant f(—, q).$$

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$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

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$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

Clearly: f is upper $\mathscr{A}$-semicont. iff it is lower $\mathscr{A}^{c}$-semicont.

f is $\mathscr{A}^{c}$-continuous iff it is $\mathscr{A}$-continuous.

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$$\boxed{\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}}$$

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

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- upper $\mathscr{A}_1$-semicontinuous functions: upper semicontinuous
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$$(f^\circ)^- = f \quad | \quad (f^-)^\circ = f$$

- upper $\mathscr{A}_2$-semicontinuous functions: normal upper semicontinuous
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$\mathscr{A}_3 = \{\mathfrak{c}(a) \colon a$ regular $G_\delta\}$

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}_3 = \{\mathfrak{c}(a) \colon a \text{ regular } G_\delta\}$

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- upper $\mathscr{A}_3$-semicontinuous functions: regular upper semicontinuous
- lower $\mathscr{A}_3$-semicontinuous functions: regular lower semicontinuous
- $\mathscr{A}_3$-continuous functions: regular continuous [Lane]

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$\boxed{\mathscr{A}_4 = \{\mathfrak{c}(\operatorname{coz} f) \colon f \in \mathsf{C}(L)\}}$

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

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- upper $\mathscr{A}_3$-semicontinuous functions: regular upper semicontinuous
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- $\mathscr{A}_3$-continuous functions: regular continuous [Lane]

$\boxed{\mathscr{A}_4 = \{\mathfrak{c}(\operatorname{coz} f) : f \in \mathsf{C}(L)\}}$

$$p < q: \quad f(-,p) \leqslant \mathfrak{c}(a_{p,q}) \leqslant f(-,q)$$

- upper $\mathscr{A}_4$-semicontinuous functions: zero upper semicontinuous
- lower $\mathscr{A}_4$-semicontinuous functions: zero lower semicontinuous
- $\mathscr{A}_4$-continuous functions: zero continuous [Stone]

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

$S, T \in \mathcal{S}(L)$

$$\boxed{S \Subset_{\mathscr{A}} T \equiv \ \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}} \colon\ S \leqslant V \leqslant U \leqslant T}$$

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$\mathscr{A}$ is a Katětov class

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$\mathscr{A}$ is a Katětov class

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice $[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$

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- a sublattice $[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$
- or closed under binary meets and $U_1, U_2 \in \mathscr{A}, U_1 \vee U_2 \leqslant V \in \mathscr{A}^{\mathsf{c}} \Rightarrow \exists\, U' \in \mathscr{A} : U_1 \vee U_2 \leqslant U' \leqslant V$ $[\mathscr{A}_2]$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

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$\mathscr{A}$ is a Katětov class

(K5) $S \Subset_{\mathscr{A}} T \Rightarrow \exists\, U \in \mathcal{S}(L)\colon S \Subset_{\mathscr{A}} U \Subset_{\mathscr{A}} T$.

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice [$\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4$]
- or closed under binary meets and
  $U_1, U_2 \in \mathscr{A}, U_1 \vee U_2 \leqslant V \in \mathscr{A}^{\mathsf{c}} \Rightarrow \exists\, U' \in \mathscr{A}\colon U_1 \vee U_2 \leqslant U' \leqslant V$ [$\mathscr{A}_2$]

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

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$\mathscr{A}$ is a Katětov class

(K5) $S \Subset_{\mathscr{A}} T \Rightarrow \exists\, U \in \mathcal{S}(L): S \Subset_{\mathscr{A}} U \Subset_{\mathscr{A}} T$. $\Leftrightarrow L$ is $\mathscr{A}$-normal

LEMMA 2

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MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

We can then apply

KATĚTOV LEMMA. Let M be a lattice and $\Subset$ a Katětov relation on M.

Let $(a_p \mid p \in \mathbb{Q}), (b_p \mid p \in \mathbb{Q}) \subseteq M$ such that

$$p < q \;\Rightarrow\; a_q \leqslant a_p, \quad b_q \leqslant b_p, \quad a_q \Subset b_p.$$

Then there exists a family $(c_p \mid p \in \mathbb{Q}) \subseteq M$ such that

$$p < q \;\Rightarrow\; c_q \Subset c_p, \quad a_q \Subset c_p, \quad c_q \Subset b_p.$$

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for $\quad M \equiv \mathcal{S}(L) \quad$ and $\quad \Subset \equiv \Subset_{\mathscr{A}}$

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for $\quad M \equiv \mathcal{S}(L) \quad$ and $\quad \Subset \equiv \Subset_{\mathscr{A}}$

whenever $\mathscr{A}$ is a Katětov class and L is $\mathscr{A}$-normal.

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-normal.

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-normal.
2. $$\underbrace{f}_{\mathscr{A}-\mathsf{USC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A}-\mathsf{C}(L) \colon \; f \leqslant h \leqslant g\,.$$

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3. Every $S, T \in \mathscr{A}$ such that $S \vee T = 1$ are completely $\mathscr{A}$-separated.

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

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$\exists$ $\mathscr{A}$-continuous $h\colon$ $h(0,\text{—}) \leqslant S$ and $h(\text{—},1) \leqslant T$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

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3. Every $S, T \in \mathscr{A}$ such that $S \vee T = 1$ are completely $\mathscr{A}$-separated.

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}^c$-normal.

2. $$\underbrace{f}_{\mathscr{A}^c-\mathrm{USC}} \leqslant \underbrace{g}_{\mathscr{A}^c-\mathrm{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A}^c - \mathsf{C}(L)\colon \; f \leqslant h \leqslant g\,.$$

3. Every $S, T \in \mathscr{A}^c$ such that $S \vee T = 1$ are completely $\mathscr{A}^c$-separated.

Then the dual result for extremal $\mathscr{A}$-disconnectedness follows just by

COMPLEMENTATION

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

COROLLARY [Stone]. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-extremally disconnected.

2. $$\underbrace{f}_{\mathscr{A}-\mathsf{LSC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{USC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A} - \mathsf{C}(L) \colon \; f \leqslant h \leqslant g\,.$$

3. Every $S, T \in \mathscr{A}$ such that $S \wedge T = 0$ are completely $\mathscr{A}$-separated.

EXTENSION: Tietze-type extension $\equiv$ Stone-type extension

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$
\begin{tikzcd}
\mathcal{S}(L) \arrow[d, "\varphi_S"] & T \arrow[d, rightsquigarrow] \\
\mathcal{S}(S) & T \vee S
\end{tikzcd}
$$

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CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] & T \arrow[d, squiggly] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f \in \mathsf{C}(S)"'] & \mathcal{S}(S) & T \vee S \end{tikzcd}$$

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(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A}_S$: Katětov class in S

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$\mathscr{A}_S$-continuous

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ADEQUATE KIND $\mathfrak{A}$ OF KATĚTOV CLASSES of L:

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localic map $f\colon L \to M$

IMAGE $f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M)$ (localic map)

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ext. disc. ext. disc.

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SKETCH OF PROOF

$$L \xrightarrow{f} M$$

$$A,\ B \in \mathscr{A}_M, \quad A \vee B = 1$$

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$f_{-1}[-]$ preserves complements

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Perfectly normal spaces: STRICT (OR STRONG) INSERTION [Michael]

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$$\Updownarrow$$

Double insertion

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Double insertion

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$\Updownarrow$ Double insertion

$\Updownarrow$ (Weak) insertion

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$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

Strict insertion $=$ Double insertion $+$ (Weak) insertion

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