Hausdorff mapping invariance theorems in Localic Topology

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— with J. Gutiérrez García (Bilbao), T. Kubiak (Poznań), A. Pultr (Prague)

CLASSICAL TOPOLOGY ⇝ POINT-FREE TOPOLOGY

topological spaces

generalized spaces: locales

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CABOOL

SUBOBJECT LATTICES

not so nice ...

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[Atomless parts of spaces, *Math. Scand.* (1972)]

AIMS

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AIM: to illustrate this idea with parts of our work, in the last few years, with J. Gutiérrez García, T. Kubiak, A. Pultr.

MOTTO Mapping Invariance Theorem (HAUSDORFF)

THEOREM: Let $f\colon X \to Y$ be a CLOSED surjection.

If X is normal then Y is also normal.

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«*(...) what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.*» R. BALL & J. WALTERS-WAYLAND

[C- and C^*-quotients in pointfree topology, *Dissert. Math.* (2002)]

THE SETTING

locales (or **frames**)

- Complete lattices L satisfying

$$\boxed{a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)}$$

(= complete Heyting algebras)

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- $$\begin{array}{ccc} L & \qquad\qquad & L \\ \big\downarrow f & & \big\uparrow h \\ M & & M \end{array}$$

preserves $\bigvee$ (incl. the bottom 0)

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frame homomorphisms taken backwards

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Each $h\colon M \to L$ in Frm has a UNIQUELY defined right adjoint

$$h_*\colon L \to M$$

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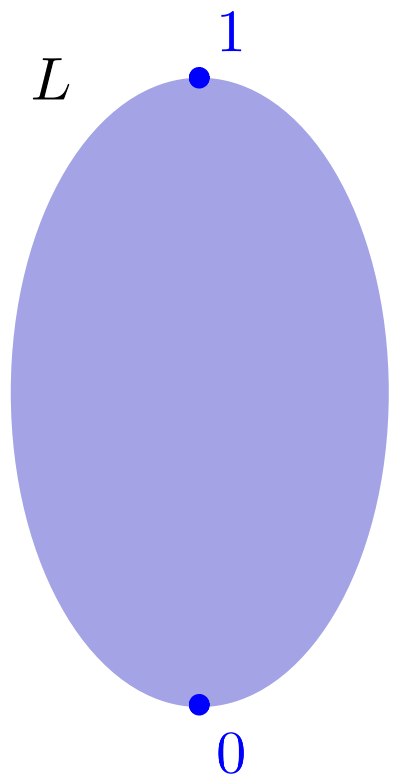
DEFINITION: $S \subseteq L$ is a SUBLOCALE of L if



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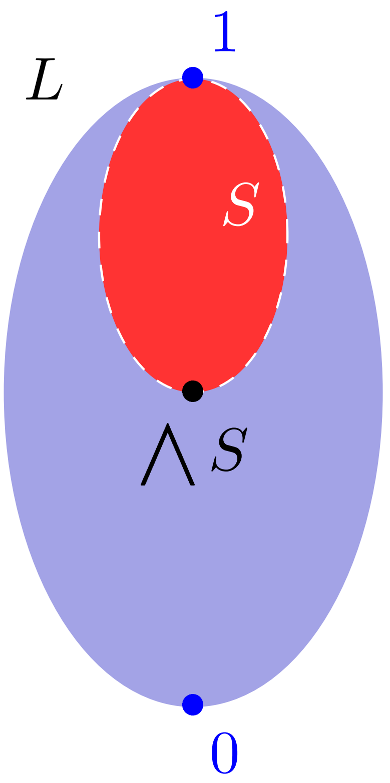

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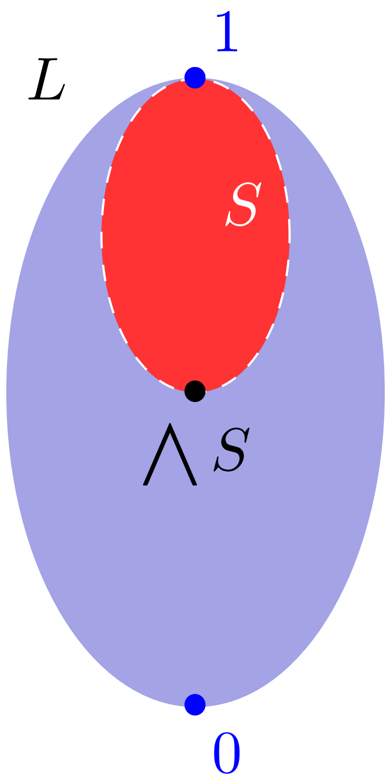

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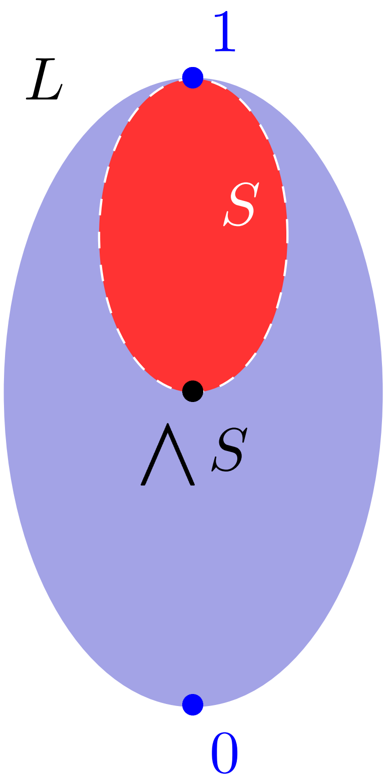

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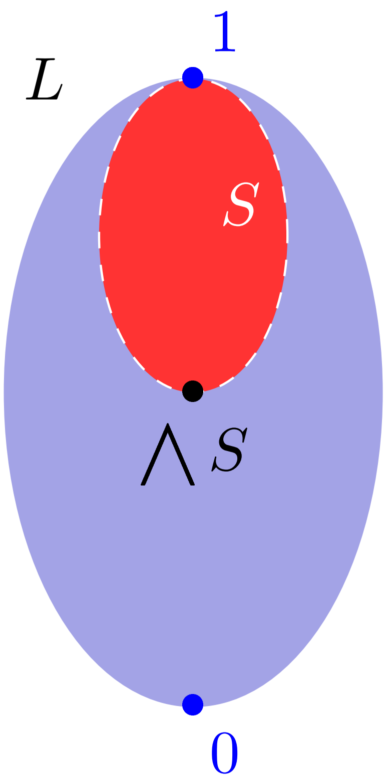

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Motivation for the definition:

PROPOSITION:

$S \subseteq L$ is a sublocale iff the embedding $j_S \colon S \subseteq L$ is a localic map.

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sublocales of L, ordered by $\subseteq$:

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complemented

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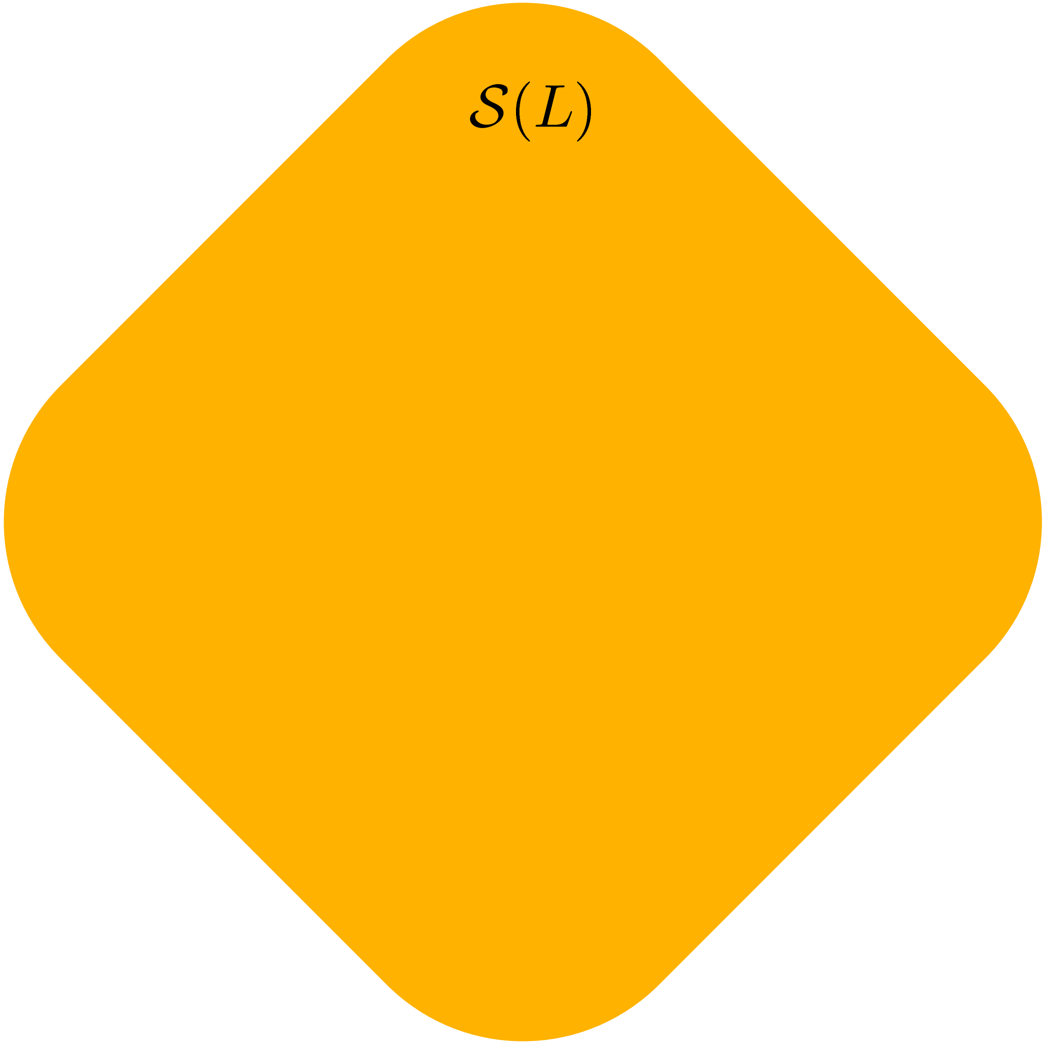
the frame of sublocales



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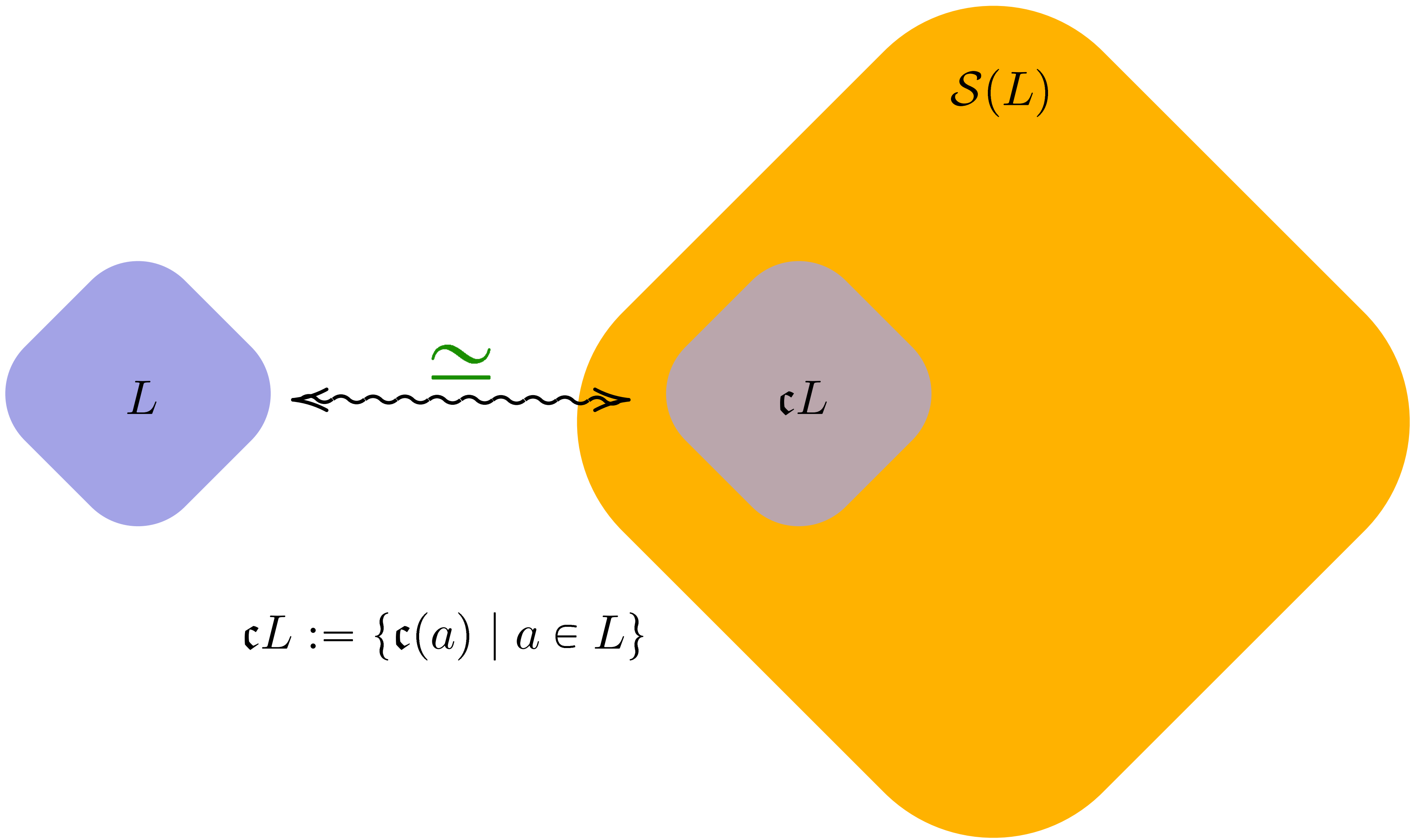

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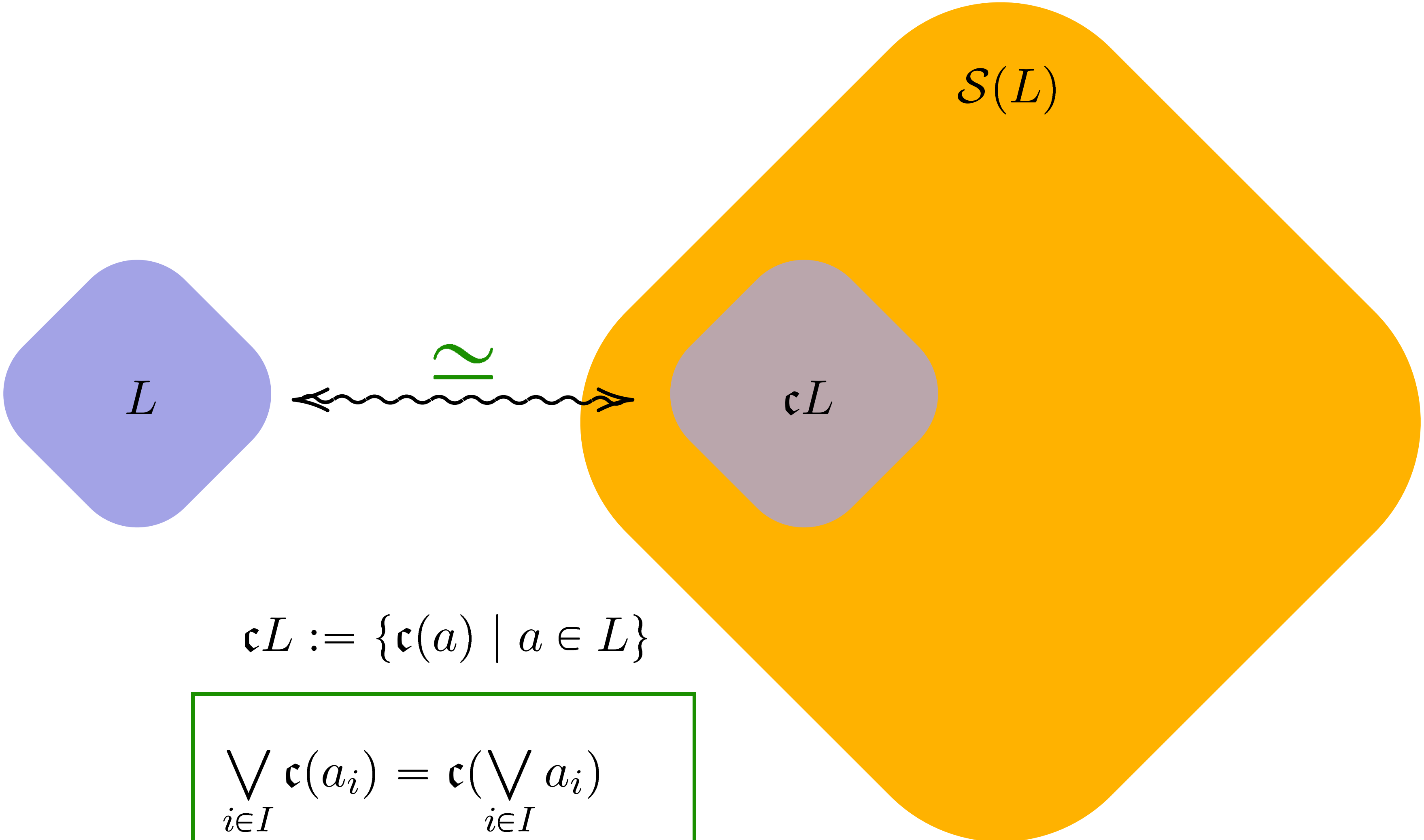

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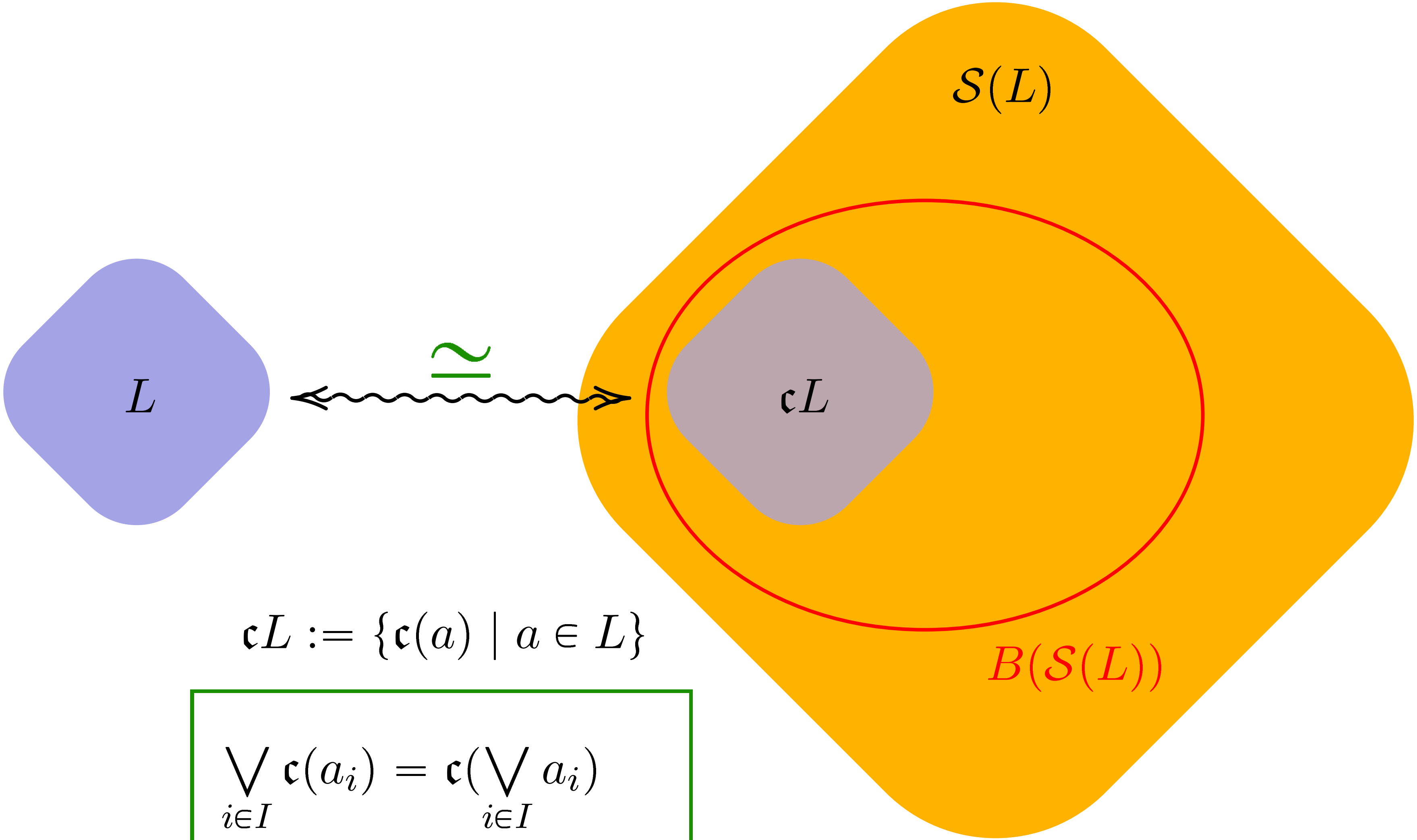

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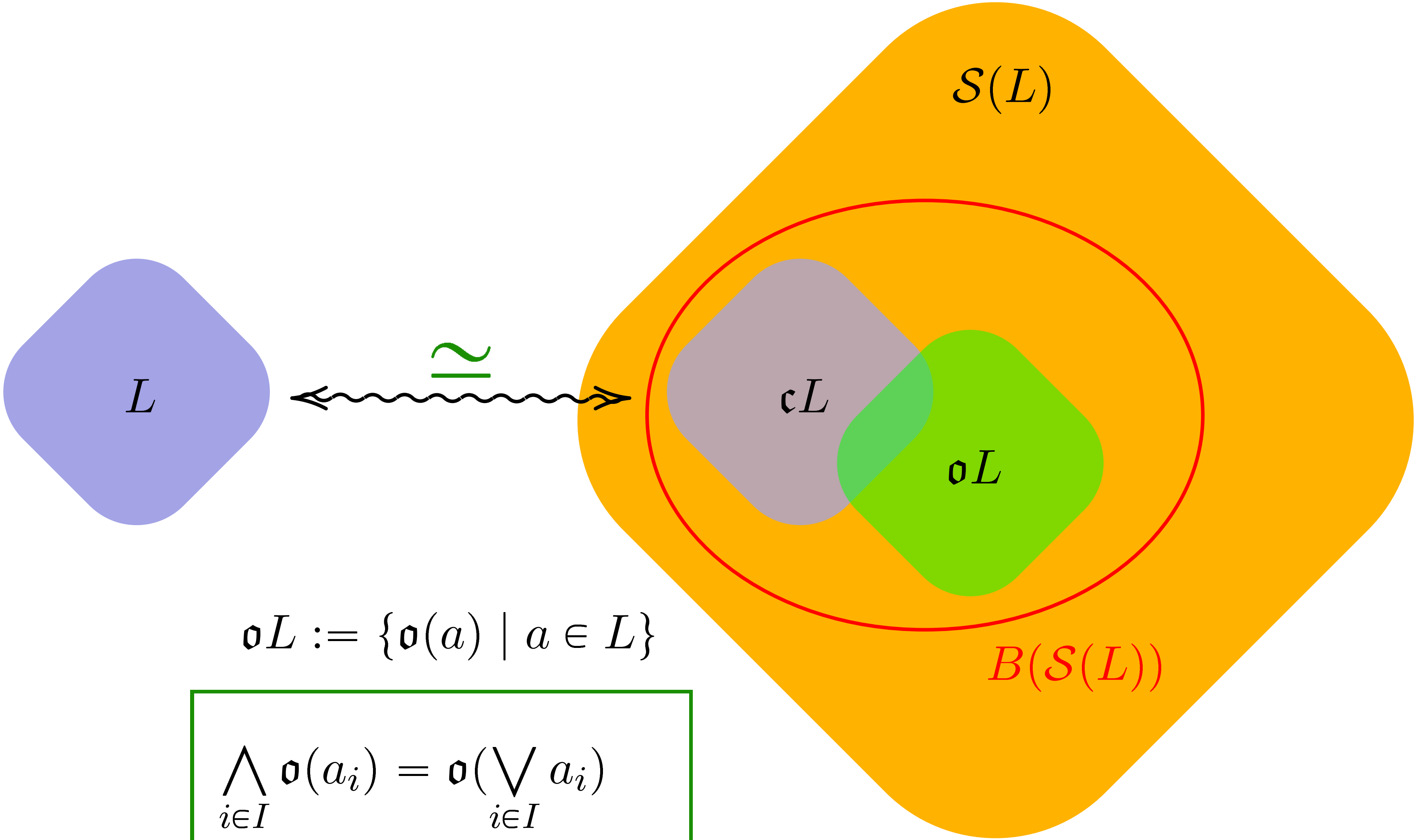

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IMAGES

localic map $f\colon L \to M$

$$\begin{array}{c} \bigcup\!| \\ S \end{array}$$

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localic map $f: L \to M$

$$\begin{array}{l} L \\ \rotatebox{90}{$\subseteq$} \\ S \end{array}$$

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$$\Leftrightarrow \quad f[\mathfrak{c}(a)] = \mathfrak{c}(f(a)) \quad \forall a \in L$$

PREIMAGES

localic map $\begin{array}{r} f: L \to M \\ \bigcup\!| \\ T \end{array}$

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So there exists the largest sublocale contained in A: A_{sloc}

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PREIMAGE MAP: $f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$ (frame homomorphism)

IMAGES AND PREIMAGES

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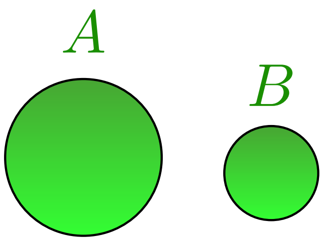
3. for surjective f: $f f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(a)$ and $f f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(a)$.

Normality

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$$



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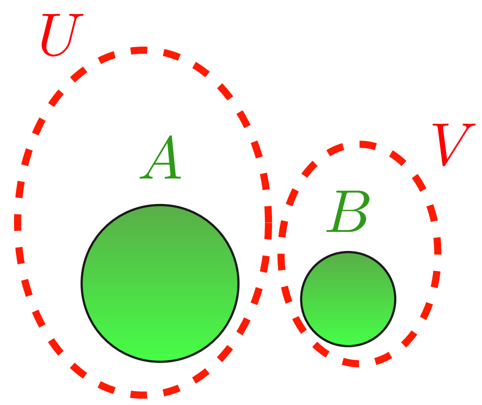
$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$$

$$\Downarrow$$

$$\exists u, v\colon \mathfrak{o}(u) \vee \mathfrak{o}(v) = 1, \mathfrak{c}(a) \geqslant \mathfrak{o}(u), \mathfrak{c}(b) \geqslant \mathfrak{o}(v).$$



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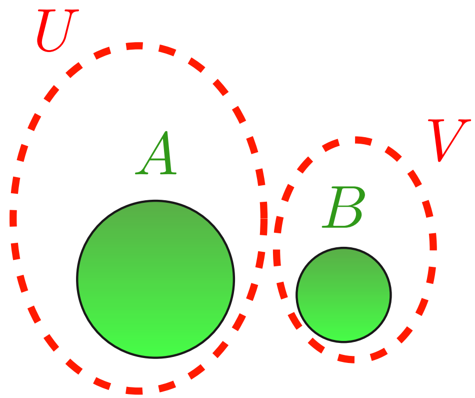

So L is normal iff

$$\boxed{\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \Rightarrow \exists u, v\colon \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 1, \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v)}$$

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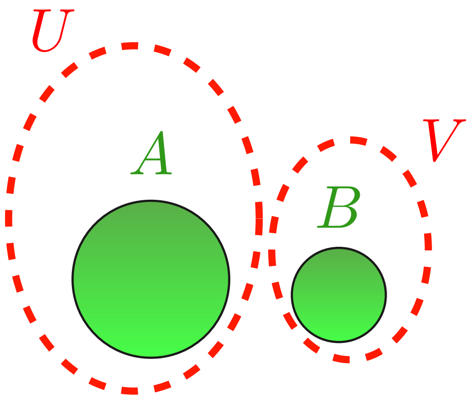

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Internally in L:
(by $\mathfrak{c}L \cong L$)

$$\boxed{a \vee b = 1 \;\Rightarrow \exists u, v \colon u \wedge v = 0, a \vee u = 1 = b \vee v}$$

(Conservative extension: X is normal iff the locale $\mathcal{O}(X)$ is normal.)

THE INVARIANCE THEOREM: first version

THEOREM: Let $f : L \to M$ be a CLOSED surjective localic map.

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PROOF: later on ...

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$\mathscr{B} : L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

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the standard model

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$\mathfrak{c}_{\mathrm{coz}}$	$\{\mathfrak{c}(\mathrm{coz} f) : f \in \mathsf{C}(L)\}$

regular G_δ element: $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$

cozero element: $\boxed{a = \bigvee_{n\in\mathbb{N}} a_n \text{ with } a_n \prec\!\prec a}$

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J. Gutiérrez García & JP, *On the parallel between normality and extremal disconnectedness*, JPAA (2014)

$\mathscr{B}$-NORMALITY

frame L

Normal:

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \Rightarrow \exists u, v\colon \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v).$$

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Selection $\mathscr{B}$	$\mathscr{B}$-normal frames
$\mathfrak{c}$	normal

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Selection $\mathscr{B}$	$\mathscr{B}$-normal frames
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Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**
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$\mathfrak{c}_{\mathrm{coz}}$	all frames

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THE INVARIANCE THEOREM: general version

localic map $f\colon L \to M$

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THEOREM: Let $f\colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-normal then M is also $\mathscr{B}$-normal.

SKETCH OF PROOF

$$L \xrightarrow{f} M$$

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$\Downarrow$ L is $\mathscr{B}$-normal

$$\exists\, U_0,\ V_0 \in \mathscr{B}(L):\ U_0 \wedge V_0 = 0,\ \ f_{-1}[A] \vee U_0 = 1 = f_{-1}[B] \vee V_0.$$

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hyp.

$U = f[U_0]$, $V = f[V_0] \in \mathscr{B}(M)$ satisfy:

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$f_{-1}[-]$ preserves complements

IMAGE AND PREIMAGE $\mathscr{B}$-PRESERVING MAPS

localic map $f\colon L \to M$

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$\mathscr{B}$	**image $\mathscr{B}$-preserving**	**preimage $\mathscr{B}$-preserving**
$\mathfrak{c}$	closed maps	all

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$\mathfrak{c}_{\mathrm{coz}}$	$f(\underset{\sim}{a} \vee f^*(b)) = f(a) \vee b$ cozero	all

ANOTHER FEATURE: take complements

$\mathscr{B}^{\mathsf{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}$ -normal:

$$\boxed{\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \ \Rightarrow \ \exists u, v\colon \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 1, \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v)}$$

ANOTHER FEATURE: take complements

$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

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need only for a, b <u>regular</u> $(a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0)$

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need only for a, b regular $\quad (a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0)$

$$\equiv (a \wedge b)^* = a^* \vee b^*$$

[De Morgan frames]

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$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**	**$\mathscr{B}$-disconnected frames**
$\mathfrak{c}$	normal	extremally disconnected

ANOTHER FEATURE: take complements

$\mathscr{B}^{\mathsf{c}} \colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	$\mathscr{B}$**-normal frames**	$\mathscr{B}$**-disconnected frames**
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$\mathfrak{c}^*$	mildly normal	extremally disconnected

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$\mathfrak{c}$	normal	extremally disconnected
$\mathfrak{c}^*$	mildly normal	extremally disconnected
$\mathfrak{c}_\delta$	δ-normal	extremally δ-disconnected
$\mathfrak{c}_{\mathrm{coz}}$	all frames	F-frames

F-frame $\equiv$ every cozero sublocale is C^*-embedded.

ANOTHER FEATURE: take complements

$\mathscr{B}^{\mathsf{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

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f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving.

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$\mathscr{B}^{\mathsf{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

COROLLARY: Let $f\colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}^{\mathsf{c}}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-~~normal~~ disconnected then M is also $\mathscr{B}$-~~normal.~~ disconnected

ANOTHER FEATURE: take complements

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If L is $\mathscr{B}$-~~normal~~ disconnected then M is also $\mathscr{B}$-~~normal~~. disconnected

- preimage $\mathscr{B}^{\mathsf{c}}$-preserving = preimage $\mathscr{B}$-preserving

(because $f_{-1}[-]$ preserves complements)

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Case $\mathscr{B} = \mathfrak{c}$: ext. disc. locales are invariant under OPEN mappings.

ANOTHER FEATURE: take complements

localic map $f: L \to M$

- image $\mathscr{B}^{\mathsf{c}}$-preserving

- preimage $\mathscr{B}^{\mathsf{c}}$-preserving $\equiv$ preimage $\mathscr{B}$-preserving

$\mathscr{B}$	**image $\mathscr{B}^{\mathsf{c}}$-preserving**	**preimage $\mathscr{B}^{\mathsf{c}}$-preserving**
$\mathfrak{c}$	open	all

ANOTHER FEATURE: take complements

- image $\mathscr{B}^c$-preserving

- preimage $\mathscr{B}^c$-preserving $\equiv$ preimage $\mathscr{B}$-preserving

$\mathscr{B}$	**image $\mathscr{B}^c$-preserving**	**preimage $\mathscr{B}^c$-preserving**
$\mathfrak{c}$	open	all
$\mathfrak{c}^*$	nearly open	f^* of type E (e.g. nearly open) [Banaschewski & Pultr]

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$\mathfrak{c}_{\mathrm{coz}}$	?	all

Perfect normality

In spaces [Michael 1956]:

$$\forall\, U \in \mathcal{O}(X)\ \exists\ (U_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}(X) \colon U = \bigcup_{n\in\mathbb{N}} U_n \text{ and } \overline{U_n} \subseteq U \ \ \forall n.$$

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In frames [Charalambous 1974]:

$$\forall a \in L\ \ \exists (a_n)_{\mathbb{N}} \subseteq L :\ a = \bigvee a_n \ \text{ and } \ a_n \prec a\ \forall n.$$

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$$\Updownarrow$$

normal $+$ perfect

THE PERFECT CASE

Perfect normality

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normal $+$ perfect every open is an F_σ-sublocale:

L is perfect $\equiv\ \forall\ \mathfrak{o}(a),\ \ \mathfrak{o}(a) = \bigwedge_{n\in\mathbb{N}} \mathfrak{c}(a_n)$

In spaces [Michael 1956]:

$$\forall\, U \in \mathcal{O}(X)\ \exists\ (U_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}(X) : U = \bigcup_{n\in\mathbb{N}} U_n \text{ and } \overline{U_n} \subseteq U\ \ \forall n.$$

In frames [Charalambous 1974]:

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$$\boxed{L \text{ is } \mathscr{B}\text{-perfect} \equiv\ \forall\ A \in \mathscr{B}^{\mathsf{c}},\ \ A = \bigwedge_{n\in\mathbb{N}} A_n \ \text{ with each } A_n \in \mathscr{B}}$$

Perfect normality

$\mathscr{B}$	$\mathscr{B}$-perfect	$\mathscr{B}$-perfectly normal	$\mathscr{B}^{c}$-perfect	$\mathscr{B}^{c}$-perfectly normal
$\mathfrak{c}$				
$\mathfrak{c}^*$				
$\mathfrak{c}_{\mathrm{coz}}$				

THE PERFECT CASE

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OZ frame $\equiv$ every regular element is a cozero.

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P-frame $\equiv$ $\mathrm{Coz}\, L$ is complemented.

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$$A_n \in \mathscr{B}(L) \longmapsto f[A_n] \in \mathscr{B}(M).$$

THE HEREDITARY CASE

Hereditary normality: normal spaces whose subspaces are all normal.

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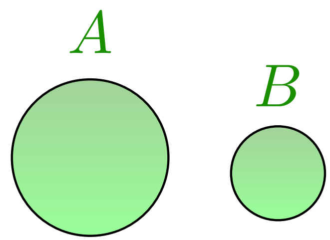

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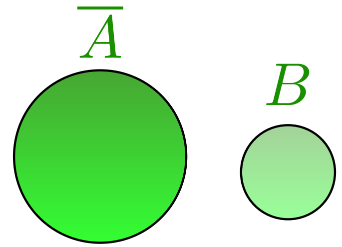

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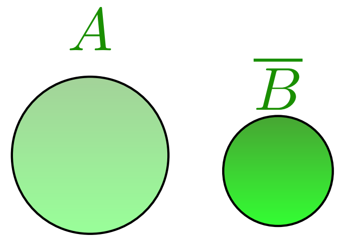

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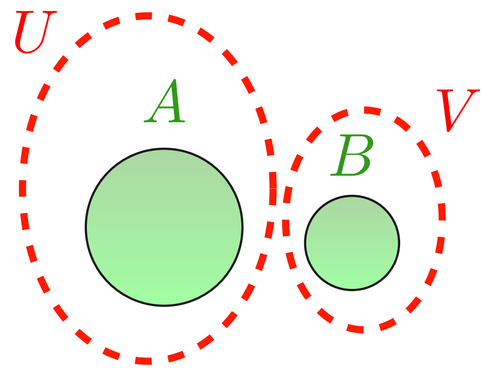

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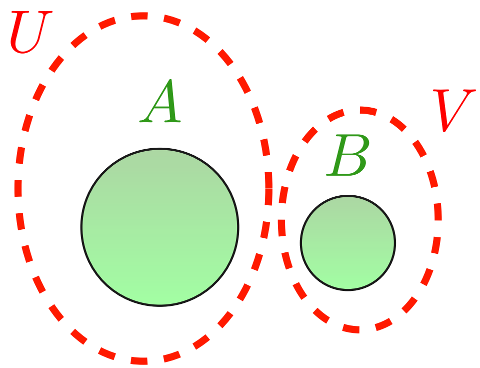

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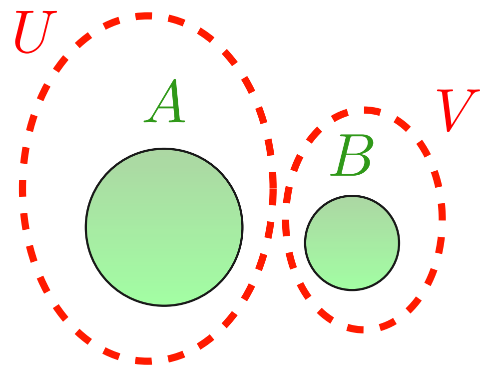

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Completely $\mathscr{B}$-disconnected frames $\equiv$ completely $\mathscr{B}^{\mathsf{c}}$-normal.

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$\mathscr{B}$ closed under arbitrary joins

$$\boxed{\mathrm{cl}_{\mathscr{B}}(S) = \bigvee\{B \in \mathscr{B} \mid B \leqslant S\} \in \mathscr{B}} \quad \boxed{\mathrm{int}_{\mathscr{B}}(S) = \bigwedge\{B \in \mathscr{B}^{\mathsf{c}} \mid B \geqslant S\} \in \mathscr{B}^{\mathsf{c}}}$$

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TFAE for any L:

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3. $S \leqslant \mathrm{cl}_{\mathscr{B}}(T)$ and $\mathrm{int}_{\mathscr{B}}(S) \leqslant T \Rightarrow \exists U, V \in \mathscr{B}\colon S \leqslant V \leqslant U^{\mathsf{c}} \leqslant T$.

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$\mathscr{B}$ closed under arbitrary joins

Sublocale S of L

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3. Each $B \in \mathscr{B}^{c}$ is $\mathscr{B}^{c}$-normal $\equiv$ Each $B \in \mathscr{B}$ is $\mathscr{B}$-disconnected.

REAL FUNCTIONS ON L

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

BACKGROUND: the frame of reals

$\mathfrak{L}(\mathbb{R}) \; := \; \mathsf{Frm}\big\langle\, (-, q), (p, -) (p, q \in \mathbb{Q}) \;\big|$

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$\mathfrak{L}(\mathbb{R})$:= Frm$\langle\, (-,q), (p,-) (p, q \in \mathbb{Q}) \mid$ (1) $(-,q) \wedge (p,-) = 0$ for $q \leqslant p$,

(2) $(-,q) \vee (p,-) = 1$ for $q > p$,

(3) $(-,q) = \bigvee_{s<q} (-,s)$,

(4) $\bigvee_{q\in\mathbb{Q}} (-,q) = 1$,

(5) $(p,-) = \bigvee_{r>p} (r,-)$,

(6) $\bigvee_{p\in\mathbb{Q}} (p,-) = 1 \,\rangle$.

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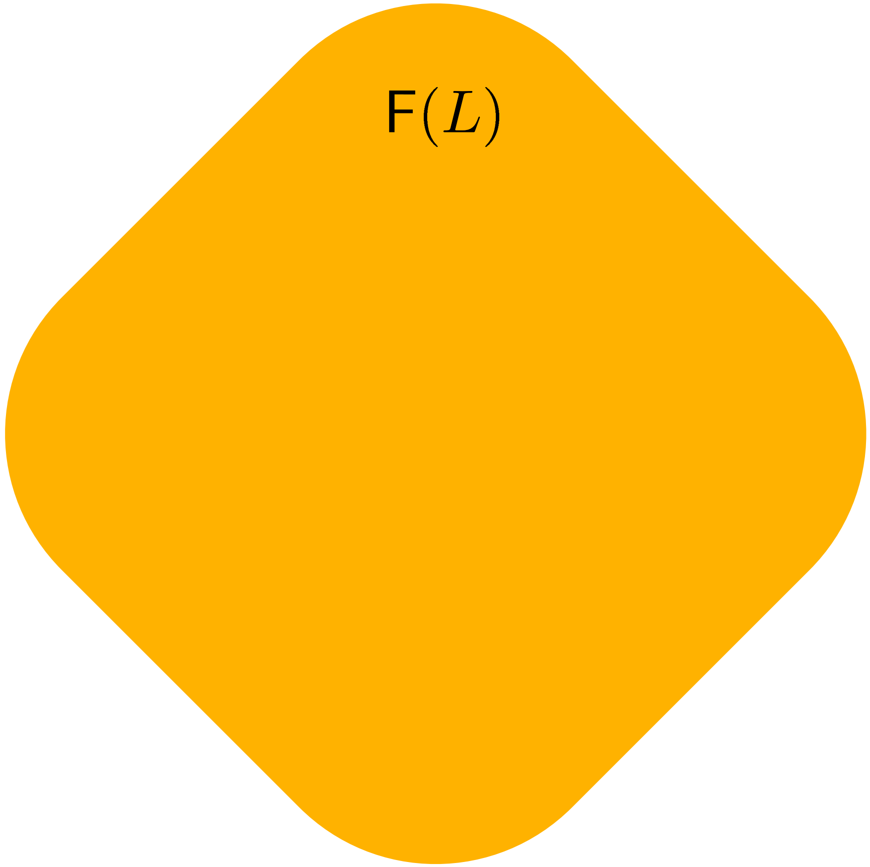

SEMICONTINUITY AND CONTINUITY

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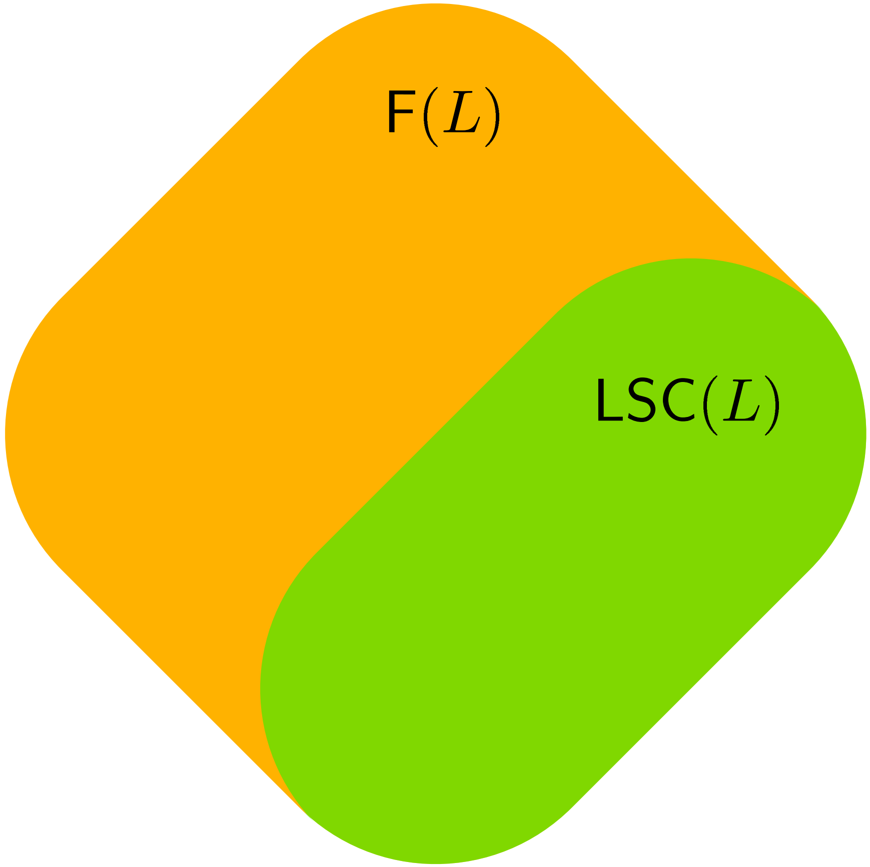

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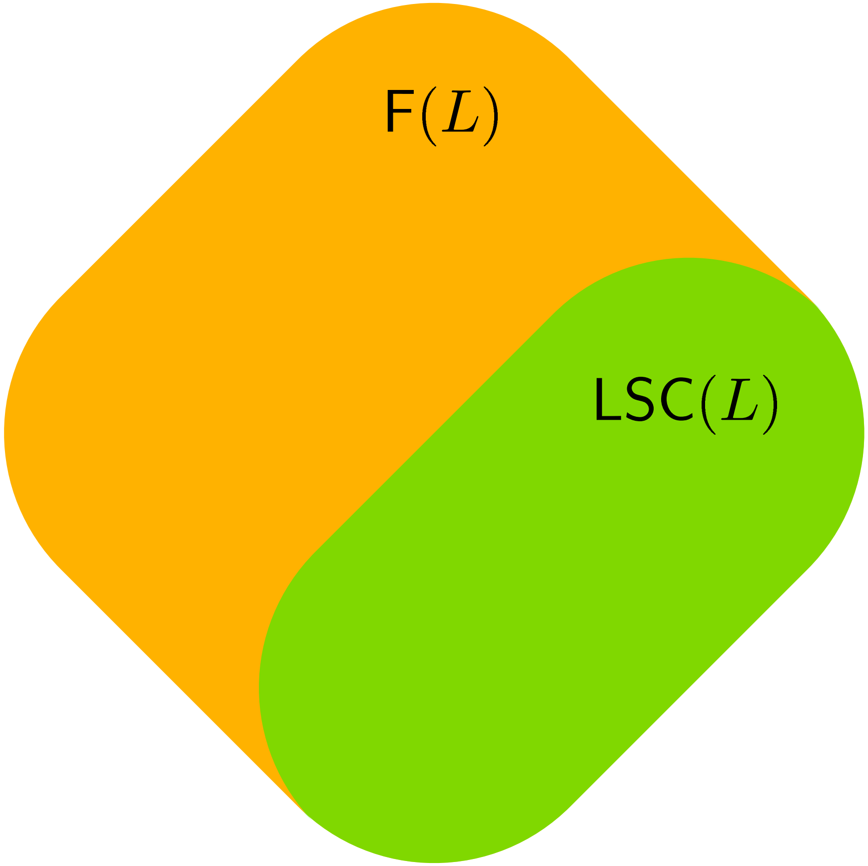

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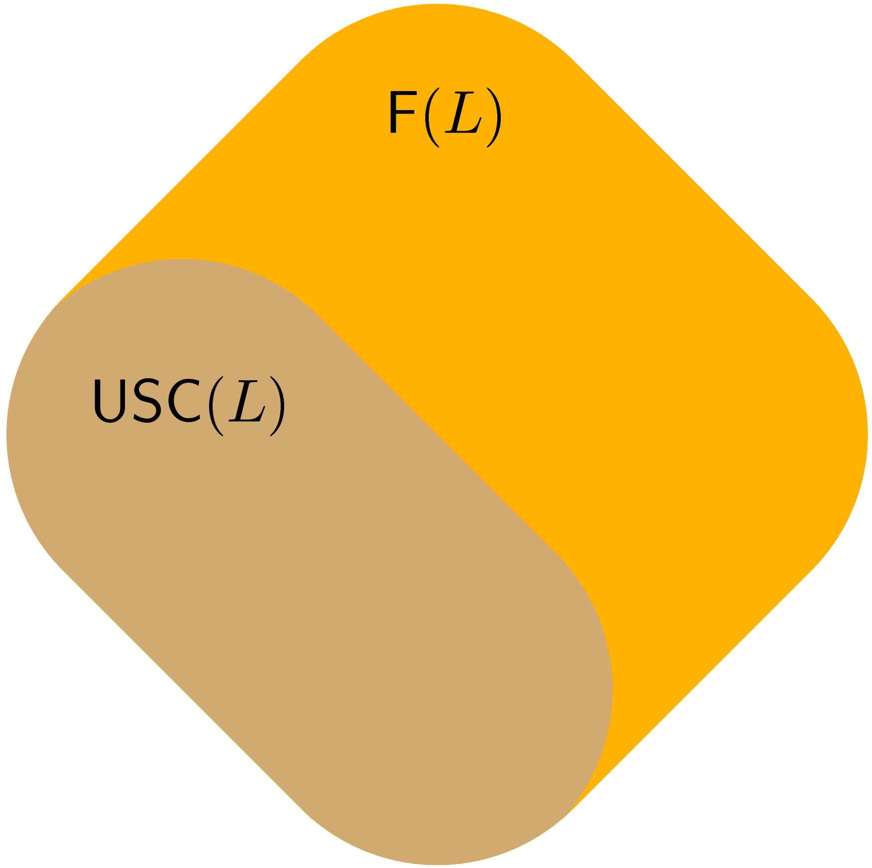

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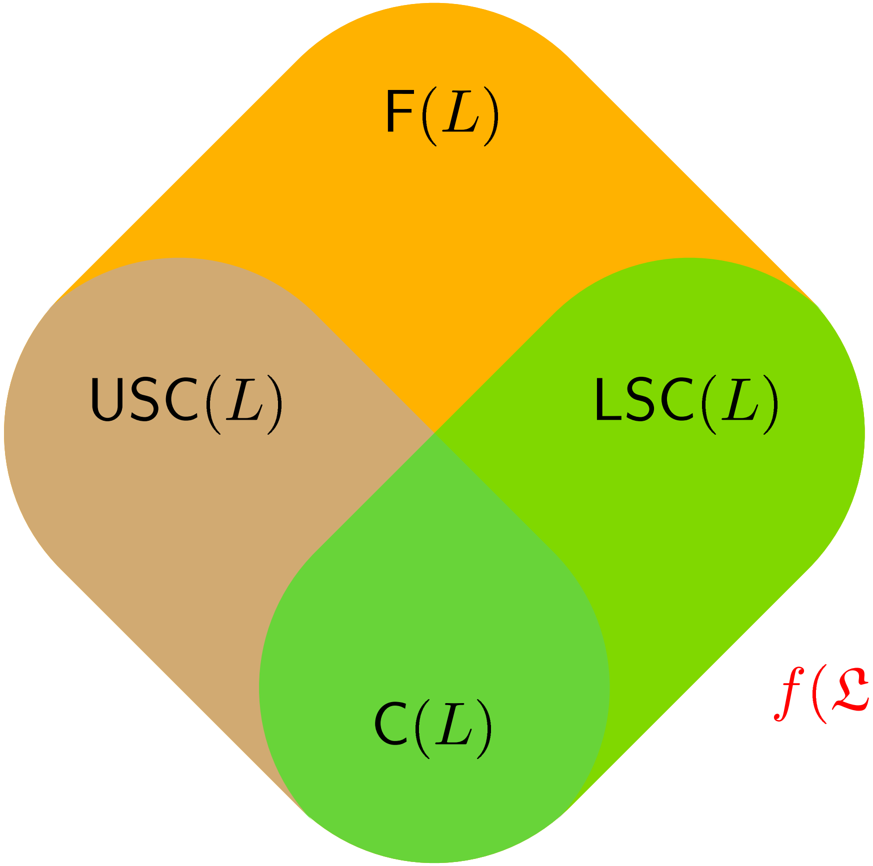

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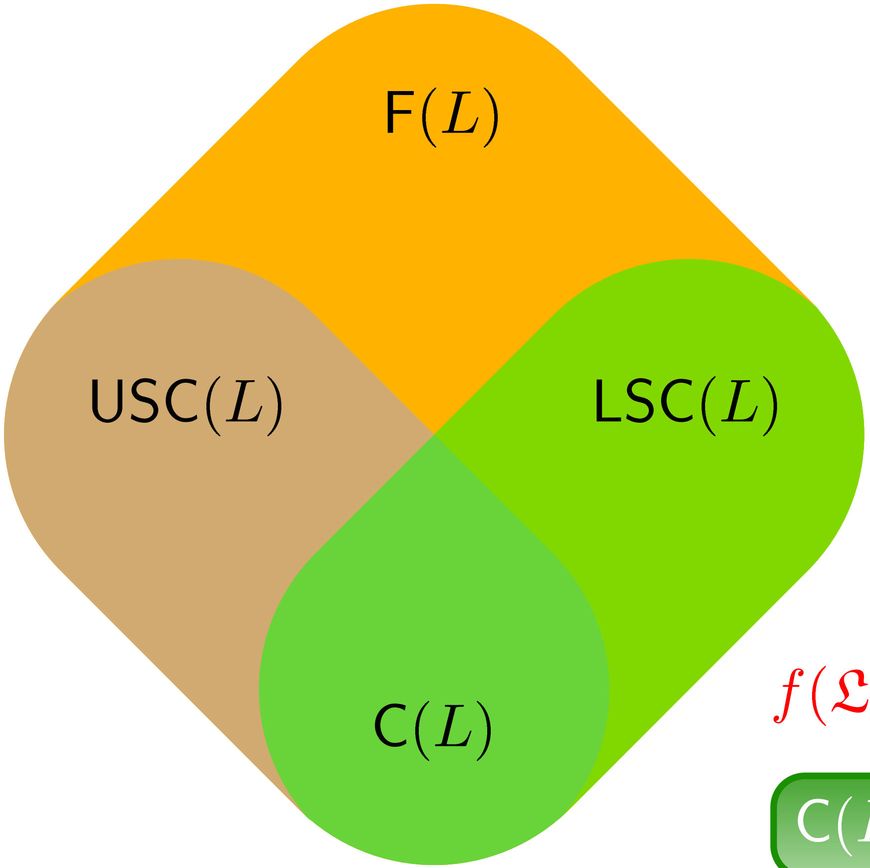

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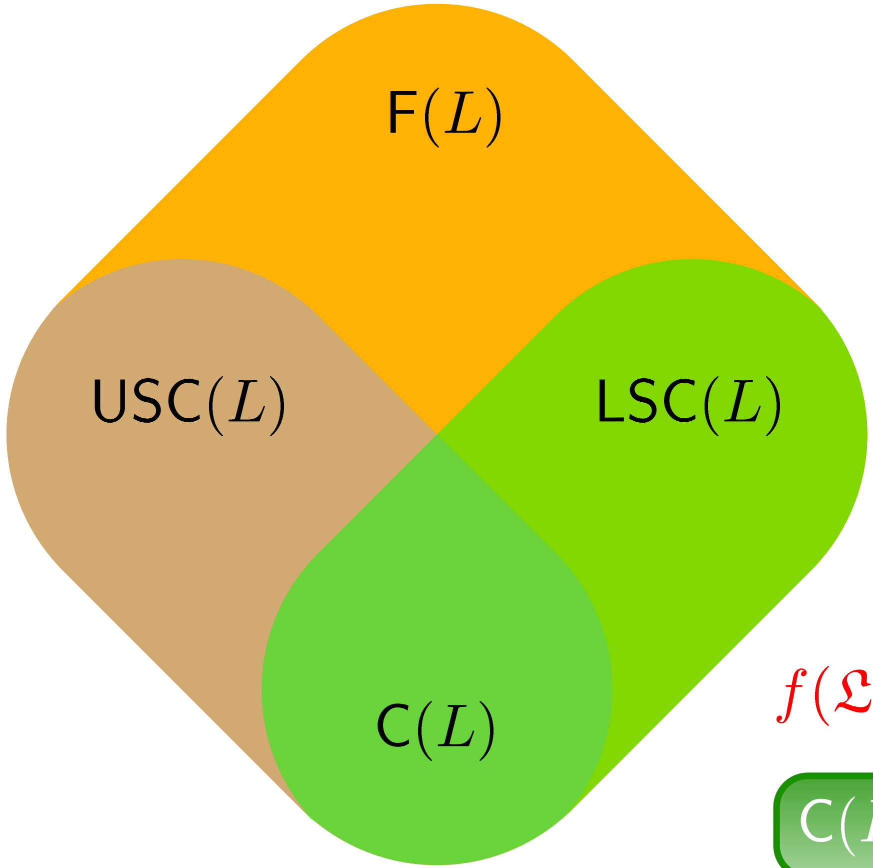

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$\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$

$$f \leqslant g \;\equiv\; f(p, -) \leqslant g(p, -), \; \forall p \in \mathbb{Q}$$

$\mathscr{B}$-SEMICONTINUITY AND $\mathscr{B}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}(L) \,:\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

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$$\mathscr{B}\text{-}\mathsf{USC}(L) \overset{\text{def}}{\equiv} \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{B} \,:\;\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

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$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

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["$\Leftarrow$": $f(\text{—}, q) = \bigvee_{r<q} f(\text{—}, r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(\text{—}, q)$.]

$$\mathscr{B}\text{-}\mathsf{USC}(L) \overset{\text{def}}{\equiv} \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{B} \;:\;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

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$$\mathscr{B}\text{-}\mathsf{C}(L) = \mathscr{B}\text{-}\mathsf{LSC}(L) \cap \mathscr{B}\text{-}\mathsf{USC}(L)$$

$\therefore\;$ f is $\mathscr{B}^{\mathsf{c}}$-continuous iff it is $\mathscr{B}$-continuous.

$\mathscr{B}$-semicontinuity and $\mathscr{B}$-continuity: EXAMPLES

$\mathscr{B}$	$\mathscr{B}$-usc	$\mathscr{B}$-lsc	$\mathscr{B}$-continuous
$\mathfrak{c}$	usc	lsc	continuous

$\mathscr{B}$-semicontinuity and $\mathscr{B}$-continuity: EXAMPLES

$\mathscr{B}$	$\mathscr{B}$-usc	$\mathscr{B}$-lsc	$\mathscr{B}$-continuous
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$\mathfrak{c}^*$	normal usc	normal lsc	normal continuous

$\mathscr{B}$-semicontinuity and $\mathscr{B}$-continuity: EXAMPLES

$\mathscr{B}$	$\mathscr{B}$-usc	$\mathscr{B}$-lsc	$\mathscr{B}$-continuous
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$\mathfrak{c}^*$	normal usc	normal lsc	normal continuous
$\mathfrak{c}_\delta$	regular usc	regular lsc	regular continuous

$\mathscr{B}$-SEMICONTINUITY AND $\mathscr{B}$-CONTINUITY: EXAMPLES

$\mathscr{B}$	$\mathscr{B}$-usc	$\mathscr{B}$-lsc	$\mathscr{B}$-continuous
$\mathfrak{c}$	usc	lsc	continuous
$\mathfrak{c}^*$	normal usc	normal lsc	normal continuous
$\mathfrak{c}_\delta$	regular usc	regular lsc	regular continuous
$\mathfrak{c}_{\mathrm{coz}}$	zero usc	zero lsc	zero continuous

APPLICATION: insertion theorems

GENERAL INSERTION THEOREM:

TFAE for any frame L and any sublocale selection $\mathscr{B}$:

1. L is completely $\mathscr{B}$-normal.

APPLICATION: insertion theorems

GENERAL INSERTION THEOREM:

TFAE for any frame L and any sublocale selection $\mathscr{B}$:

1. L is completely $\mathscr{B}$-normal.

2. $$\underbrace{f_1, f_2}_{\mathsf{F}(L)}: \ f_1 \leqslant \underbrace{g_1}_{\mathscr{B}-\mathsf{LSC}(L)} \leqslant f_2\,, \ f_1 \leqslant \underbrace{g_2}_{\mathscr{B}-\mathsf{USC}(L)} \leqslant f_2$$

$$\Downarrow$$

$\exists\, l \in \mathscr{B}$-$\mathsf{LSC}(L), u \in \mathscr{B}$-$\mathsf{USC}(L):\ f_1 \leqslant l \leqslant u \leqslant f_2$.

APPLICATION: insertion theorems

COROLLARY 1 (case $\mathscr{B} = \mathfrak{c}$). TFAE for any frame L:

1. L is completely normal.
2. $\underbrace{f_1,\ f_2}_{\mathsf{F}(L)} \colon f_1 \leqslant f_2^\circ\,,\ f_1^- \leqslant f_2 \quad \Rightarrow\ \exists\, l \in \mathsf{LSC}(L) \colon f_1 \leqslant l \leqslant l^- \leqslant f_2$.

APPLICATION: insertion theorems

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COROLLARY 2 (case $\mathscr{B} = \mathfrak{o}$). TFAE for any frame L:

1. L is completely extremally disconnected.
2. $\underbrace{f_1,\ f_2}_{\mathsf{F}(L)} : f_1 \leqslant f_2^{\circ}\,,\ f_1^{-} \leqslant f_2 \quad \Rightarrow\ \exists\, l \in \mathsf{LSC}(L),\ \exists\, u \in \mathsf{USC}(L) :$
$f_1 \leqslant u \leqslant l \leqslant f_2$.

APPLICATION: insertion theorems

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$L = \mathcal{O}(X)$

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$$f_1 \leqslant u \leqslant l \leqslant f_2.$$

APPLICATION: insertion theorems

COROLLARY 3:

TFAE for any frame L:

1. L is completely normal and extremally disconnected.
2. L is normal and completely extremally disconnected.
3. $\underbrace{f,\ g}_{\mathsf{F}(L)},\ f \leqslant g^{\circ},\ f^{-} \leqslant g \quad \Rightarrow \quad \exists\, h \in \mathsf{C}(L)\colon f \leqslant h \leqslant g.$

MAIN REFERENCES

- J. Gutiérrez García & JP, *On the parallel between normality and extremal disconnectedness*, JOURNAL OF PURE AND APPLIED ALGEBRA (2014).
- J. Gutiérrez García, T. Kubiak & JP, *On extremal disconnectedness and its hereditary property*, IN PREPARATION.
- JP & A. Pultr, *Frames and locales: topology without points*, SPRINGER BASEL (2012).

$f \in \mathsf{F}(L)$ s.t. $\{g \in \mathsf{LSC}(L) \mid g \leqslant f\} \neq \varnothing$

- lower regularization f°

$$\boxed{f^\circ(p,—) = \bigvee_{q>p} \overline{f(q,—)}} \qquad \boxed{f^\circ(—,q) = \bigvee_{p<q} \overline{f(p,—)}^{\,*}}$$

Then: $f^\circ \in \mathsf{LSC}(L)$

$f^\circ \leqslant f$

$f^\circ = \bigvee \{g \in \mathsf{LSC}(L) \mid g \leqslant f\}$

- Dually: the upper regularization $f^- = -(-f)^\circ$.