A pointfree extension of the Fletcher construction

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FLETCHER CONSTRUCTION IN TOP

$(X, \mathcal{T})$ [P. Fletcher, 1970]

$\mathfrak{A}$ = family of interior-preserving open covers

For each open part $A \in \mathcal{A} \in \mathfrak{A}$ let

$$E_A := (A \times A) \cup (X \setminus A \times X)$$

$$R_{\mathcal{A}} := \bigcap_{A \in \mathcal{A}} E_A \quad (\mathcal{A} \in \mathfrak{A})$$

$$S_{\mathfrak{A}} := \{R_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}\}.$$

$S_{\mathfrak{A}}$ is a subbase for a (transitive) quasi-unif. $\mathcal{E}_{\mathfrak{A}}$ on X such that $\underbrace{\mathcal{T}_1(\mathcal{E}_{\mathfrak{A}}) = \mathcal{T}}_{\mathcal{E}_{\mathfrak{A}} \text{ is compatible with the given } \mathcal{T}}$

PROBLEM [Brümmer, 2001]:

How can one express the Fletcher construction in quasi-uniform frames/locales?

POINTFREE TOPOLOGY

Frm

- frames (locales) L ‖ complete lattice $a \wedge \bigvee b_i = \bigvee (a \wedge b_i)$
- homomorphisms $f : L \to M$ ‖ $f(\bigvee a_i) = \bigvee f(a_i)$, $f(a \wedge b) = f(a) \wedge f(b)$

TOPOLOGY $\overset{\text{abstraction}}{\rightsquigarrow}$ POINTFREE TOPOLOGY

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, squiggly, "\mathcal{O}"] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, squiggly] & (\mathcal{O}Y, \subseteq) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$\boxed{\text{Loc}}$=Frm^{op}

locale $X \leftrightsquigarrow$ frame L

THE CATEGORY OF QUASI-UNIFORM FRAMES

[J. P., 1995]

Entourages

$E \in L \oplus L$ $\quad\equiv\quad$ $E \subseteq (L \times L, \leq)$

$$(x, y) \leq (z, w) \in E \Rightarrow (x, y) \in E$$

$$\{x\} \times S \subseteq E \Rightarrow (x, \bigvee S) \in E$$

$$S \times \{x\} \subseteq E \Rightarrow (\bigvee S, x) \in E$$

such that $\displaystyle\bigvee_{(x,x)\in E} x = 1.$

$$x \oplus y := \downarrow (x, y) \cup \{(0, a), (a, 0) \mid a \in L\}$$

$$E \circ F := \bigvee\{x \oplus y \mid \exists z \neq 0 : x \oplus z \subseteq E, z \oplus y \subseteq F\}$$

$\boxed{\text{Objects}}$ $(L, \mathcal{E})$ $\mathcal{E} \neq \emptyset$ filter of $(Ent(L), \subseteq)$

(Q1) $\forall\, E \in \mathcal{E} \quad \exists F \in \mathcal{E} : F \circ F \subseteq E$

(Q2) $\forall\, x \in L \quad x = \bigvee \{ y \in L \mid \underbrace{y \overset{\overline{\mathcal{E}}}{\lhd_1} x}\}$

$\boxed{\exists\, E \in \overline{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}^{-1} : E \circ (y \oplus y) \subseteq x \oplus x}$

$\boxed{y \overset{\overline{\mathcal{E}}}{\lhd_2} x \;\equiv\; \exists\, E \in \overline{\mathcal{E}} : (y \oplus y) \circ E \subseteq x \oplus x}$

$$(L, \mathcal{E}) \in QUFrm \xrightarrow{\mathcal{L}_i(\mathcal{E}) := \{x \in L \mid x = \bigvee \{y \in L \mid y \overset{\mathcal{E}}{\lhd_i} x\}\}} (L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$$

is a biframe

$\boxed{\text{Morphisms}}$ $h : (L, \mathcal{E}_1) \longrightarrow (M, \mathcal{E}_2)$

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "h"'] & L \oplus L \arrow[d, dashed, "h \oplus h"] & L \arrow[l] \arrow[d, "h"] \\
M \arrow[r] & M \oplus M & M \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
E \in \mathcal{E}_1 \arrow[d, mapsto, "h \oplus h"] \\
(h \oplus h)(E) \in \mathcal{E}_2
\end{tikzcd}$$

FRM

PROBLEM

PROBLEM

$\boxed{\text{THE SUBOBJECT LATTICE}}$ Locale X

$$\begin{array}{ll} \mathcal{S}(X) & \text{is a CO-FRAME} \\ \Big\updownarrow \cong & \\ (\mathfrak{C}L)^{op} & \text{dual of the congruence lattice of } L = \mathcal{O}X \end{array}$$

$$\left.\begin{array}{ll} \nabla_a := \{(x,y) \mid x \vee a = y \vee a\} & \text{CLOSED} \\ \Delta_a := \{(x,y) \mid x \wedge a = y \wedge a\} & \text{OPEN} \end{array}\right\} \text{complem.}$$

$\nabla L := \{\nabla_a \mid a \in L\}$ subframe of $\mathfrak{C}L$

$L \to \nabla L : a \mapsto \nabla_a$ isomorphism

$\Delta L :=< \{\Delta_a \mid a \in L\} >$ subframe of $\mathfrak{C}L$

$L \to \Delta L : a \mapsto \Delta_a$ dual poset embedding

$\boxed{(\mathfrak{C}L, \nabla L, \Delta L) \text{ is a biframe}}$ SKULA BIFRAME of L

$\boxed{\text{MOTIVATING EXAMPLE}}$ [W. Hunsaker, J.P., 2002]

Frame L

$(\mathfrak{C}L, \nabla L, \Delta L)$

$E_a := (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a) \ (a \in L)$

The E_a $(a \in L)$ generate a quasi-uniformity $\mathcal{P}$ on $\mathfrak{C}L$

$$\begin{tikzcd}
(\mathfrak{C}L, \mathcal{P}) \arrow[d, mapsto, squiggly] \\
\mathcal{L}_1(\mathcal{P}) = \nabla L \cong L
\end{tikzcd}$$

compatible

$\boxed{\text{The pointfree Császár-Pervin quasi-unif. } \mathcal{P}}$

FLETCHER CONSTRUCTION IN FRM

$$E_a := (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a) \qquad a \in L$$

$\mathcal{A}$: family of covers of L Interior-preserving covers

LOC $\overset{dual}{\longleftrightarrow}$ FRM

$$int(\bigwedge_I U_i) = \bigwedge_I U_i \qquad\qquad int(\bigvee_I \Delta_{a_i}) = \bigvee_I \Delta_{a_i}$$

$$\| $$

$$\Delta_{\bigwedge_I a_i}$$

$$R_A := \bigcap_{a \in A} E_a \in Ent(\mathfrak{C}L)? \qquad A \in \mathcal{A}$$

$$\Updownarrow$$

$$\boxed{\bigvee_{A_1 \cup A_2 = A} (\bigwedge_{A_1} \nabla_a \wedge \bigwedge_{A_2} \Delta_a) = 1}$$

Fletcher covers

EXAMPLES

Finite covers

Locally finite covers $A \subseteq L$

- $\exists$ cover $C \subseteq L$ s.t., for every $c \in C$,

$$A_c := \{a \in A \mid a \wedge c \neq 0\} < \infty.$$

Spectra covers $A = \{a_n \mid n \in \mathbb{Z}\} \subseteq L$

- $a_n \leq a_{n+1}$
- $\bigvee_{n \in \mathbb{Z}} \Delta_{a_n} = 1$ (in particular, $\bigwedge_{n \in \mathbb{Z}} a_n = 0$).

Well-monotone covers $A \subseteq L$,

- well-ordered by the partial order $\leq$ of L.

$\boxed{\text{THE CONSTRUCTION}}$

$\mathcal{A} =$ family of int.-pres. Fletcher covers of L

$\mathcal{S}_{\mathcal{A}} := \{R_A \mid A \in \mathcal{A}\}$

$\mathcal{E}_{\mathcal{A}} :=$ the filter of $Ent(\mathfrak{C}L)$ generated by $\mathcal{S}_{\mathcal{A}}$

$\boxed{\text{LEMMA. } \bigcup \mathcal{A} \text{ subbase of } L \Rightarrow \mathcal{L}_1(\mathcal{E}_{\mathcal{A}}) = \nabla L}$

$\boxed{\text{PROBLEM}}$ But, in general, $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}}) \subseteq \Delta L$

so $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}}))$ may not be a biframe!

$\boxed{\text{SOLUTION}}$ $\mathfrak{C}L' = <\nabla L \cup \mathcal{L}_2(\mathcal{E}_{\mathcal{A}})>$ subframe of $\mathfrak{C}L$

$$R'_A := R_A \cap (\mathfrak{C}L' \times \mathfrak{C}L')$$

$$\mathcal{S}'_{\mathcal{A}} := \{R'_A \mid A \in \mathcal{A}\}$$

$\boxed{(\mathfrak{C}L', \mathcal{E}'_{\mathcal{A}}) \text{ is a } \underbrace{\text{quasi-uniform}}_{\text{transitive}} \text{ frame compatible with } L}$

EXAMPLES

Subbase	Quasi-unif.
$\{R_A \mid A$ finite cover of $L\}$	$\mathcal{P}$
$\{R_A \mid A$ int.-pres. Fletcher cover of $L\}$	$\mathcal{FT}$
$\{R_A \mid A$ locally finite cover of $L\}$	$\mathcal{LF}$
$\{R_A \mid A$ cover of L, well-ordered by $\leq\}$	$\mathcal{W}$
$\{R_A \mid A$ open spectrum of $L\}$	$\mathcal{SC}$

THE CONSTRUCTION ACCOUNTS FOR ALL TRANSITIVE QUASI-UNIFORMITIES

$\mathcal{E}$ a transitive quasi-unif. on a subframe $\mathfrak{C}L' \subseteq \mathfrak{C}L$, compatible with L

$\mathcal{S}$ transitive subbase, $E \in \mathcal{S}$

$$st_1(\theta, E) := \bigvee\{\alpha \in \mathfrak{C}L' \,|\, (\alpha, \beta) \in E, \beta \wedge \theta \neq 0\} \in \underbrace{\mathcal{L}_1(\mathcal{E})}_{=\nabla L}$$

$$st_1(\theta, E) = \nabla_{E[\theta]} \text{ for some } E[\theta] \in L$$

$$CovE := \{E[\theta] \,|\, (\theta, \theta) \in E\}$$

PROPOSITION. For each $E \in \mathcal{S}$:

(1) $CovE$ is an int.-pres. Fletcher cover of L.

(2) $\bigcup_{E \in \mathcal{S}} CovE$ is a subbase for L.

$\mathcal{A} \subseteq CovL$ induces $\mathcal{E}$ if $\{R'_A \mid A \in \mathcal{A}\}$ is a subbase of $\mathcal{E}$

THM. 1. Each compatible transitive quasi-unif. on a subframe of $\mathfrak{C}L$ is induced by a set $\mathcal{A}$ of int.-pres. Fletcher covers of L s.t. $\bigcup \mathcal{A}$ is a subbase for L.

THM 2. Let $\mathcal{E}$ be a compatible transitive quasi-unif. on a subframe of $\mathfrak{C}L$ and let

$$\mathcal{A} = \{A \mid A \in CovL, R'_A \in \mathcal{E}\}.$$

Then:

(1) $\mathcal{A}$ is the largest subset of $CovL$ that induces $\mathcal{E}$.

(2) Each $A \in \mathcal{A}$ is an int.-pres. Fletcher cover of L.

(3) $\bigcup \mathcal{A}$ is a base for L.