Frames and locales: topology without points

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OUTLINE

- AIM: to give an overview of the basics of point-free topology.

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- Part I. Frames: the algebraic facet of spaces
- Part II. Locales: the geometric facet of frames
- Part III. Categorical aspects of **Frm** and **Loc**
- Part IV. Doing topology in **Loc**

WHAT IS POINT-FREE TOPOLOGY?

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- The techniques may hide some geometrical intuition, but often offers powerful algebraic tools and opens new perspectives.

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'topology itself'

> *«The topological structure of a locale cannot live in its points: the points, if any, live on the open sets rather than the other way about.»*
>
> P. T. Johnstone
>
> [The art of pointless thinking, *Category Theory at Work* (1991)]

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'lattice theory applied to topology' 'topology itself'

> «(...) *what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.*» R. BALL & J. WALTERS-WAYLAND
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> [C- and C*-quotients in pointfree topology, *Dissert. Math.* (2002)]

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MORE: different categorical properties with advantage to the point-free side.

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- RAMIFICATIONS: category theory, topos theory, logic and computer science.

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4. J. Isbell (1972). Introduced the category of locales, as a substitute for (and in many ways an improvement on) the category of topological spaces. Revolutionary step of turning the arrows around.

MAIN BASIC REFERENCES

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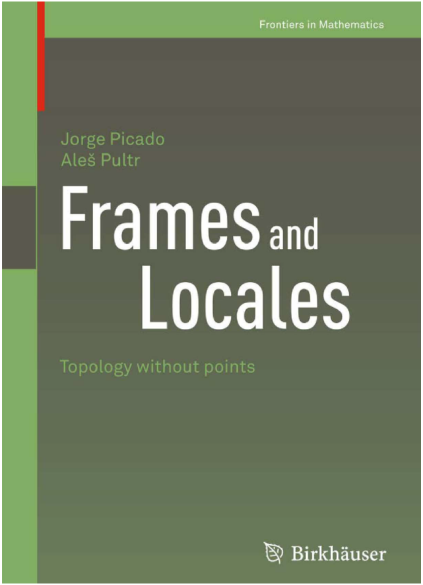

PART I.

Frames: the algebraic facet of spaces

FROM SPACES TO FRAMES

Top

$(X, \Omega(X))$

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• complete lattice:

$$\bigvee U_i = \bigcup U_i, \quad 0 = \emptyset$$

$$U \wedge V = U \cap V, \quad 1 = X$$

$$\bigwedge U_i = \mathrm{int}(\bigcap U_i)$$

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more:

$$\boxed{U \wedge \bigvee_I V_i = \bigvee_I (U \wedge V_i)}$$

Top

$$
\begin{tikzcd}
(X, \Omega(X)) \arrow[r, squiggly] \arrow[d, "f"'] & (\Omega(X), \subseteq) \\
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\end{tikzcd}
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$$\begin{array}{ccc} (X, \Omega(X)) & \rightsquigarrow & (\Omega(X), \subseteq) \\ \Big\downarrow f & & \Big\uparrow f^{-1}[-] \\ (Y, \Omega(Y)) & \rightsquigarrow & (\Omega(Y), \subseteq) \end{array}$$

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- $f^{-1}[-]$ preserves $\bigvee$ and $\wedge$

FROM SPACES TO FRAMES

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frame:

$$\boxed{a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)}$$

• frame homomorphisms: $h\colon M \to L$ preserves $\bigvee$ and $\wedge$

FROM SPACES TO FRAMES

$$\boxed{\mathbf{Top}} \xrightarrow{\quad\Omega\quad} \boxed{\mathbf{Frm}}$$

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The algebraic nature of the category **Frm** is obvious.

More about that later on...

MORE EXAMPLES of frames

- Finite distributive lattices, complete Boolean algebras, complete chains.

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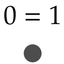
1: 

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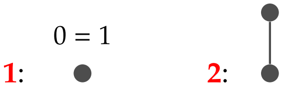

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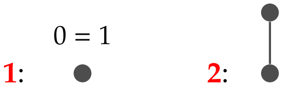

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- intervals of a frame L: $a, b \in L, a \leq b$
  $[a, b] = \{x \in L \mid a \leq x \leq b\}$, $\quad {\downarrow}b = [0, b]$, $\quad {\uparrow}a = [a, 1]$.

MORE EXAMPLES of frames: relatives of $\mathbf{Frm}$

- For any $\wedge$-semilattice $(A, \wedge, 1)$, $\mathfrak{D}(A) = \{\text{down-sets of } A\}$ is a frame:

$$\bigwedge = \bigcap, \quad \bigvee = \bigcup.$$

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$$\mathbf{SLat} \xrightarrow{\mathfrak{D}} \mathbf{Frm}$$

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(forgetful functor)

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$$\boxed{\begin{array}{ccc} \mathbf{Hom_{Frm}}(\mathfrak{D}(A), L) & \simeq & \mathbf{Hom_{SLat}}(A, G(L)) \\ h & \mapsto & (\widetilde{h}\colon a \mapsto h(\downarrow a)) \\ (\overline{g}\colon S \mapsto \bigvee g[S]) & \leftarrow\!\shortmid & g \end{array}}$$

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(inclusion as a non-full subcategory)

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$$\bigvee : \mathfrak{I}(L) \to L, \quad J \mapsto \bigvee J \qquad\qquad \bigvee : \mathfrak{D}(L) \to L, \quad S \mapsto \bigvee S$$

BACKGROUND: POSETS AS CATEGORIES

$(A, \leq)$ as a thin category

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OBJECTS: $a \in A$

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$(A, \leq)$ as a thin category $\left[\begin{array}{l} \text{OBJECTS: } a \in A \\ \\ \text{MORPHISMS: } a \xrightarrow{\exists!} b \text{ whenever } a \leq b \end{array}\right.$

(there is at most one arrow between any pair of objects)

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$$
\begin{tikzcd}
a \arrow[loop above, "1_a"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[loop above, "1_b"] \arrow[d, "g"] \\
& c \arrow[loop below, "1_c"]
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In fact, a preorder suffices:

(1) reflexivity: provides the identity morphisms 1_a.

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(2) transitivity: provides the composition of morphisms $g \circ f$.

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$$\begin{tikzcd} a \arrow[r, squiggly] \arrow[d, "\leq"'] & f(a) \arrow[d, "\leq"] \\ a' \arrow[r, squiggly] & f(a') \end{tikzcd}$$

order-preserving maps

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(binary) PRODUCTS: $a \longleftarrow \quad ? \quad \longrightarrow b$

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order-preserving maps

(binary) PRODUCTS:

$$\begin{tikzcd} a & ? \arrow[l] \arrow[r] & b \\ & c \arrow[ul] \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

$(A, \leq)$ as a thin category

- OBJECTS: $a \in A$
- MORPHISMS: $a \xrightarrow{\exists!} b$ whenever $a \leq b$

FUNCTORS: $f : A \longrightarrow B$

$$\begin{tikzcd} a \arrow[r, squiggly] \arrow[d, "\leq"'] & f(a) \arrow[d, "\leq"] \\ a' \arrow[r, squiggly] & f(a') \end{tikzcd}$$

order-preserving maps

(binary) PRODUCTS:

$$\begin{tikzcd} a & a \wedge b \arrow[l] \arrow[r] & b \\ & c \arrow[ul] \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

meets

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order-preserving maps

(binary) COPRODUCTS:

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & a \vee b \arrow[d, dotted] & b \arrow[l] \arrow[dl] \\ & c & \end{tikzcd}$$

joins

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From this point of view:

CATEGORY THEORY is an extension of LATTICE THEORY

(GALOIS) ADJUNCTIONS

$$(A, \leq) \underset{g}{\overset{f}{\rightleftarrows}} (B, \leq)$$

$$\boxed{f(a) \leq b \ \ \text{iff} \ \ a \leq g(b)}$$

(GALOIS) ADJUNCTIONS

$$\mathbf{Hom}_B(f(a), b) \cong \mathbf{Hom}_A(a, g(b))$$

$$\begin{tikzcd}(A, \leq) \arrow[r, shift left, "f"] & (B, \leq) \arrow[l, shift left, "g"]\end{tikzcd}$$

$$f \dashv g \qquad \boxed{f(a) \leq b \ \text{ iff } \ a \leq g(b)}$$

(GALOIS) ADJUNCTIONS

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$$\begin{tikzcd} (A, \leq) \arrow[r, shift left, "f"] & (B, \leq) \arrow[l, shift left, "g"] \end{tikzcd}$$

$f \dashv g$ $\boxed{f(a) \leq b \text{ iff } a \leq g(b)}$

$\Leftrightarrow$ $\boxed{fg \leq \mathrm{id} \text{ and } \mathrm{id} \leq gf}$

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$$g[B] = \{a \in A \mid gf(a) = a\}$$

$$f[A] = \{b \in B \mid fg(b) = b\}$$

ADJOINT FUNCTOR THEOREM

Let $f : A \longrightarrow B$ be an order-preserving map between posets. Then:

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FRAMES as HEYTING ALGEBRAS

Heyting algebra: lattice L with an extra $\rightarrow$ satisfying

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$\therefore$ frames = cHa. BUT different categories (morphisms).

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H1 $$a \rightarrow (\bigwedge b_i) = \bigwedge (a \rightarrow b_i).$$ Properties

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H1 $$a \to (\bigwedge b_i) = \bigwedge (a \to b_i).$$

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H1 $a \to (\bigwedge b_i) = \bigwedge(a \to b_i)$.

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H4 $a \to b = a \to (a \wedge b)$.

H5 $a \wedge (a \to b) = a \wedge b$.

H6 $a \wedge b = a \wedge c$ iff $a \to b = a \to c$.

H7 $(a \wedge b) \to c = a \to (b \to c) = b \to (a \to c)$.

H8 $a = (a \vee b) \wedge (b \to a)$.

H9 $a \le (a \to b) \to b$.

H10 $((a \to b) \to b) \to b = a \to b$.

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P3 $(\bigvee a_i)^* = \bigwedge a_i^*$. [De Morgan law] (Caution: not for $\bigwedge$)

PART II.

Locales: the geometric facet of frames

MAKING THE PICTURE COVARIANT: the category of locales

$$\mathbf{Top} \xrightarrow{\Omega} \mathbf{Frm}$$

Contravariant

$$\begin{tikzcd}
(X,\Omega(X)) \arrow[r, squiggly] \arrow[d, "f"'] & (\Omega(X),\subseteq) \\
(Y,\Omega(Y)) \arrow[r, squiggly] & (\Omega(Y),\subseteq) \arrow[u, "f^{-1}[-]"']
\end{tikzcd}$$

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$$\mathbf{Loc} = \mathbf{Frm}^{op}$$

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- MORPHISMS:

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$$\begin{tikzcd} L \arrow[d, "f"'] \\ M \end{tikzcd} \qquad \begin{tikzcd} L \\ M \arrow[u, "h"'] \end{tikzcd}$$

frame homomorphisms taken backwards

preserves $\bigvee$ (incl. 0)

$\wedge$ (incl. 1)

Loc **Frm**

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Each $h\colon M \longrightarrow L$ in **Frm** has a UNIQUELY defined right adjoint

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LOCALIC MAP:
a map $f\colon L \longrightarrow M$ that has a left adjoint f^* in **Frm**, i.e., preserving finite meets:

(1) $f^*(1) = 1$.

(2) $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$.

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Let $f : L \to M$ have a left adjoint f^*. Then:

(1) $f^*(1) = 1$ iff $f[L \smallsetminus \{1\}] \subseteq M \smallsetminus \{1\}$.

(2) $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$ $\forall a, b \in L$ iff

$$f(f^*(a) \to b) = a \to f(b) \ \ \forall a, b \in L.$$

Proof: (2) $\Rightarrow$: $x \le f(f^*(a) \to b)$ iff $f^*(x) \le f^*(a) \to b$

iff $f^*(x \wedge a) \le b$

iff $x \wedge a \le f(b)$

iff $x \le a \to f(b)$.

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PROOF: (2) $\Rightarrow$: $x \leq f(f^*(a) \to b)$ iff $f^*(x) \leq f^*(a) \to b$

iff $f^*(x \wedge a) \leq b$

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iff $x \leq a \to f(b)$.

$\Leftarrow$: $f^*(a \wedge b) \leq x$ iff $a \leq b \to f(x) = f(f^*(b) \to x)$

iff $f^*(a) \wedge f^*(b) \leq x$. ∎

Loc

- OBJECTS: locales = frames (=cHa)

- MORPHISMS: $L \xrightarrow{f} M$

  - $f(\bigwedge S) = \bigwedge f[S]$
  - $f(a) = 1 \Rightarrow a = 1$
  - $f(f^*(a) \to b) = a \to f(b)$

MAKING THE PICTURE COVARIANT: the category of locales

$\mathbf{Top} \xrightarrow{\Omega} \mathbf{Frm}$ is immediately modifiable to a functor

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$\mathbf{Top} \xrightarrow{\Omega} \mathbf{Frm}$ is immediately modifiable to a functor

$$\mathbf{Top} \xrightarrow{\mathsf{Lc}} \mathbf{Loc}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \Omega(X) \\
Y \arrow[r, mapsto] & \Omega(Y) \arrow[u, bend left, "\Omega(f)"]
\end{tikzcd}$$

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$$U \rightsquigarrow Y \smallsetminus \overline{f[X \smallsetminus U]}$$

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\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, squiggly] \\
Y \smallsetminus \overline{f[X \smallsetminus U]}
\end{tikzcd}$$

Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$

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$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathsf{c}}]^{\mathsf{c}}$)

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Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathsf{c}}]^{\mathsf{c}}$)

iff $V \subseteq \operatorname{int}(Y \smallsetminus f[X \smallsetminus U]) = Y \smallsetminus \overline{f[X \smallsetminus U]}$.

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map $\{*\} \longrightarrow X$

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Abstraction: a point of a *general* locale L is a localic map

$$p : 2 \quad \to \quad L$$

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$$\begin{aligned} p\colon 2 &\to L \\ 1 &\mapsto 1 \end{aligned}$$

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$\rightsquigarrow$ Lc

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$a \neq 1$

$\wedge$-IRREDUCIBLES

(PRIME ELEMENTS)

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Loc

$$\begin{tikzcd} \{*\} \arrow[r] \arrow[d, squiggly, "\mathsf{Lc}"] & X \\ \Omega(\{*\}) = 2 \arrow[r] & \Omega(X) \end{tikzcd}$$

Abstraction: a point of a *general* locale L is a localic map

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iff $x \leq p(0) = a$ or $y \leq p(0) = a$.

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$\wedge$-IRREDUCIBLES

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$\mathsf{Pt}(L)$

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$a \in L$, $\Sigma_a = \{p \in \mathsf{Pt}(L) \mid a \not\leq p\}$.

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$$\mathbf{Loc} \xrightarrow{\mathsf{Pt}} \mathbf{Top}$$

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$$\begin{tikzcd}
\mathbf{Loc} \arrow[r, "\mathsf{Pt}"] & \mathbf{Top} \\
L \arrow[r, squiggly] \arrow[d, "f"'] & \mathsf{Pt}(L) \arrow[d, "f_{|\mathsf{Pt}(L)}"] \\
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Localic maps send points to points

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\end{tikzcd}$$

Localic maps send points to points

$$\mathsf{Pt}(f)^{-1}(\Sigma_b) = \{p \in \mathsf{Pt}(L) \mid b \not\leq f(p)\} = \{p \mid f^*(b) \not\leq p\} = \Sigma_{f^*(b)}.$$

SPACES AND LOCALES

$$\begin{tikzcd} \textbf{Top} \arrow[r, shift left, "\mathsf{Lc}"] & \textbf{Loc} \arrow[l, shift left, "\mathsf{Pt}"] \end{tikzcd}$$

$$\mathbf{Top} \underset{\mathsf{Pt}}{\overset{\mathsf{Lc}}{\rightleftarrows}} \mathbf{Loc}$$

A frame is SPATIAL if it is isomorphic to some topology.

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A frame is SPATIAL if it is isomorphic to some topology.

$\mathsf{Lc}(X)$ is always spatial.

A space X is SOBER if every meet-irreducible open is of the form

$$X \setminus \overline{\{x\}}$$

for a $\underline{\text{unique}}$ $x \in X$.

$$\boxed{\mathbf{T_2} \subset \mathbf{Sob} \subset \mathbf{T_0}}$$

no relation with $\mathbf{T_1}$

SPACES AND LOCALES

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no relation with $\mathbf{T_1}$

$\mathsf{Pt}(L)$ is always sober.

SPACES AND LOCALES

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathsf{Lc}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, shift left, "\mathsf{Pt}"] \end{tikzcd}$$

UNIT:
$$\begin{aligned} \eta_X \colon X &\longrightarrow \mathsf{Pt}\,\mathsf{Lc}(X) \\ x &\longmapsto X \smallsetminus \overline{\{x\}} \end{aligned}$$

$$\mathbf{Top} \underset{\mathsf{Pt}}{\overset{\mathsf{Lc}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Loc}$$

UNIT:

$$\begin{aligned} \eta_X : X &\longrightarrow \mathsf{Pt}\,\mathsf{Lc}(X) \\ x &\longmapsto X \setminus \overline{\{x\}} \end{aligned}$$

PROPOSITION. η_X is a homeomorphism iff X is sober.

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COUNIT:

$$\begin{aligned} \varepsilon_L : \mathsf{Lc}\,\mathsf{Pt}(L) &\to L \\ \Sigma_a &\mapsto \bigvee\{b \in L \mid \Sigma_b \subseteq \Sigma_a\} \end{aligned}$$

$$\mathbf{Top} \underset{\mathsf{Pt}}{\overset{\mathsf{Lc}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Loc}$$

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PROPOSITION. ε_L is an isomorphism iff L is spatial.

SPACES AND LOCALES

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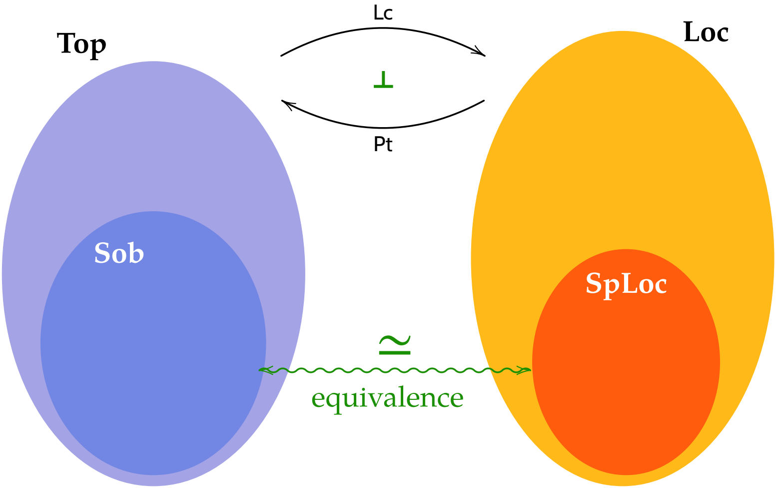
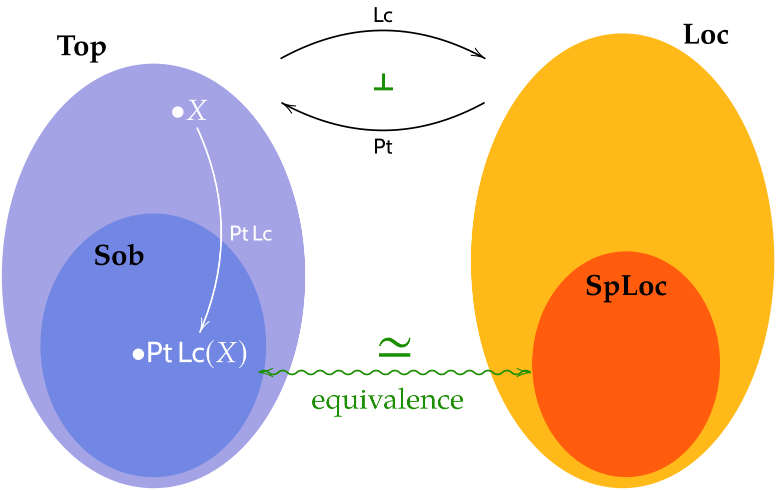
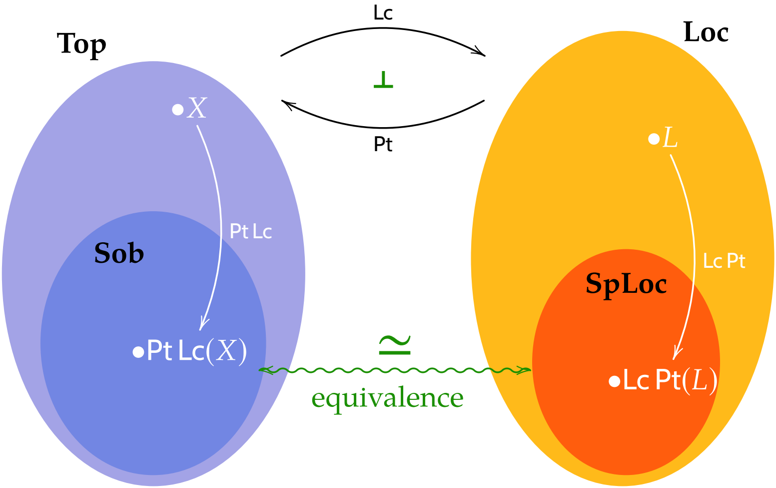


Perception: **Sob** more representative of all of **Top** than **SpLoc** of **Loc**.



“sobrification” of a space

SPACES AND LOCALES



"sobrification" of a space

"spatialization" of a locale

Each SOBER space can be reconstructed from the lattice $\Omega(X)$.

Moreover:

POINT-FREE REPRESENTATION OF TOPOLOGICAL DATA

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Moreover:

$$\mathbf{Top}(X, Y) \xrightarrow{\ \Omega\ } \mathbf{Loc}(\Omega(X), \Omega(Y))$$

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(less well-known: this characterizes the sobriety of Y)

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Moreover:

$$\mathsf{Top}(X, Y) \xrightarrow{\ \Omega\ } \mathsf{Loc}(\Omega(X), \Omega(Y))$$

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MORE: axiom T_D characterizes other facts...

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Moreover:

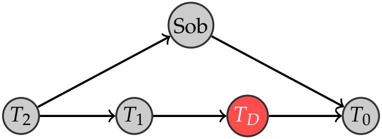
$$\mathbf{Top}(X, Y) \xrightarrow{\Omega} \mathbf{Loc}(\Omega(X), \Omega(Y))$$

for SOBER Y: $\simeq$

MORE: axiom T_D characterizes other facts...

Singletons are locally closed i.e. each $\{x\}$ is closed in some open U:

$\forall x \in X \;\; \exists$ open $U \ni x \colon \{x\} = U \cap \overline{\{x\}}$.



THE BOOLEAN CASE: non-spatial locales

PROPOSITION. L is spatial iff each $a \neq 1$ is a meet of points of L.

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L has "enough points"

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COROLLARY. In a Boolean algebra, every meet-irreducible is a co-atom. Therefore a Boolean locale B is spatial iff it is atomic.

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PROOF: Let $p < x$, p meet-irreducible.

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$$0 = x \wedge \neg x \leq p$$

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B spatial $\Rightarrow$ each $a \neq 1$ in B is a meet of co-atoms

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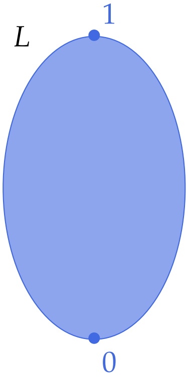
$\Leftrightarrow$ each $a \neq 1$ in B is a join of atoms (by complement.). ■

Generalized subspaces: SUBLOCALES

$S \subseteq L$ is a SUBLOCALE of L if:



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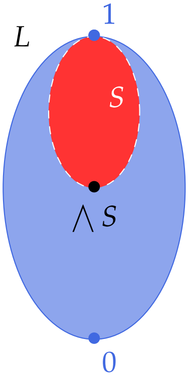
(S1) $\forall A \subseteq S,\ \bigwedge A \in S$.



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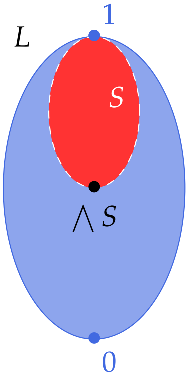

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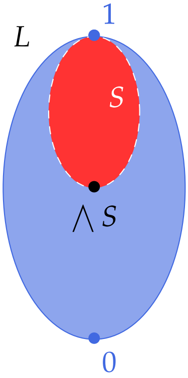

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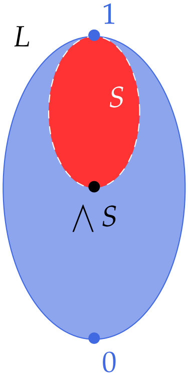

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Motivation for the definition:

Proposition

$S \subseteq L$ is a sublocale iff the embedding $j_S \colon S \subseteq L$ is a localic map.

THE SUBLOCALE LATTICE

$\mathcal{S}(L)$: sublocales of L, ordered by $\subseteq$

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PROPOSITION. $\mathcal{S}(L)$ is a co-frame.

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PROOF: $\forall i,\ x = a_i \wedge b_i\ (a_i \in A, b_i \in B_i)$. Let $a = \bigwedge a_i \in A$.

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Then $x = a \wedge b_i,\ \forall i \overset{(H6)}{\Longrightarrow} \underbrace{a \to b_i}_{b \in \bigcap B_i}$ does not depend on i.

$$\overset{(H5)}{=} a \wedge (a \to b_i) = a \wedge b \in A \vee (\bigcap B_i). \qquad \blacksquare$$

PART III.

Categorical aspects of $\mathbf{Frm}$

1. $\mathbf{Frm}$ is equationally presentable i.e.

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Objects are described by a (proper class of) operations and equations:

OPERATIONS:

- 0-ary: $0, 1 \colon L^0 \longrightarrow L$
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EQUATIONS:

- $(L, \wedge, 1)$ is an idempotent commutative monoid
- with a zero 0 sat. the absorption law $a \wedge 0 = 0 = 0 \wedge a\ \forall a$.
- $\bigvee_0 a_i = 0, a_j \wedge \bigvee_\kappa a_i = a_j, a \wedge \bigvee_\kappa a_i = \bigvee_\kappa (a \wedge a_i)$.

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COROLLARY

Frm has all (small) limits (i.e., it is a COMPLETE category)
and they are constructed exactly as in **Set**
(i.e., the forgetful functor **Frm** $\longrightarrow$ **Set** preserves them).

ALGEBRAIC ASPECTS OF Frm

2. **Frm** has free objects: there is a free functor **Set** $\longrightarrow$ **Frm** (i.e., a left adjoint of the forgetful functor **Frm** $\longrightarrow$ **Set**):

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CONSTRUCTION (in two steps):

$$\begin{tikzcd} \mathbf{SLat} & \mathbf{Frm} \arrow[l, "\text{forgets } \bigvee"'] \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=huge] \mathbf{Set} \arrow[r, bend left, "F"] & \mathbf{SLat} \arrow[l, "\text{forgets } \wedge"] & \mathbf{Frm} \arrow[l, "\text{forgets } \bigvee"] \end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & F(X) \arrow[d, "F(f)"] \\
Y \arrow[r, mapsto] & F(Y)
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\qquad F(X) = \big(\{S \subseteq X \mid S \text{ finite}\}, \supseteq\big)
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f[S]
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THE UNIT: $X \xrightarrow{\eta_X} F(X)$

$$x \rightsquigarrow \{x\}$$

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$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[ddrr, bend right, "f"'] & & F(X) \arrow[dd, dotted, bend left, "\exists!\ \varphi \in \mathbf{SLat}"] \\ & x \arrow[r, squiggly] & \{x\} \\ & & A \end{tikzcd}$$

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 & A
\end{tikzcd}
\qquad
x \rightsquigarrow \{x\}
\qquad
\begin{tikzcd}
\{x_1, x_2, \ldots, x_n\} \arrow[d, mapsto] \\
\bigwedge_{i=1}^{n} f(x_i)
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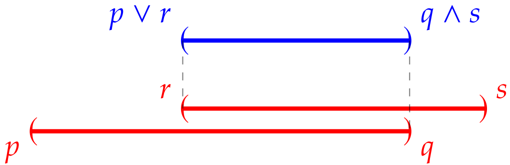
(6) And there exist presentations by generators and relations: just take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

Frame of reals $\mathfrak{L}(\mathbb{R})$

Generators: ordered pairs (p, q), $p, q \in \mathbb{Q}$,

Relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2)
(R3)
(R4)



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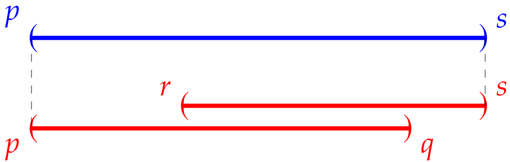
(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \le r < q \le s$,
(R3)
(R4)

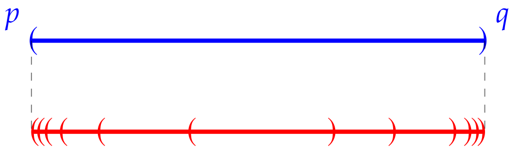


Frame of reals $\mathfrak{L}(\mathbb{R})$

Generators: ordered pairs (p,q), $\quad p,q \in \mathbb{Q}$,

Relations:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,
(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,
(R4)



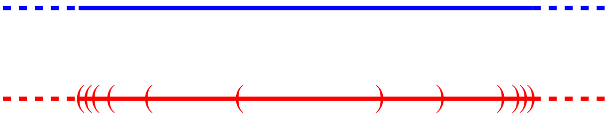
EXAMPLE 1: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

Generators: ordered pairs (p,q), $p,q \in \mathbb{Q}$,

Relations:

$$
\begin{aligned}
&\text{(R1)} \quad (p,q) \wedge (r,s) = (p \vee r, q \wedge s),\\
&\text{(R2)} \quad (p,q) \vee (r,s) = (p,s) \text{ whenever } p \le r < q \le s,\\
&\text{(R3)} \quad (p,q) = \bigvee\{(r,s) \mid p < r < s < q\},\\
&\text{(R4)} \quad \bigvee_{p,q\in\mathbb{Q}} (p,q) = 1.
\end{aligned}
$$



EXAMPLE 1: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

Nice features: (1) Rings $\mathsf{C}(L)$ of continuous real functions,

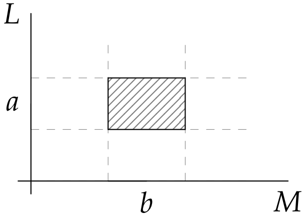
(2) Semicontinuous real functions, ...

EXAMPLE 2: PRESENTATIONS

Loc

The *product* $L \times M$ of L and M

Generators: pairs $a \otimes b$, $\quad a \in L, b \in M$



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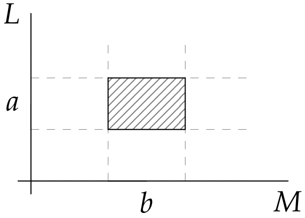

Relations:

(R1) $1 \otimes 1 = 1,$

Loc The *product* $L \times M$ of L and M

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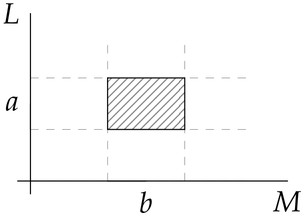

EXAMPLE 2: PRESENTATIONS

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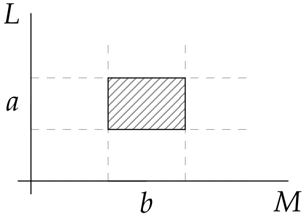


Generators: pairs $a \otimes b$, $a \in L, b \in M$

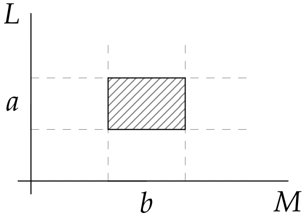
Relations:

(R1) $1 \otimes 1 = 1$,

(R2) $a \otimes 0 = 0 \otimes b = 0$,

(R3) $(a \otimes b) \wedge (a' \otimes b') = (a \wedge a') \otimes (b \wedge b')$, Loc

The *product* $L \times M$ of L and M



Generators: pairs $a \otimes b$, $a \in L, b \in M$

Relations:

(R1) $1 \otimes 1 = 1$,

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(R3) $(a \otimes b) \wedge (a' \otimes b') = (a \wedge a') \otimes (b \wedge b')$,

(R4) $\bigvee(a_i \otimes b) = (\bigvee a_i) \otimes b$, $\bigvee(a \otimes b_i) = a \otimes (\bigvee b_i)$.

COPRODUCTS in $\mathbf{Frm}$ concretely

like tensor products... ☺

Frm

The *coproduct* $L \oplus M$ of L and M:

COPRODUCTS in $\mathbf{Frm}$ concretely

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Frm The *coproduct* $L \oplus M$ of L and M:

cp-ideals of $L \times M$

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$R \subseteq L \times M$

COPRODUCTS in Frm concretely

like tensor products...

Frm

The *coproduct* $L \oplus M$ of L and M:

- $\downarrow R = R$ (down-sets)

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COPRODUCTS in $\mathbf{Frm}$ concretely

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Frm The *coproduct* $L \oplus M$ of L and M:

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$$a \oplus b := \downarrow(a,b) \cup \downarrow(1,0) \cup \downarrow(0,1)$$

COPRODUCTS in Frm concretely

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$$L \xrightarrow{u_L} L \oplus M \xleftarrow{u_M} M$$

COPRODUCTS in Frm concretely

like tensor products... ☺

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$$1 \oplus b \leftsquigarrow b$$

PRODUCTS in Loc

like tensor products... ☺

Nice features: (1) Tychonoff's Theorem is Choice-free,

fully constructive (in the sense of topos theory).

(2) Paracompactness and Lindelöfness are

productive properties, ...

... behave better than products of spaces!

PART IV.

Doing topology in $\mathbf{Loc}$

SPECIAL SUBLOCALES

$a \in L, \qquad \mathfrak{c}(a) = \uparrow a$ CLOSED

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$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN

SPECIAL SUBLOCALES

$$a \in L, \qquad \left.\begin{array}{ll} \mathfrak{c}(a) = {\uparrow} a & \text{CLOSED} \\[1em] \mathfrak{o}(a) = \{a \to x \mid x \in L\} & \text{OPEN} \end{array}\right\} \text{complemented}$$

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- $y \in \mathfrak{c}(a) \cap \mathfrak{o}(a)$: $a \leq y = a \to x$

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- $\forall x \in L,\ x \overset{\mathbf{H8}}{=} (x \vee a) \wedge (a \to x) \in \mathfrak{c}(a) \vee \mathfrak{o}(a).$

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Properties

(1) $a \leq b$ iff $\mathfrak{c}(a) \supseteq \mathfrak{c}(b)$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$.

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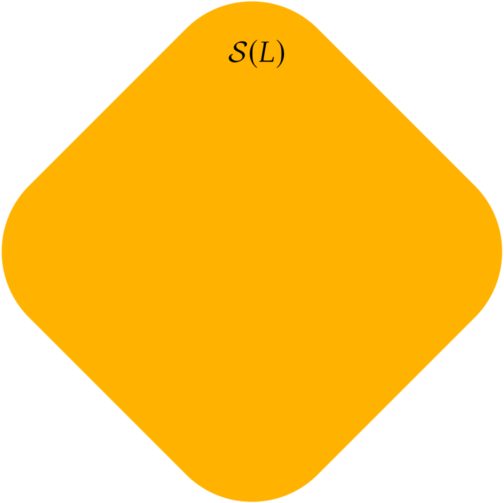
SPECIAL SUBLOCALES

It is a co-frame!



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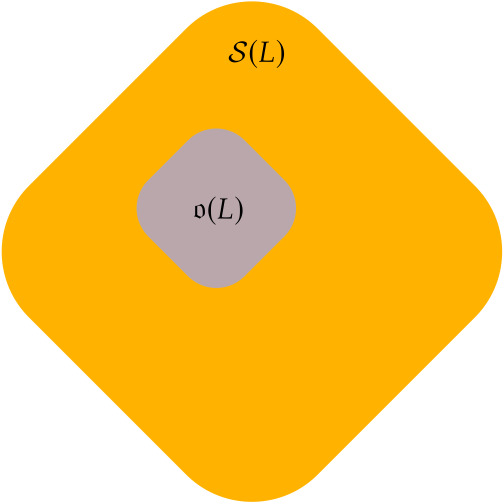

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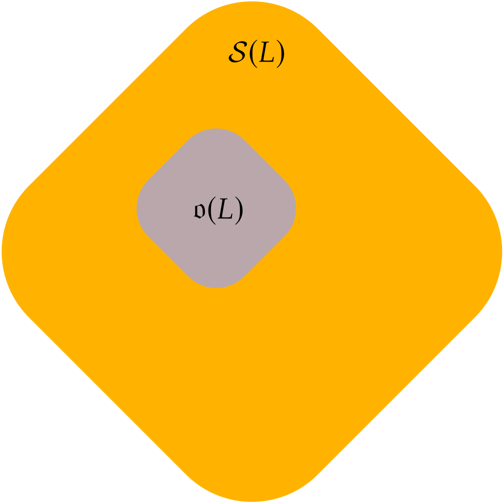

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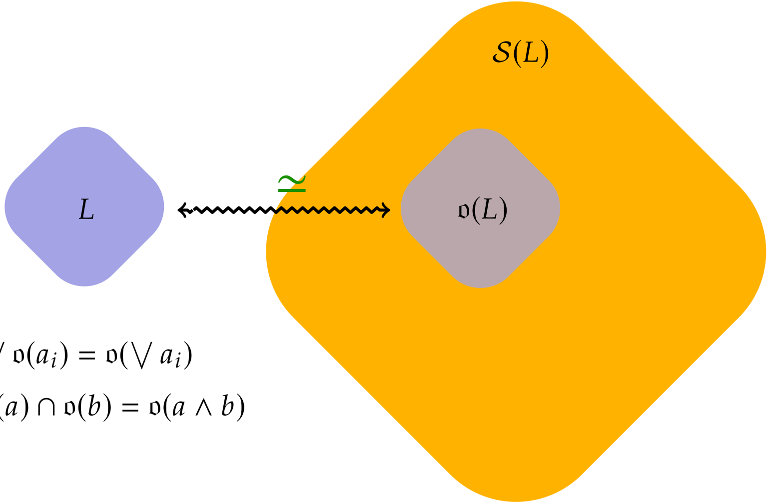

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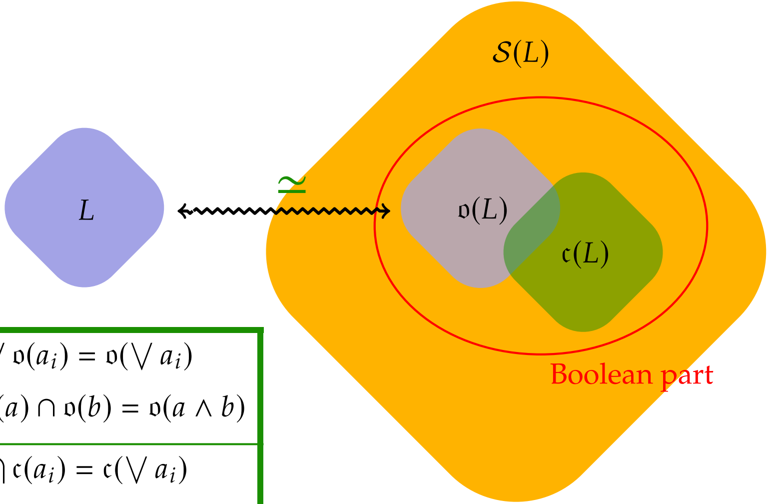

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Starting doing topology: CLOSURE and INTERIOR operators

CLOSURE: $\overline{S} = \bigwedge\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\}$

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$$\text{CLOSURE:}\quad \overline{S} = \bigwedge\{\overset{\uparrow a}{\mathfrak{c}(a)} \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \mid a \leq \bigwedge S\}) = \mathfrak{c}(\bigwedge S).$$

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INTERIOR: $\operatorname{int} S = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}$.

Starting doing topology: CLOSURE and INTERIOR operators

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EXAMPLE

$$\overline{\mathfrak{o}(b)} = \mathfrak{c}(\bigwedge \mathfrak{o}(b)) = \mathfrak{c}(b \to 0) = \mathfrak{c}(b^*).$$

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EXAMPLE

$\overline{\mathfrak{o}(b)} = \mathfrak{c}(\bigwedge \mathfrak{o}(b)) = \mathfrak{c}(b \to 0) = \mathfrak{c}(b^*)$.

By complementation, $\operatorname{int} \mathfrak{c}(b) = \mathfrak{o}(b^*)$.

[M.M. Clementino, PhD Thesis, 1992]

ISBELL'S DENSITY THEOREM!

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S is DENSE: $\overline{S} = L$

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$\overset{\uparrow a}{}$

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Hence: intersections of dense sublocales are dense,

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Hence: intersections of dense sublocales are dense,

i.e., there exists <u>the</u> smallest dense sublocale of a locale!

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THEOREM [J. Isbell, 1972]

$\mathfrak{B}_L = \{x^* \mid x \in L\} = \{x \mid x^{**} = x\}$ is the least dense sublocale.

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Proof: • $\mathfrak{B}_L$ is a sublocale: $\bigwedge x_i^* = (\bigvee x_i)^*$.

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$$a \to x^* = a \to (x \to 0) = a \wedge x \to 0 = (a \wedge x)^*.$$

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- $\mathfrak{B}_L$ is a sublocale: $\bigwedge x_i^* = (\bigvee x_i)^*$.

  $a \longrightarrow x^* = a \longrightarrow (x \longrightarrow 0) = a \wedge x \longrightarrow 0 = (a \wedge x)^*$.

- $0 \in \mathfrak{B}_L$ so $\mathfrak{B}_L$ is dense.

ISBELL'S DENSITY THEOREM!

CLOSURE: $\overline{S} = \bigwedge\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \mid \overset{\uparrow a}{a \leq \bigwedge S}\}) = \mathfrak{c}(\bigwedge S)$.

INTERIOR: $\operatorname{int} S = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}$.

S is DENSE: $\overline{S} = L \Leftrightarrow \uparrow(\bigwedge S) = L \Leftrightarrow \bigwedge S = 0 \Leftrightarrow 0 \in S$.

THEOREM [J. Isbell, 1972]

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- S dense $\Rightarrow \mathfrak{B}_L \subseteq S$: $x^* = x \to \underbrace{0}_{\in S} \in S$. ■

ISBELL'S DENSITY THEOREM: consequences

$$j\colon Y \subseteq X \quad \dashrightarrow \quad h\colon \begin{array}{ccc} \Omega(X) & \to & \Omega(Y) \\ U & \mapsto & U \cap Y \end{array} \quad \text{in } \mathsf{Frm}$$

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BUT, of course, $S_{Y_1} \cap S_{Y_2}$ is a pointless sublocale:

$\mathsf{Pt}(S_{Y_1} \cap S_{Y_2}) \subseteq Y_1 \cap Y_2 = \emptyset.$

INTERMEZZO: points as sublocales

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if $x \wedge y \le a$ and $x \not\le a$ (i.e. $x \to a \not\le 1$) then $y \le x \to a = a$.

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IMAGES

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PREIMAGES

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$$\begin{tikzcd} L \arrow[r, "f"] & M \\ \underbrace{f^{-1}[T]}_{} & T \arrow[u, phantom, "\cup\!\mid"] \end{tikzcd}$$

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Preimage Map: $f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$

Properties of the PREIMAGE MAP

[M.M. Clementino, PhD Thesis, 1992]

1. $\boxed{f_{-1}[\mathfrak{c}(b)] = f^{-1}[\mathfrak{c}(b)] = \mathfrak{c}(f^*(b))}$

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$$(f^*(a) \to s) \to s \in S \subseteq f^{-1}[\mathfrak{o}(a)] \Rightarrow f((f^*(a) \to s) \to s) =$$
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$$\Rightarrow (f^*(a) \to s) \to s = 1, \text{ that is, } f^*(a) \to s = s.$$

THE BIG PICTURE

$$f[S] \subseteq T \iff S \subseteq f^{-1}[T] \iff S \subseteq f_{-1}[T].$$

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$$\begin{tikzcd} L \arrow[d, "f"] \arrow[d, phantom, shift right=3, "\dashv"] \\ M \arrow[u, bend left, "f^*"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{S}(L) \arrow[d, bend right, "{f[-]}"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S}(M) \arrow[u, bend right, "{f_{-1}[-]}"'] \end{tikzcd}$$

AS IT SHOULD BE!

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AS IT SHOULD BE!

co-frame homomorphism

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$$\begin{tikzcd}
L \arrow[d, "f"] \\
M \arrow[u, bend left=50, "f^*"] \arrow[u, phantom, shift right=12, "\dashv"]
\end{tikzcd}
\qquad\qquad
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AS IT SHOULD BE!

co-localic map co-frame homomorphism

JOYAL-TIERNEY THEOREM

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TFAE for a localic map $f\colon L \to M$:

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(4) f^* admits a left adjoint $f_!$ that satisfies the identity

$$f(a \to f^*(b)) = f_!(a) \to b \quad \forall a \in L, b \in M.$$

PROOF. (3) $\Leftrightarrow$ (2):

$$\boxed{f_!(a \wedge f^*(b)) = f_!(a) \wedge b}$$

$$\begin{tikzcd} L \arrow[r, "f_!"] \arrow[d, "(-)\wedge f^*(b)"'] \arrow[dr, phantom, "\#"] & M \arrow[d, "(-)\wedge b"] \\ L \arrow[r, "f_!"'] & M \end{tikzcd}$$

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RIGHT ADJOINTS

$$\begin{tikzcd}
L \arrow[dr, phantom, "\#"] & M \arrow[l, "f^*"'] \\
L \arrow[u, "f^*(b) \to (-)"] & M \arrow[l, "f^*"] \arrow[u, "b \to (-)"']
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RIGHT ADJOINTS

$$\begin{tikzcd} L & M \arrow[l, "f^*"'] \\ L \arrow[u, "f^*(b) \to (-)"] \arrow[ur, phantom, "\#"] & M \arrow[l, "f^*"] \arrow[u, "b \to (-)"'] \end{tikzcd}$$

$$\boxed{f^*(b) \to f^*(y) = f^*(b \to y)}$$

Proof. (3) $\Leftrightarrow$ (4):

$$\boxed{f_!(a \wedge f^*(b)) = f_!(a) \wedge b}$$

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L \arrow[d, "a \wedge (-)"'] \arrow[dr, phantom, "\#"] & M \arrow[l, "f^*"'] \arrow[d, "f_!(a) \wedge (-)"] \\
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$$\boxed{f(a \to f^*(b)) = f_!(a) \to b}$$

PROOF. (1) $\Rightarrow$ (3): By assumption, $\forall a \in L \;\; \exists b \in M\colon \;\; f[\mathfrak{o}(a)] = \mathfrak{o}(b)$.

The b is necessarily unique so we have a map $f_! \colon L \to M \;\; (a \mapsto b)$.

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JOYAL-TIERNEY THEOREM

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$$\begin{aligned} \text{Then } y = b \to y = b \to f(a \to x) &= f(f^*(b) \to (a \to x)) \\ &= f((f^*(b) \wedge a) \to x) \end{aligned}$$

JOYAL-TIERNEY THEOREM

PROOF. (1) $\Rightarrow$ (3): By assumption, $\forall a \in L\ \ \exists b \in M\colon\ \ f[\mathfrak{d}(a)] = \mathfrak{d}(b)$.

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Then $y = b \to y = b \to f(a \to x) = f(f^*(b) \to (a \to x))$

$$= f(\underbrace{(f^*(b) \wedge a) \to x}) \quad \text{in } \mathfrak{d}(a \wedge f^*(b)) = \mathfrak{d}(a) \cap \mathfrak{d}(f^*(b))$$

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PROOF. (4) $\Rightarrow$ (1):

Suffices: $f[\mathfrak{o}(a)] = \mathfrak{o}(f_!(a))$ for every $a \in L$.

JOYAL-TIERNEY THEOREM

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$$
\begin{aligned}
f_!(a) \to f(a \to x) &\overset{(4)}{=} f(a \to f^* f(a \to x)) \\
&\le f(a \to (a \to x)) \\
&= f(a \to x).
\end{aligned}
$$

A REMARK: Localic maps as continuous maps

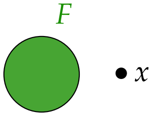
THEOREM

TFAE for any mapping $f\colon L \longrightarrow M$ between locales:

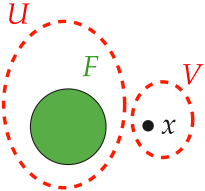
(1) f is localic.

(2) f is a right adjoint and $f_{-1}[\mathfrak{o}(b)] = \mathfrak{o}(f^*(b))$ $\forall b \in M$.

(3) $\forall b \in M\ \exists a \in L\colon \quad f^{-1}[\mathfrak{c}(b)] = \mathfrak{c}(a)$ and $f_{-1}[\mathfrak{o}(b)] = \mathfrak{o}(a)$.

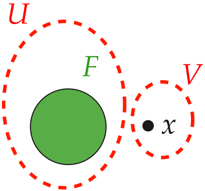

F
x

Doing topology in Loc: REGULARITY



$A = X \smallsetminus F$

$\forall A \in \Omega(X),\ \forall x \in A,\ \exists V \in \Omega(X):\ x \in V \subseteq \overline{V} \subseteq A.$



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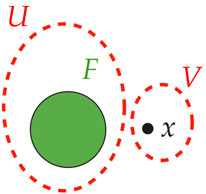

$\Leftrightarrow \quad \boxed{\forall A \in \Omega(X),\ \ A = \bigcup\{V \in \Omega(X) \mid \overline{V} \subseteq A\}}$

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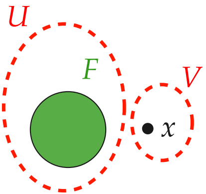

$\Leftrightarrow \quad \boxed{\forall A \in \Omega(X),\ \ A = \bigcup\{V \in \Omega(X) \mid \underbrace{\overline{V} \subseteq A}_{V \prec A}\}}$

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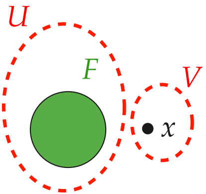

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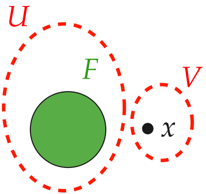

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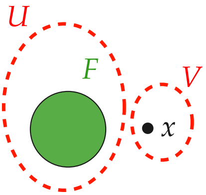

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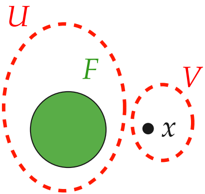
Doing topology in Loc: REGULARITY



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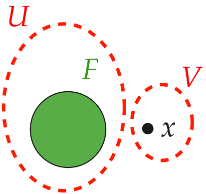

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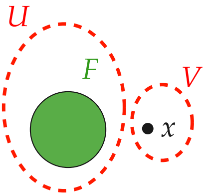

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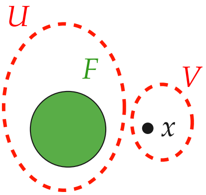

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(Conservative extension: X is regular iff the locale $\Omega(X)$ is regular)

partial order $\prec$

Properties

1. $a \prec b \;\Rightarrow\; a \leq b.$

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2. $$a \leq b \prec c \leq d \;\Rightarrow\; a \prec d.$$

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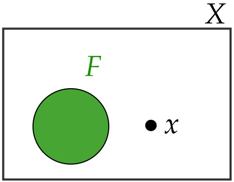
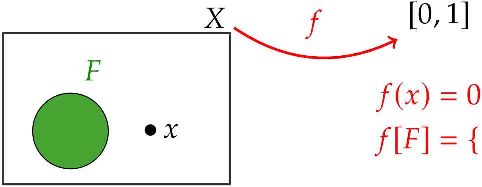

X
F
x

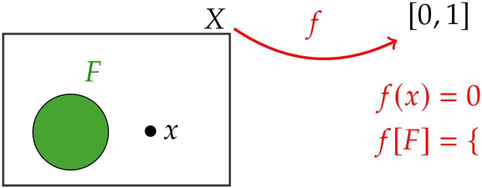
Doing topology in Loc: COMPLETE REGULARITY



$f(x) = 0$

$f[F] = \{1\}$

Doing topology in Loc: COMPLETE REGULARITY



$$f(x) = 0$$
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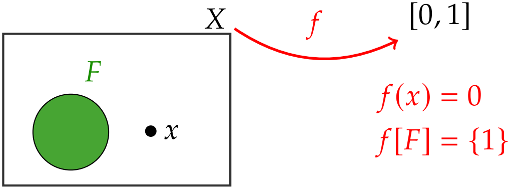
By Urysohn's Lemma,

X is completely regular iff $\boxed{\forall U \in \Omega(X),\ U = \{V \in \Omega(X) \mid V \lll U\}}$



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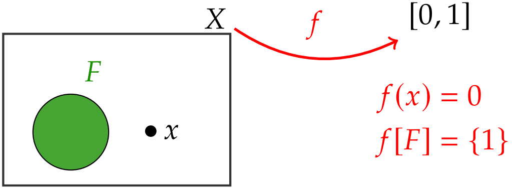
$V \lll U \ \equiv\ \exists (W_q)_{q \in \mathbb{Q} \cap [0,1]} :\ W_0 = V,\ W_1 = U,\ p < q \Rightarrow W_p \prec W_q.$

[B. Banaschewski (1953)]

Doing topology in Loc: COMPLETE REGULARITY



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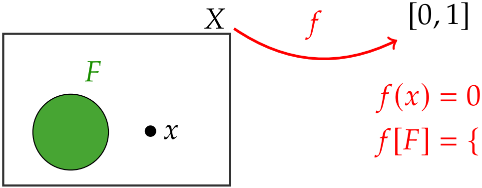

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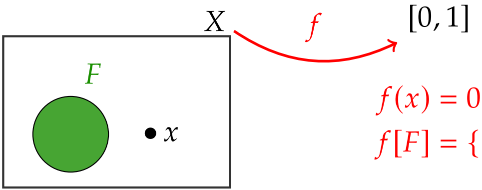
Thus, for a general locale L:

L is completely regular if $\boxed{\forall a \in L, a = \bigvee\{b \in L \mid b \ll a\}}$



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X is completely regular iff $\boxed{\forall U \in \Omega(X),\ U = \{V \in \Omega(X) \mid V \ll U\}}$

$\ll \ \equiv$ the largest INTERPOLATIVE relation contained in $\prec$

Thus, for a general locale L:

L is completely regular if $\boxed{\forall a \in L, a = \bigvee\{b \in L \mid b \ll a\}}$

(Conservative extension: X is c. regular iff $\Omega(X)$ is c. regular)

Doing topology in Loc: COMPLETE REGULARITY

partial order $\lll$

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3. $\lll$ is the largest interpolative partial order contained in $\prec$.

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$$\Big\downarrow \text{compactness}$$

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On the other hand

$$x_i \prec b \ (i = 1, \ldots, n) \implies c \prec b. \qquad \blacksquare$$

The (constructive) STONE-ČECH compactification of locales

Ideals of L: $\mathfrak{I}(L)$ (I1) $b \le a \in J \Rightarrow b \in J$, (I2) $a, b \in J \Rightarrow a \vee b \in J$

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  $\supseteq$: obvious

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$$\subseteq:\ x = x_1 \vee \cdots \vee x_n \in (\bigvee J_i) \cap K \quad (x_j \in J_{i_j})$$

$$\text{(I1)} \Big\Downarrow {\scriptstyle x_j \le x \in K}$$

$$x_j \in J_{i_j} \cap K \Rightarrow x \in \bigvee (J_i \cap K)$$

- $\bigvee J_i = L \ni 1 \Rightarrow 1 = x_1 \vee \cdots \vee x_n$ (some $x_j \in J_{i_j}$).

Then $1 \in \bigvee_{j=1}^{n} J_{i_j} \;\Rightarrow\; L = \bigvee_{j=1}^{n} J_{i_j}$. ∎

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(Just by the interpolation property of $\ll$)

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- subframes of compact frames are (obviously) compact. ■

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Proof: By the Proposition it suffices to show that $\mathfrak{R}(L)$ is regular.

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For each $J \in \Re(L)$,

$$J = \bigcup\{\downarrow\!\!\downarrow a \mid a \in J\} = \bigvee\{\downarrow\!\!\downarrow a \mid a \in J\}$$

The (constructive) STONE-ČECH compactification of locales

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$$a \ll b \text{ in } L \quad \Rightarrow \quad {\downarrow}a \prec {\downarrow}b \text{ in } \mathfrak{R}(L)$$

(easy to check...) ∎

The (constructive) STONE-ČECH compactification of locales

LEMMA 4

For each completely regular locale L,

$$\begin{aligned} \beta_L : L &\longrightarrow \beta(L) := \mathfrak{R}(L) \\ a &\longmapsto {\downarrow\!\!\downarrow} a \end{aligned}$$

is a dense embedding.

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Localic embedding: Let $v_L : J \in \mathfrak{R}(L) \mapsto \bigvee J \in L$. Clearly:

- $v_L \beta_L(a) = a$ and $\beta_L v_L(J) \supseteq J$. In particular: $v_L \dashv \beta_L$; β_L is injective.
- $v_L(L) = 1$.
- $v_L(J_1) \wedge v_L(J_2) = \bigvee\{x \wedge y \mid x \in J_1, y \in J_2\} \leq \bigvee\{z \mid z \in J_1 \cap J_2\} = = v_L(J_1 \cap J_2)$.

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THEOREM

There is a functor $\beta\colon \mathsf{CRegLoc} \to \mathsf{CRegLoc}$

$$\begin{tikzcd} L \arrow[r, mapsto] \arrow[d, "f"'] & \beta(L) \arrow[d, "\beta(f)"] \\ M \arrow[r, mapsto] & \beta(M) \end{tikzcd}$$

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and a natural transformation $\mathrm{Id} \xrightarrow{\bullet} \beta$

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(3) β_L is an isomorphism iff L is compact.

PROOF

$\Longleftarrow$:

$$L \xhookrightarrow[\beta_L]{} \beta(L)$$

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PROOF

$\Leftarrow$: If L is compact then $\beta_L v_L(J) \subseteq J$ and v_L is the inverse of β_L:

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$$\Rightarrow x^* \vee \underbrace{a_1 \vee \cdots \vee a_n = 1}_{a \in J} \;\Rightarrow\; x \prec a \in J \;\Rightarrow\; x \in J. \qquad \blacksquare$$

$$L \xhookrightarrow[\beta_L]{} \beta(L)$$

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