Variants of normality and their duals: a pointfree unification of insertion and extension theorems for real-valued functions

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— joint work with J. Gutiérrez García (UPV-EHU, Bilbao, Spain)

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This shapes the idea that the two notions are somehow dual

to each other and may therefore be studied in parallel.

AIMS

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- there is a variety of classical insertion type results
  (for several variants of normality).

  Can we unify them under a single general result ?


YES, WE CAN.

NORMALITY vs EXTREMAL DISCONNECTEDNESS

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$$\Leftrightarrow (a \wedge b)^* = a^* \vee b^*$$

[De Morgan frames]

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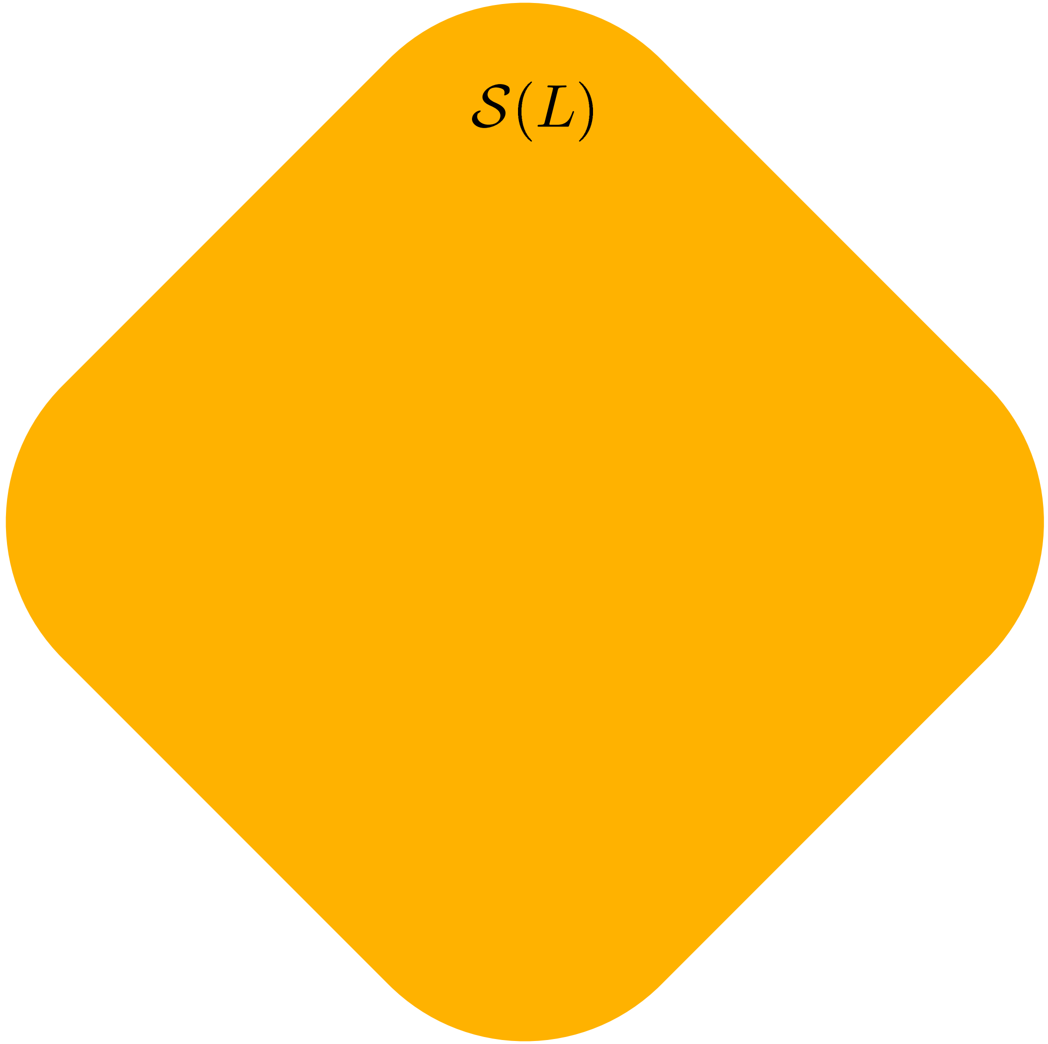
as lattices: L is normal iff L^{op} is extremally disconnected.

(FRAME) (CO-FRAME)

IDEA: go to $\mathcal{S}(L)$, take complements

the frame of sublocales



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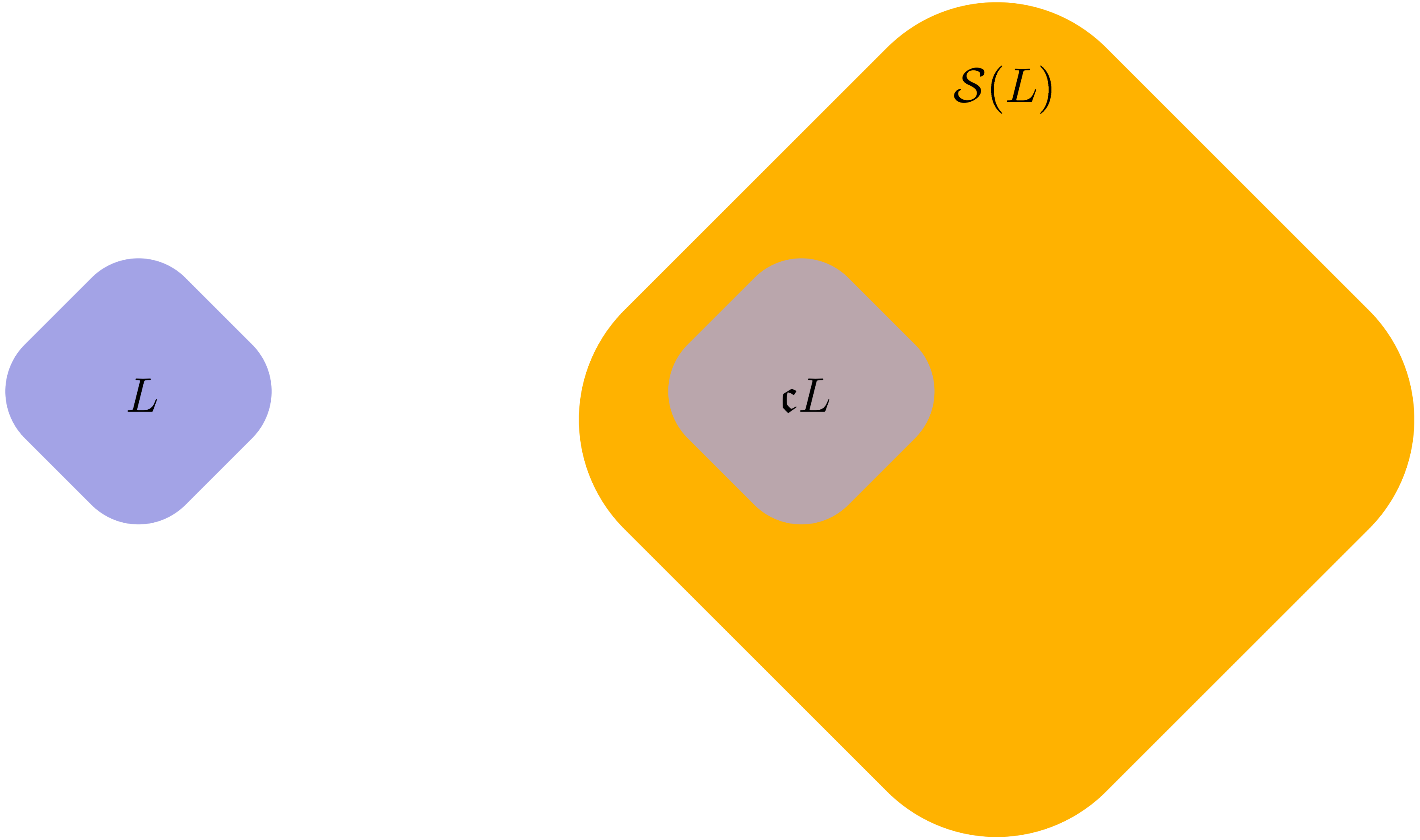

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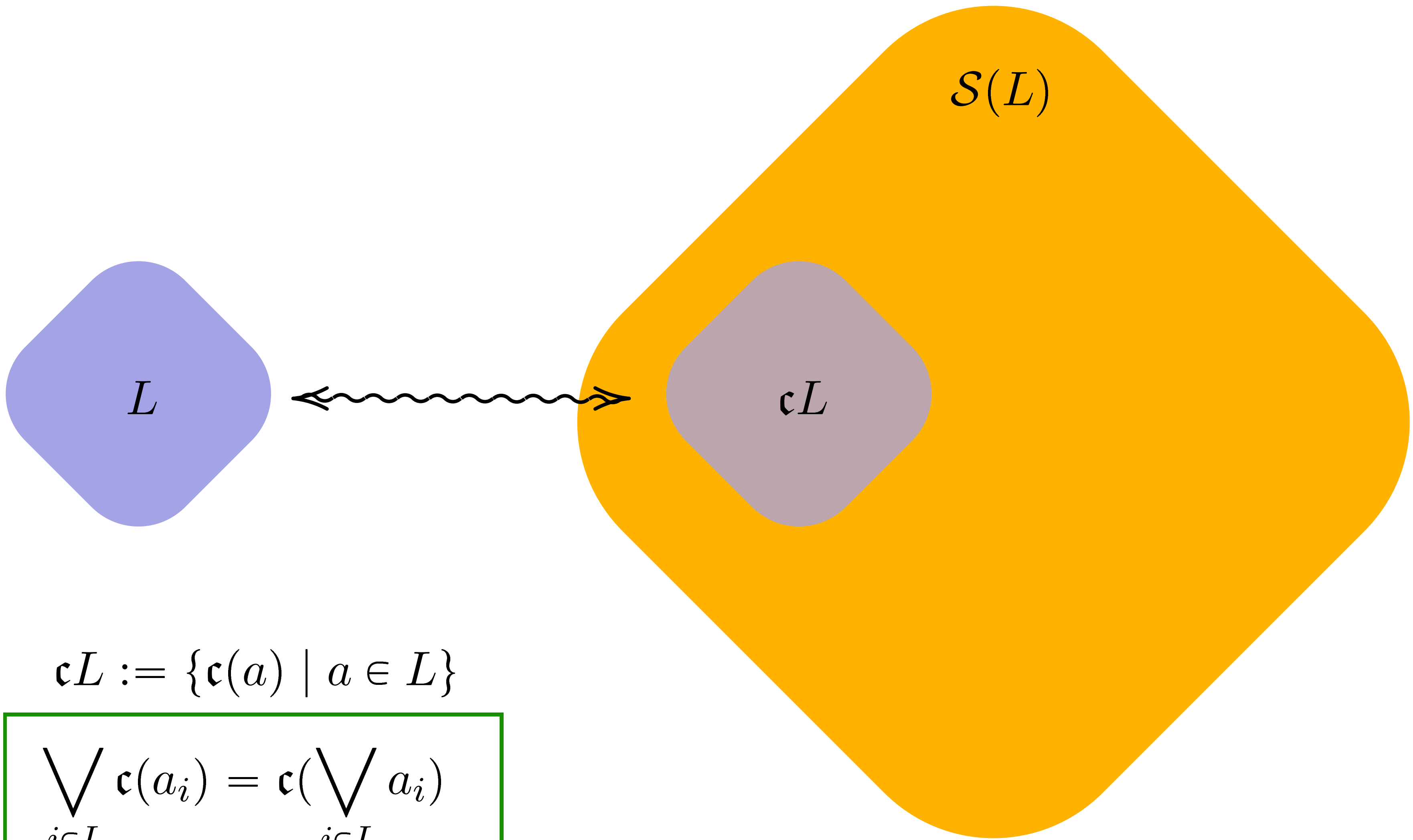

$$\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\boxed{\begin{array}{l} \bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in I} a_i) \\ \mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b) \end{array}}$$

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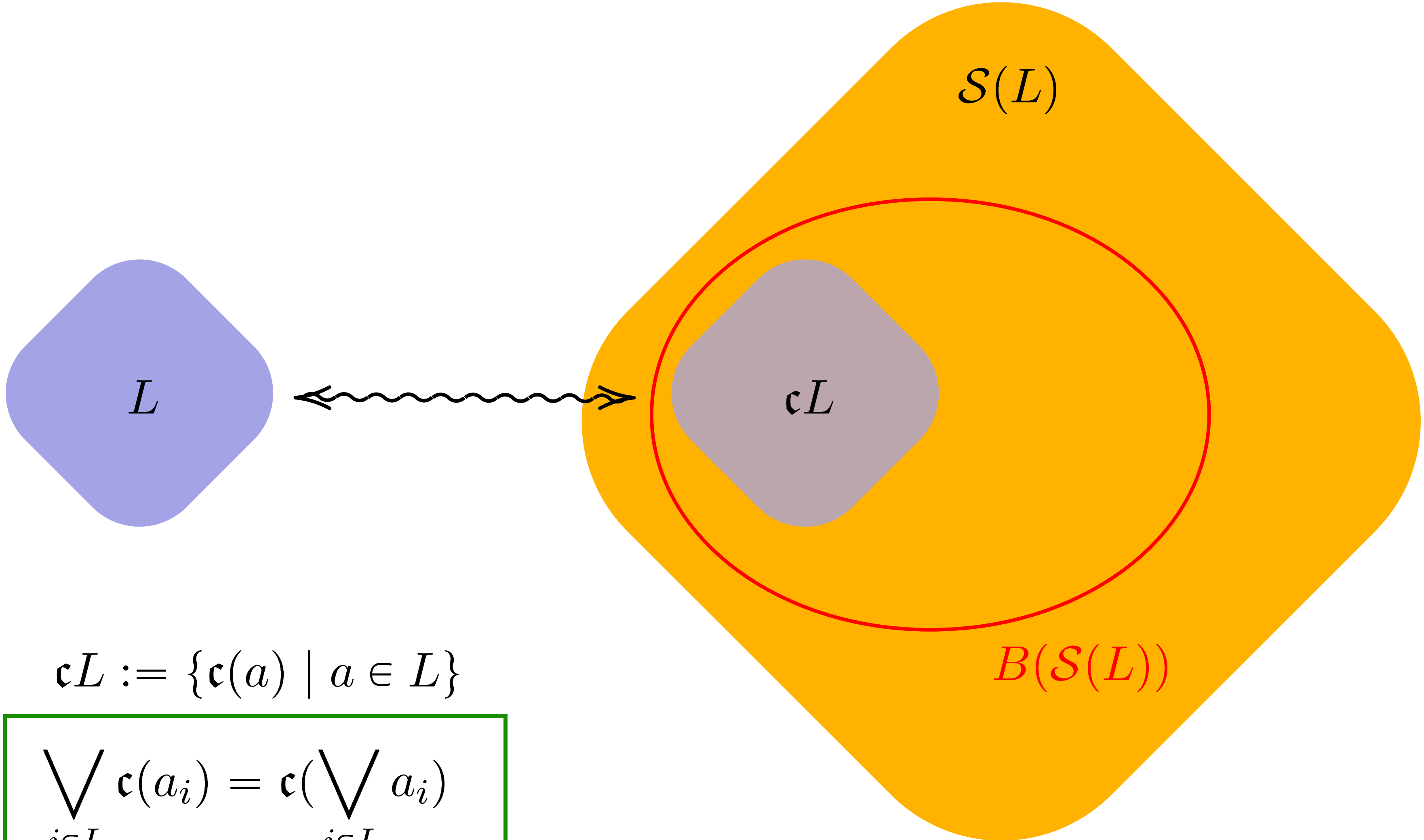

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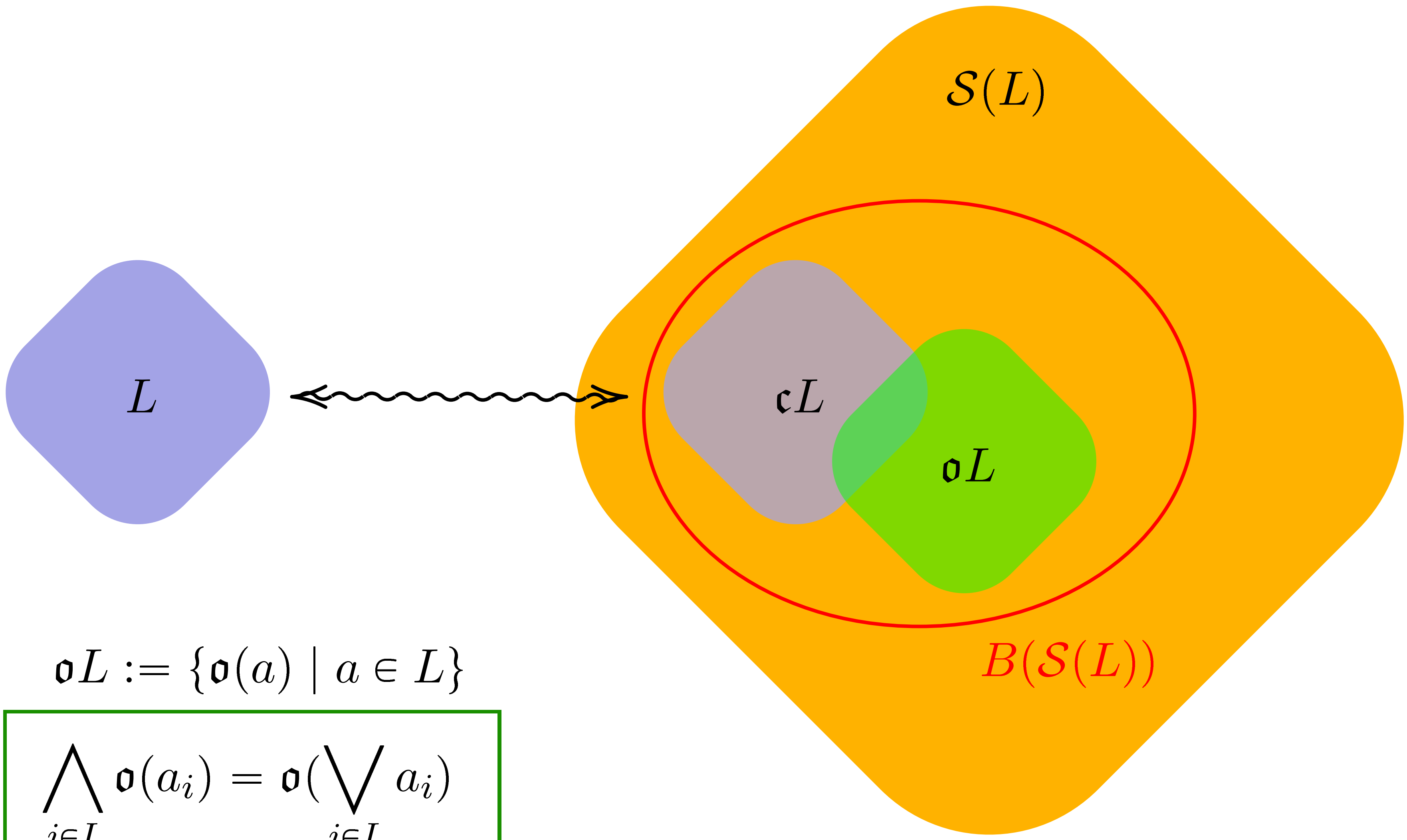

$$\mathfrak{o}L := \{\mathfrak{o}(a) \mid a \in L\}$$

$$\bigwedge_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i)$$

$$\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

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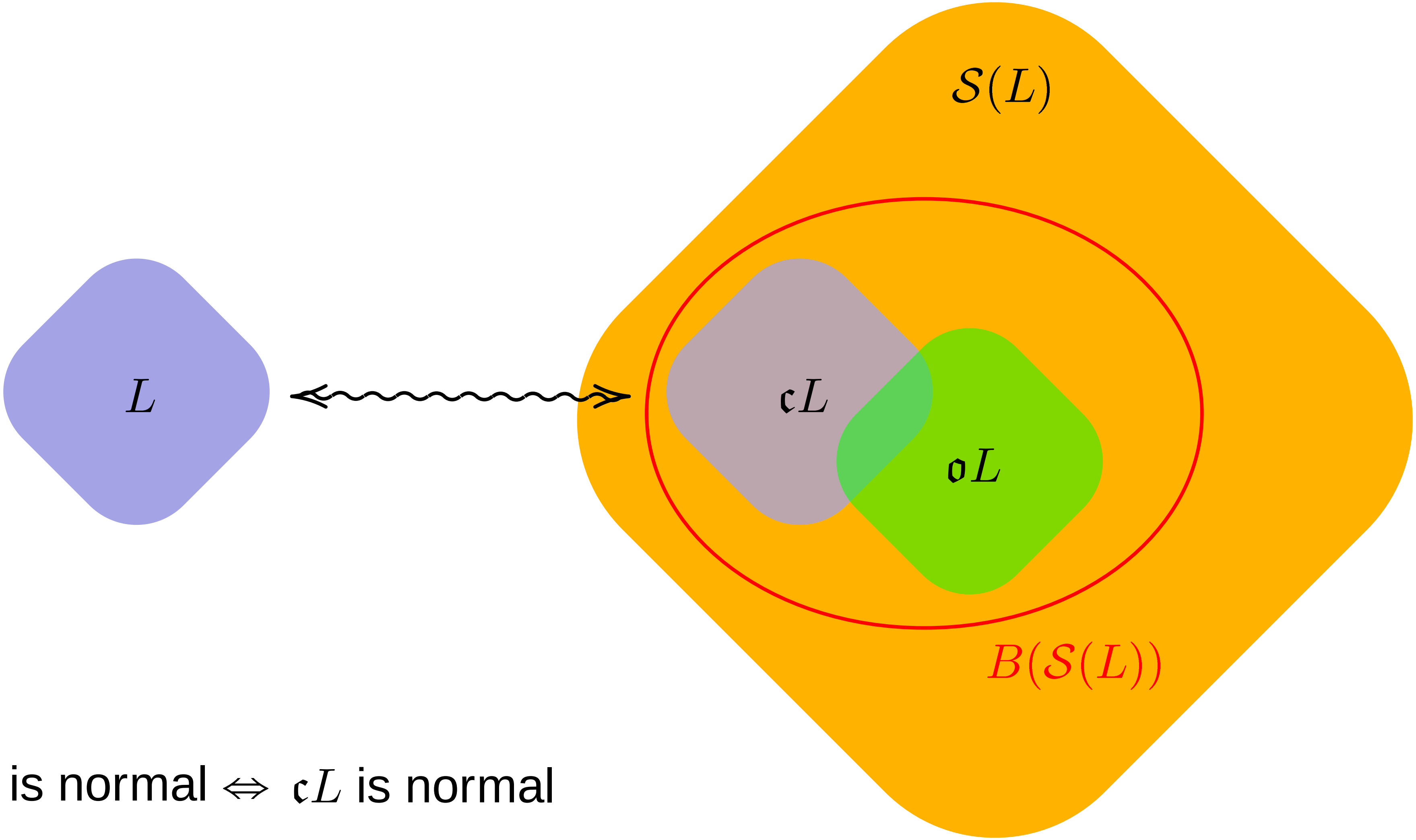

L is normal $\Leftrightarrow$ $\mathfrak{c}L$ is normal

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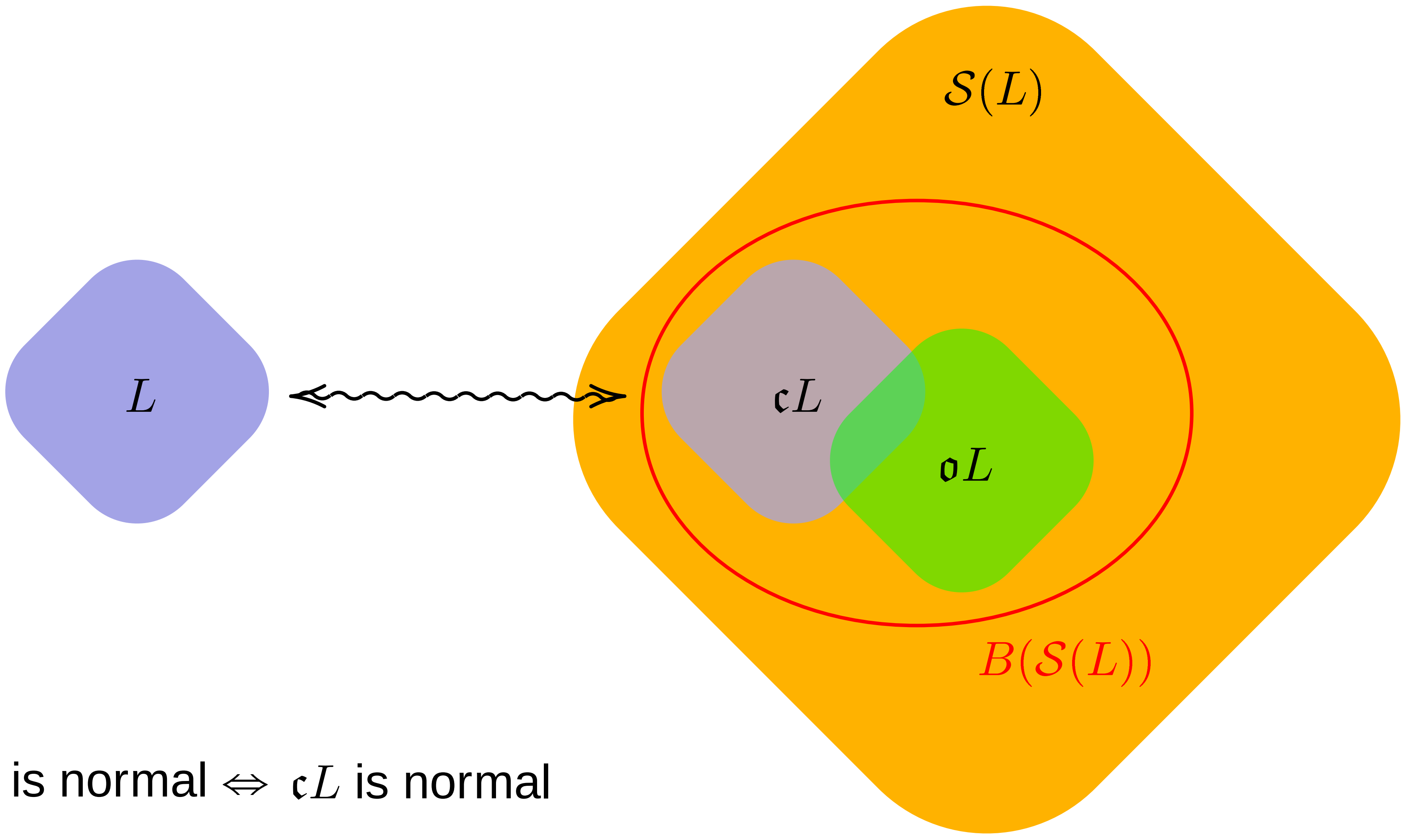

L is normal $\Leftrightarrow$ $\mathfrak{c}L$ is normal

$\Leftrightarrow$ $\mathfrak{o}L$ is extremally disconnected

$\mathscr{A}$-normality vs $\mathscr{A}$-extremal disconnectedness

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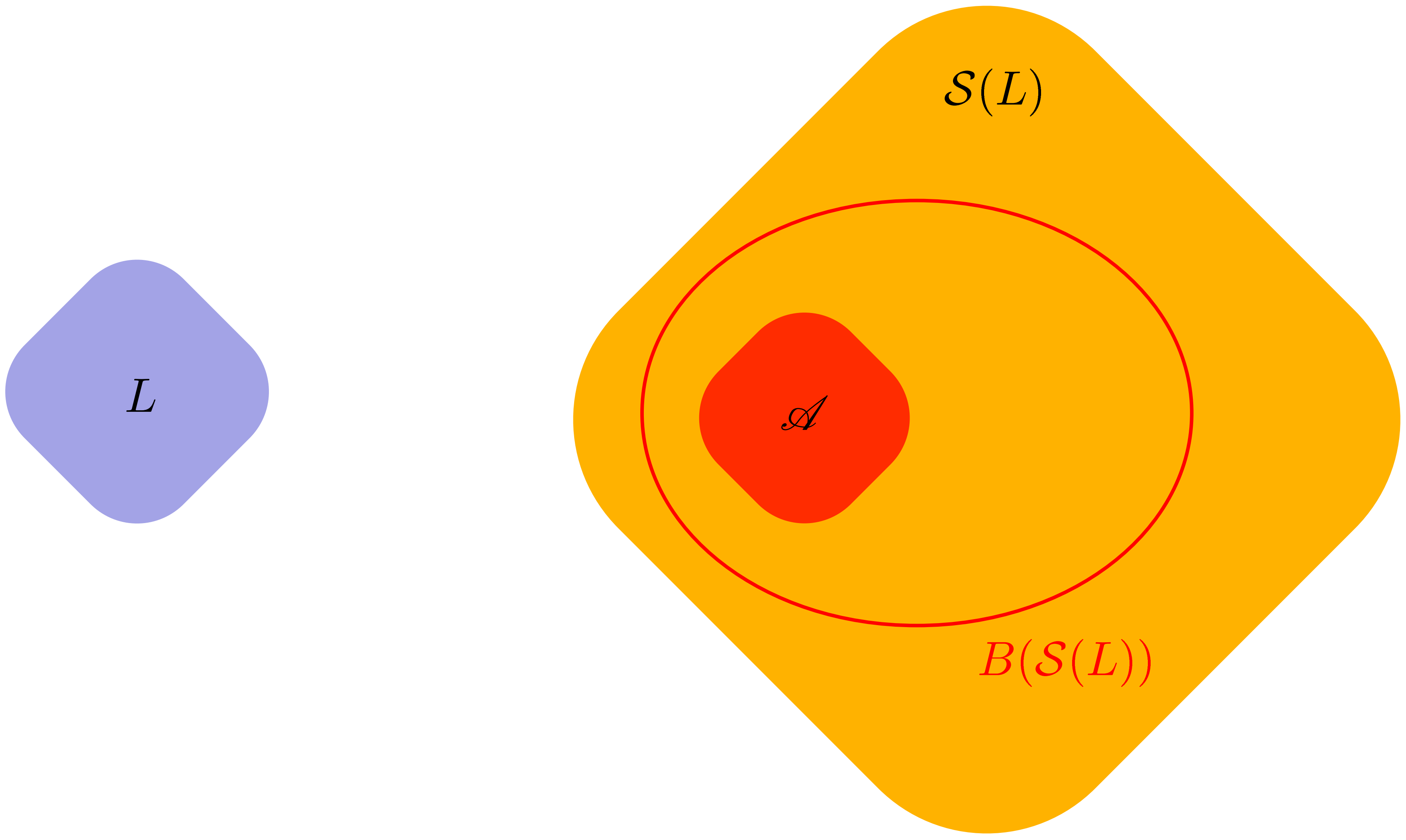

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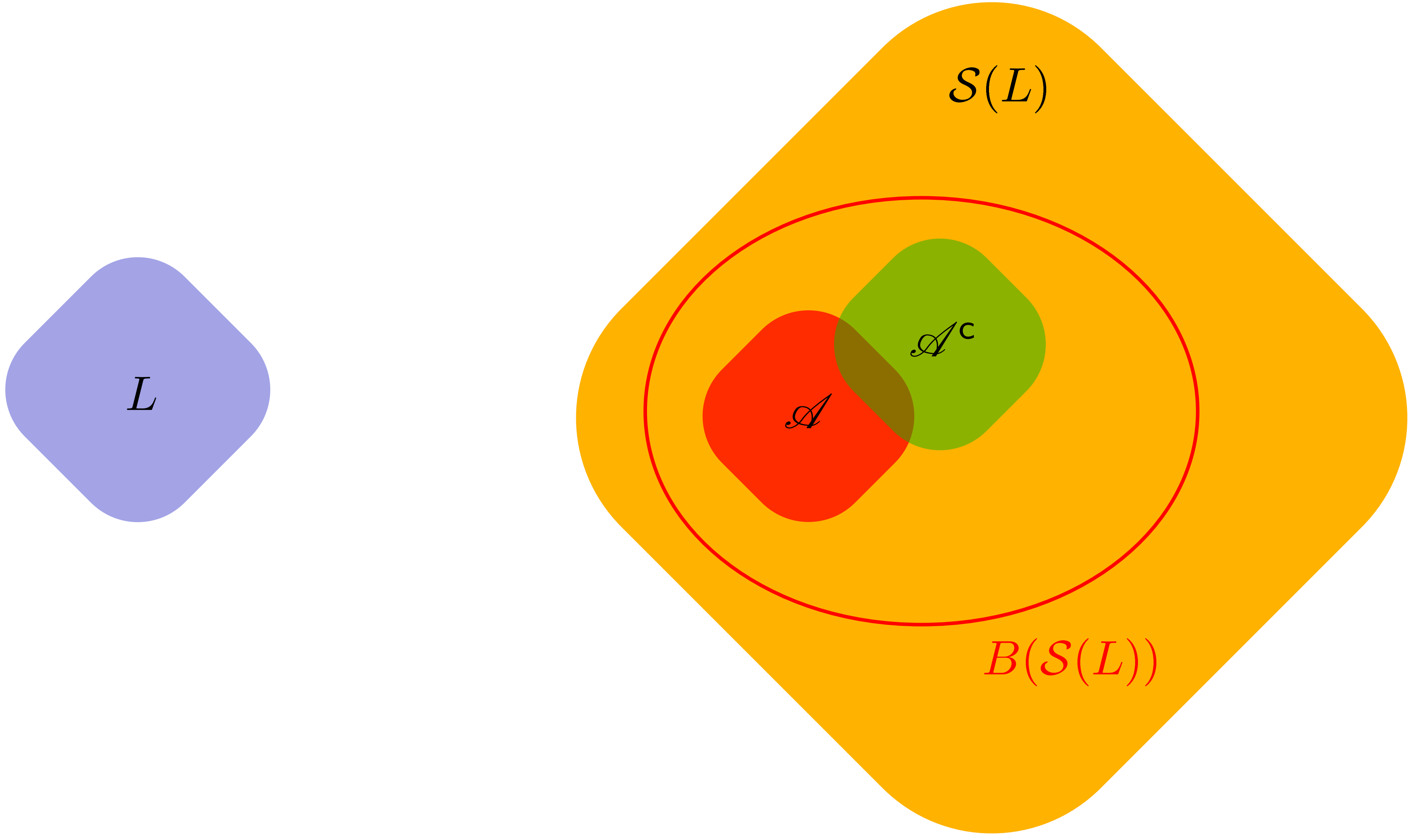

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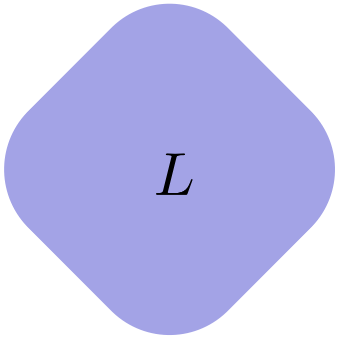

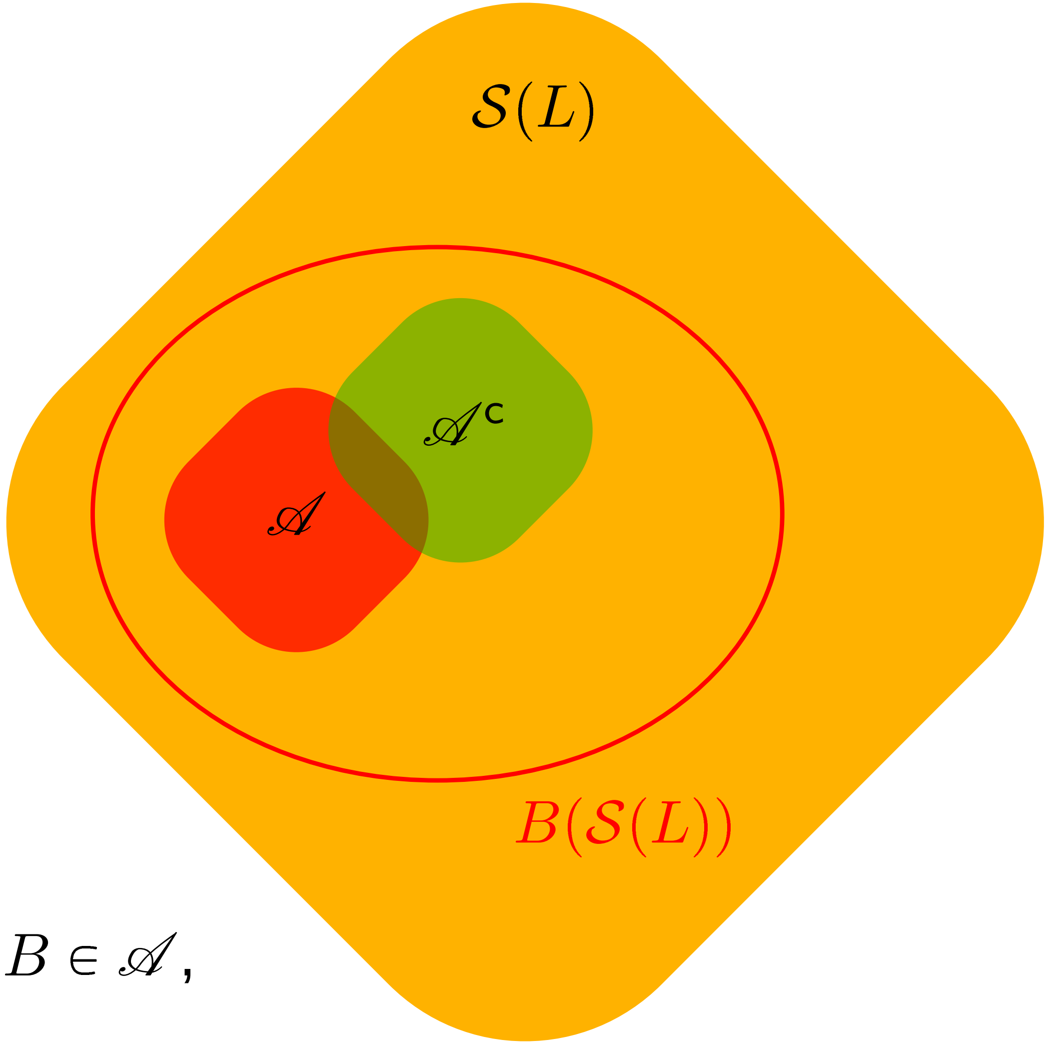

L is $\mathscr{A}$-normal $\equiv$ For any $A, B \in \mathscr{A}$,

$$A \vee B = 1 \;\Rightarrow\; \exists\, U, V \in \mathscr{A} : U \wedge V = 0,\; A \vee U = 1 = B \vee V.$$

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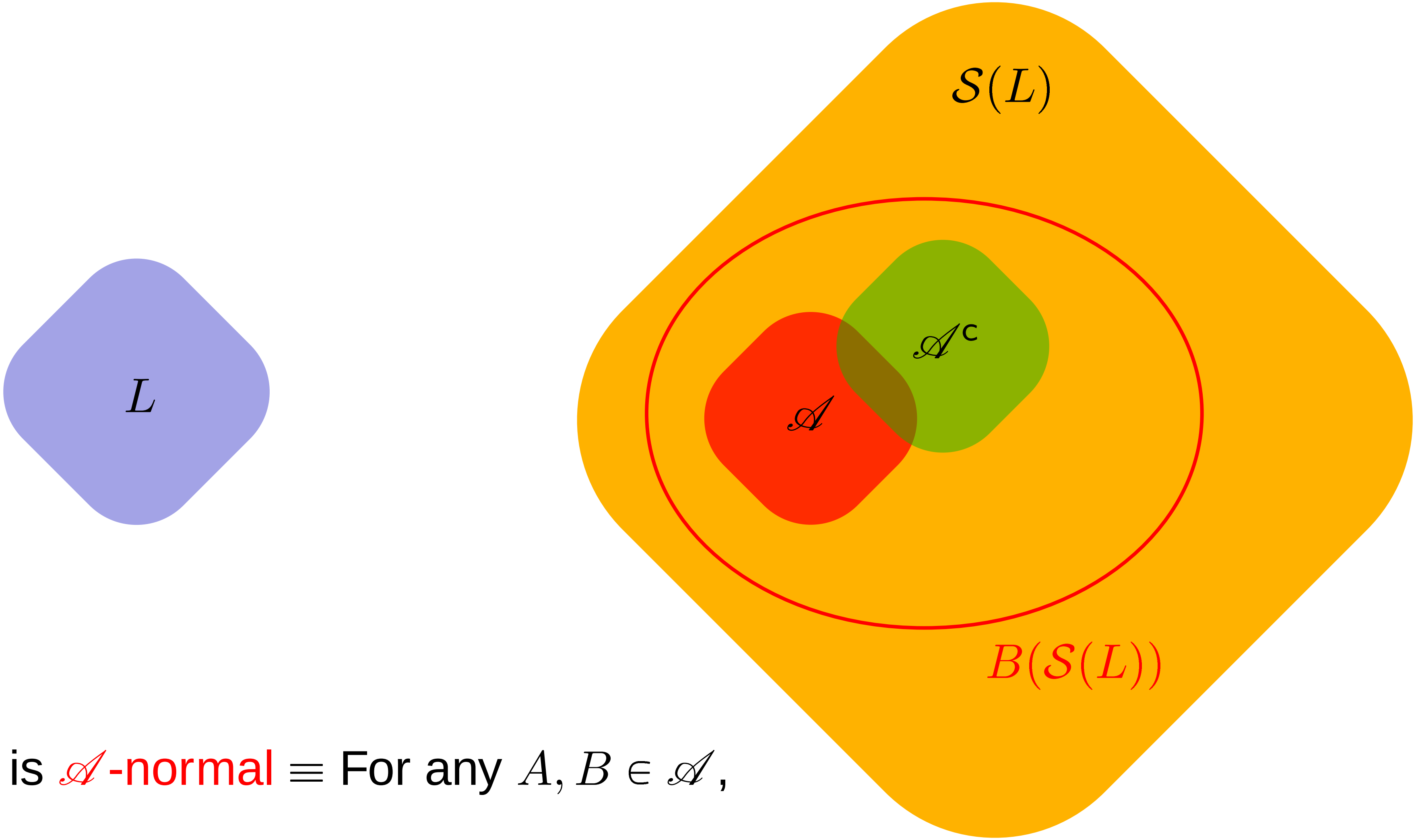

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L is $\mathscr{A}$-extremally disconnected $\equiv$ L is $\mathscr{A}^c$-normal.

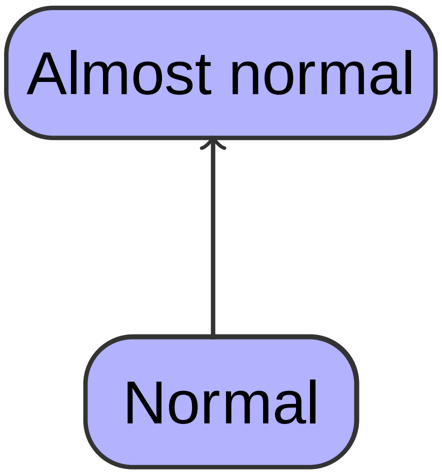
VARIANTS OF NORMALITY

Normal

$$a \vee b = 1 \Rightarrow \exists u, v \in L\colon u \wedge v = 0, a \vee u = 1 = b \vee v.$$

$(a, b \in L)$

VARIANTS OF NORMALITY



a: regular

$$a \vee b = 1 \Rightarrow \exists u, v \in L \colon u \wedge v = 0, a \vee u = 1 = b \vee v.$$

$(a, b \in L)$

VARIANTS OF NORMALITY

Mildly normal $\quad a, b\colon$ regular

$\uparrow$

Almost normal $\quad a\colon$ regular

$\uparrow$

Normal

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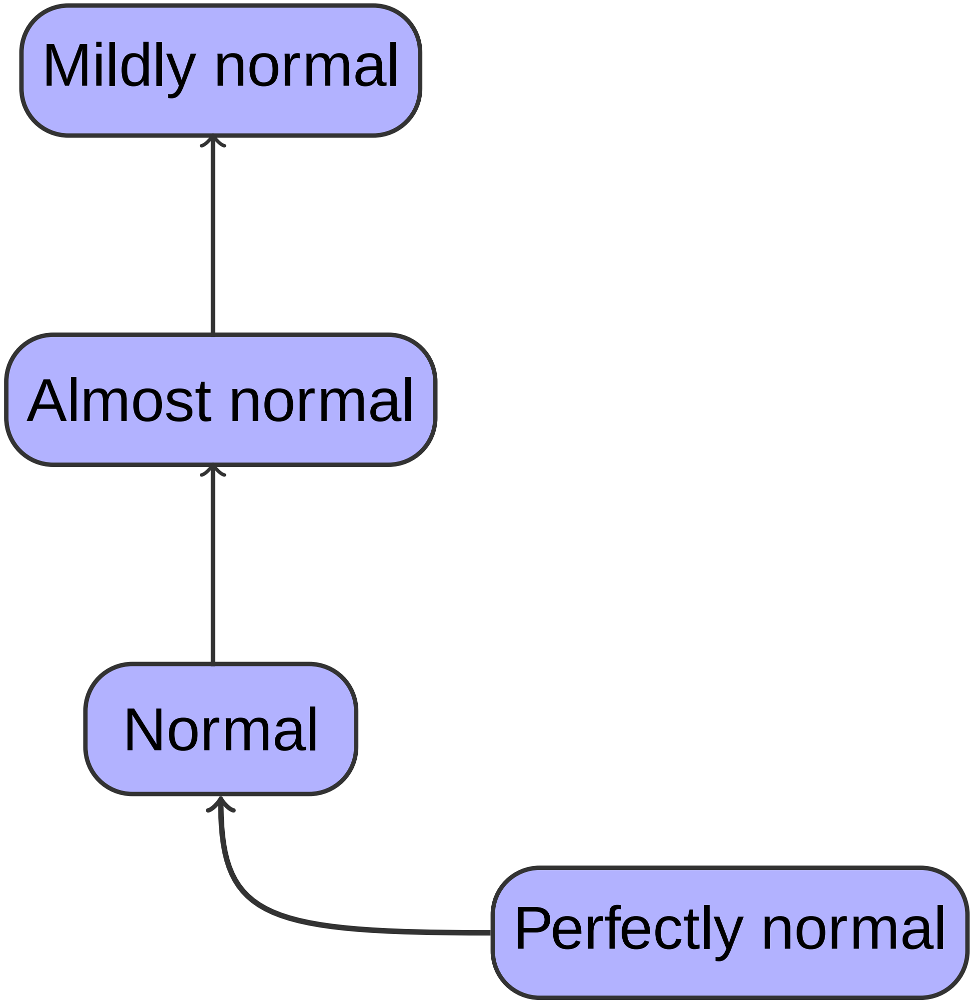
VARIANTS OF NORMALITY



Every $x \in L$ is regular G_δ

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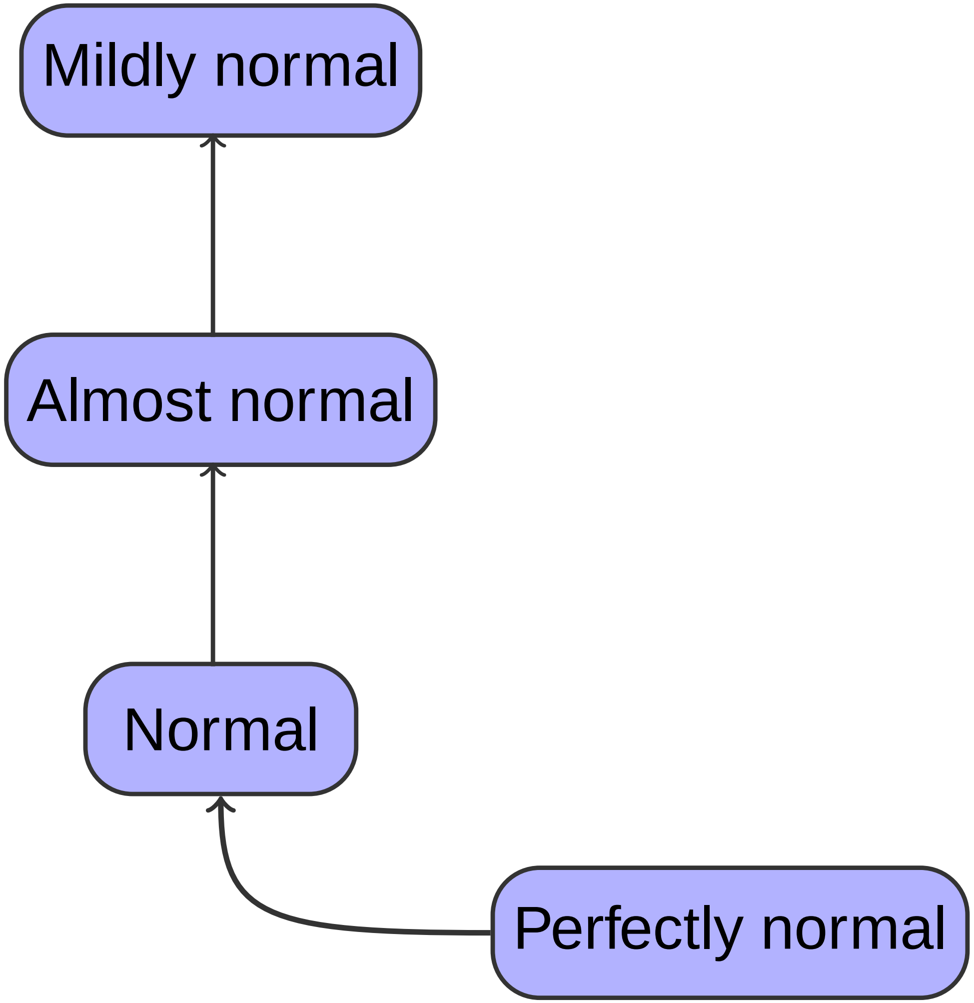

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$$\boxed{x = \bigvee_{n\in\mathbb{N}} x_n \text{ with } x_n \prec x}$$

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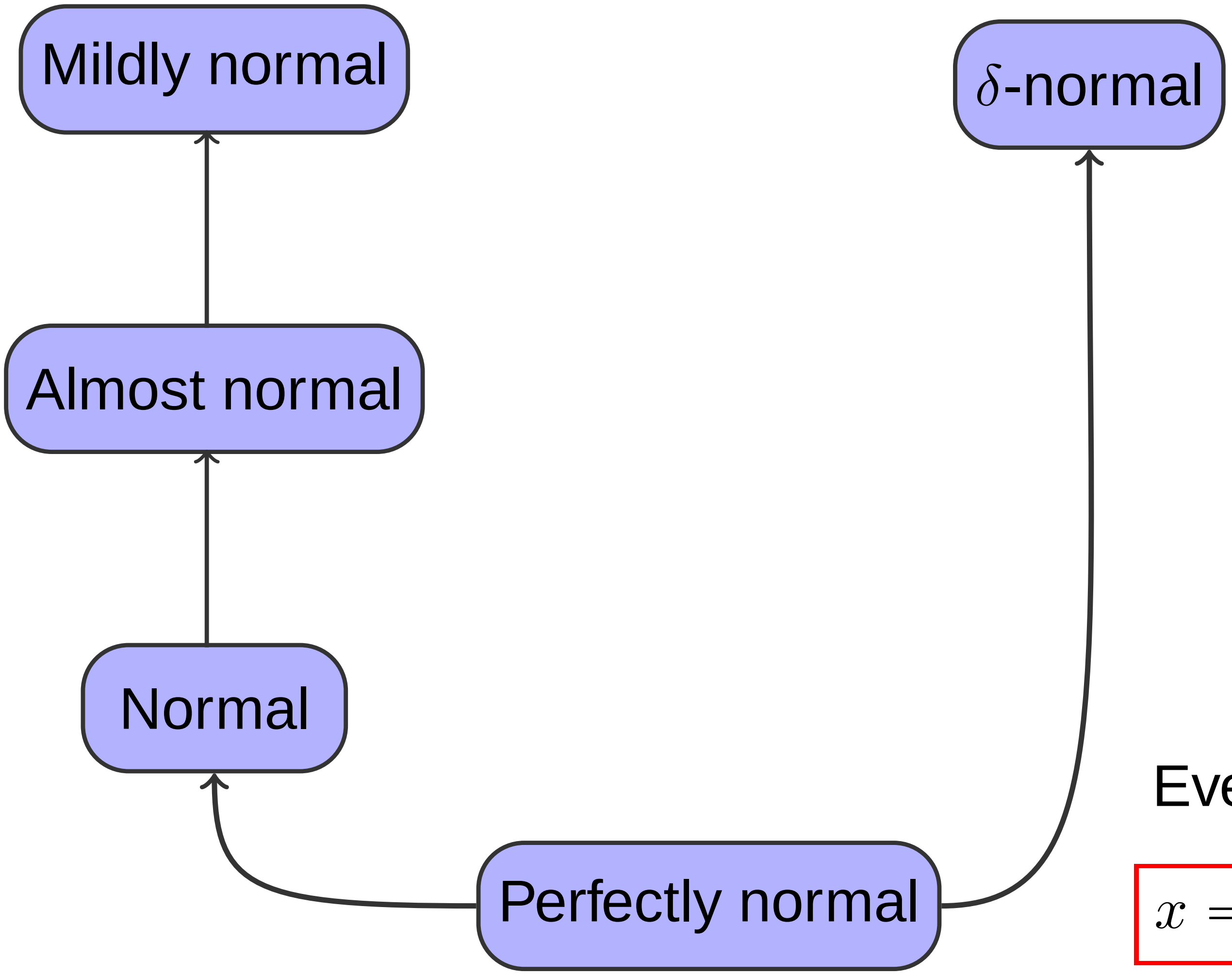

$a, b,\ u, v:$ regular G_δ

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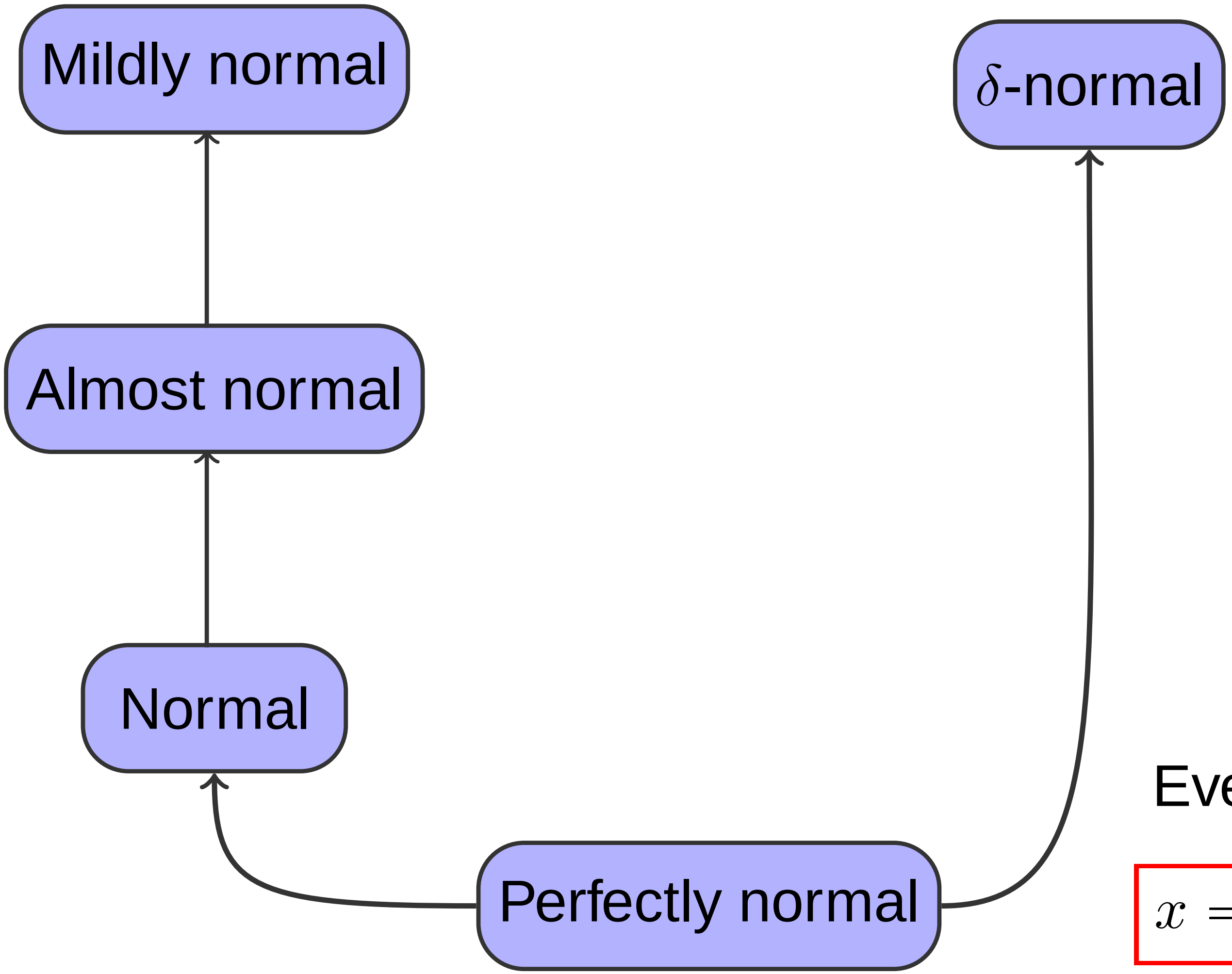
VARIANTS OF NORMALITY

$(x = \bigvee_{n\in\mathbb{N}} x_n$ with $x_n \ll x)$

$a, b,\ u, v\colon\ \underline{\mathrm{Coz}\, L}$



$a, b,\ u, v\colon\ \underline{\text{regular } G_\delta}$

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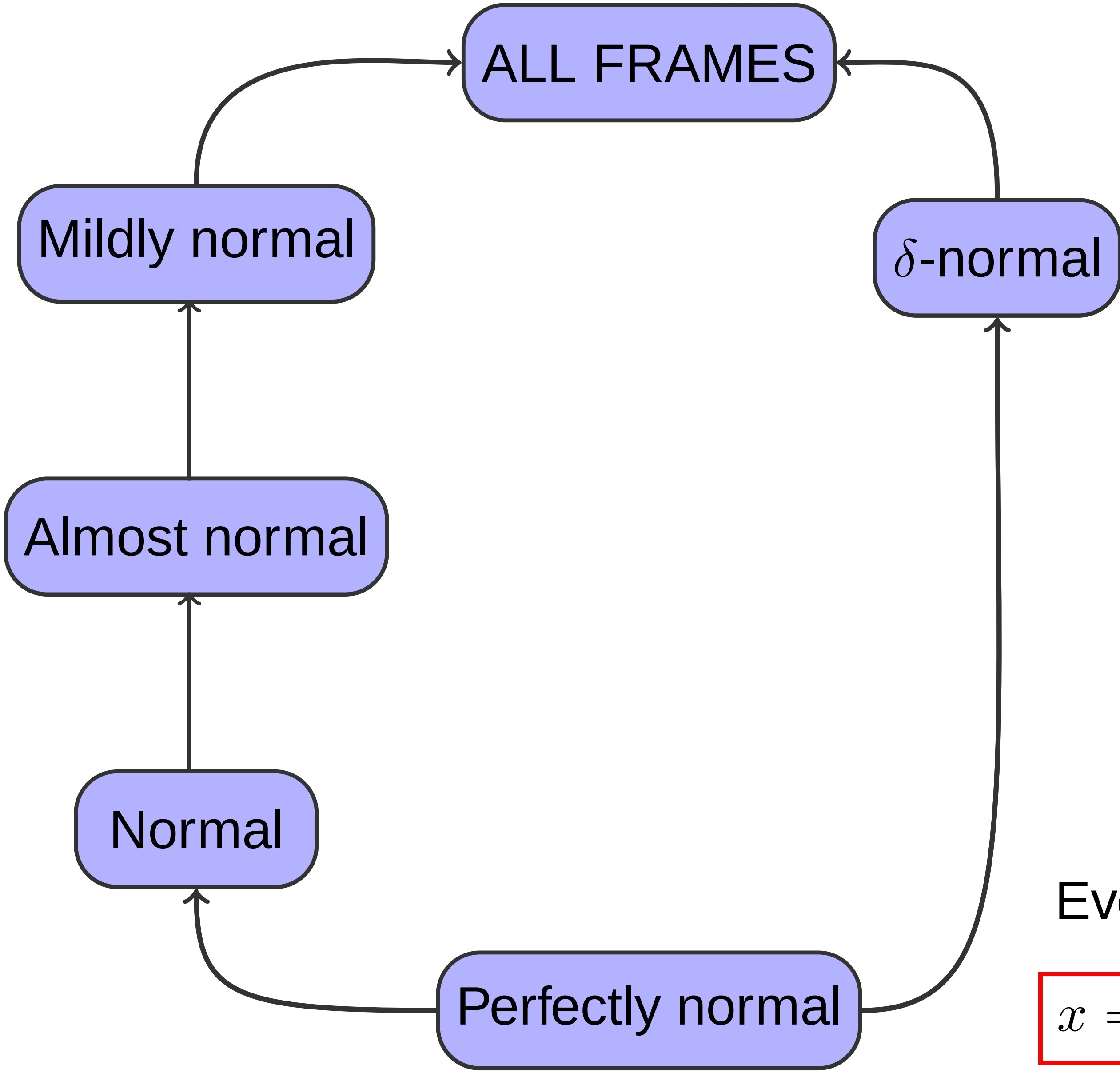

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$\mathscr{A}$-normality: EXAMPLES

$\mathscr{A}$	$\mathscr{A}$-normal frames	$\mathscr{A}$-disconnected frames
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$\mathscr{A}$-normality: EXAMPLES

$\mathscr{A}$	$\mathscr{A}$**-normal frames**	$\mathscr{A}$**-disconnected frames**
$\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}$	normal	extremally disconn.
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$\mathscr{A}_3 = \{\mathfrak{c}(a) \colon a$ regular $G_\delta\}$	δ-normal	extremally δ-disconn.
$\mathscr{A}_4 = \{\mathfrak{c}(\mathrm{coz} f) \colon f \in \mathsf{C}(L)\}$	all frames	F-frames

(F-frame: every cozero sublocale is C^*-embedded.)

REAL FUNCTIONS ON L

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

BACKGROUND: the frame of reals

$\mathfrak{L}(\mathbb{R})$ $=$ Frm$\big\langle\ (—, q), (p, —)\, (p, q \in \mathbb{Q})\ \big|$

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$\mathfrak{L}(\mathbb{R}) =$ Frm $\langle\ (-, q), (p, -)(p, q \in \mathbb{Q}) \mid$ (1) $(-, q) \wedge (p, -) = 0$ for $q \leqslant p$,

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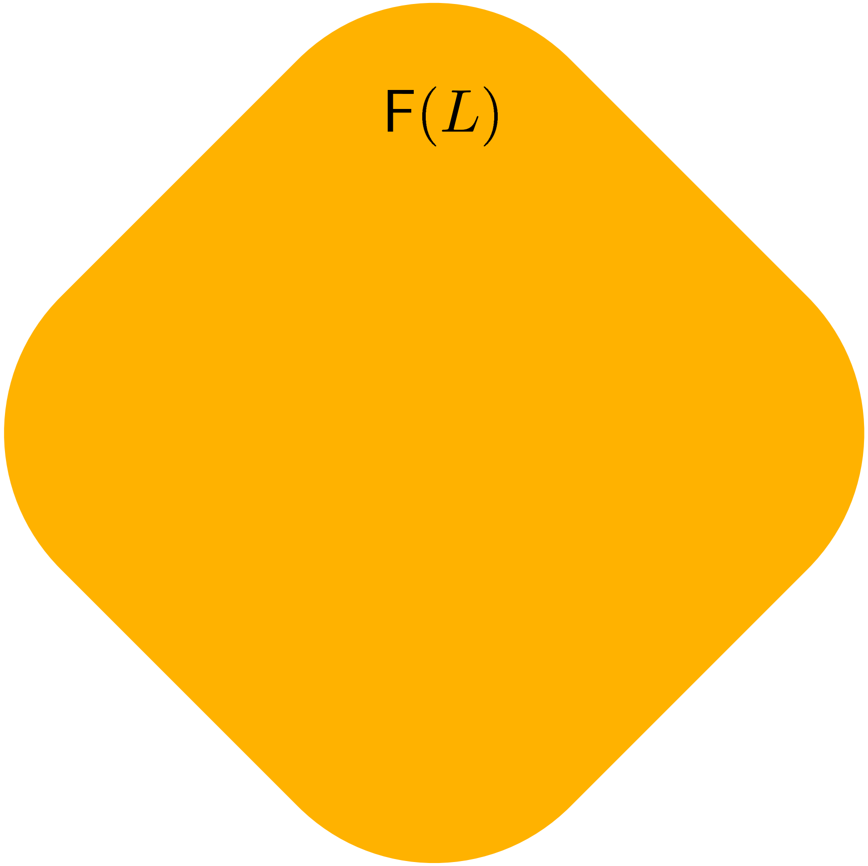
(6) $\bigvee_{p \in \mathbb{Q}} (p, —) = 1 \,\rangle$.

REAL FUNCTIONS ON L

$f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$



SEMICONTINUITY AND CONTINUITY

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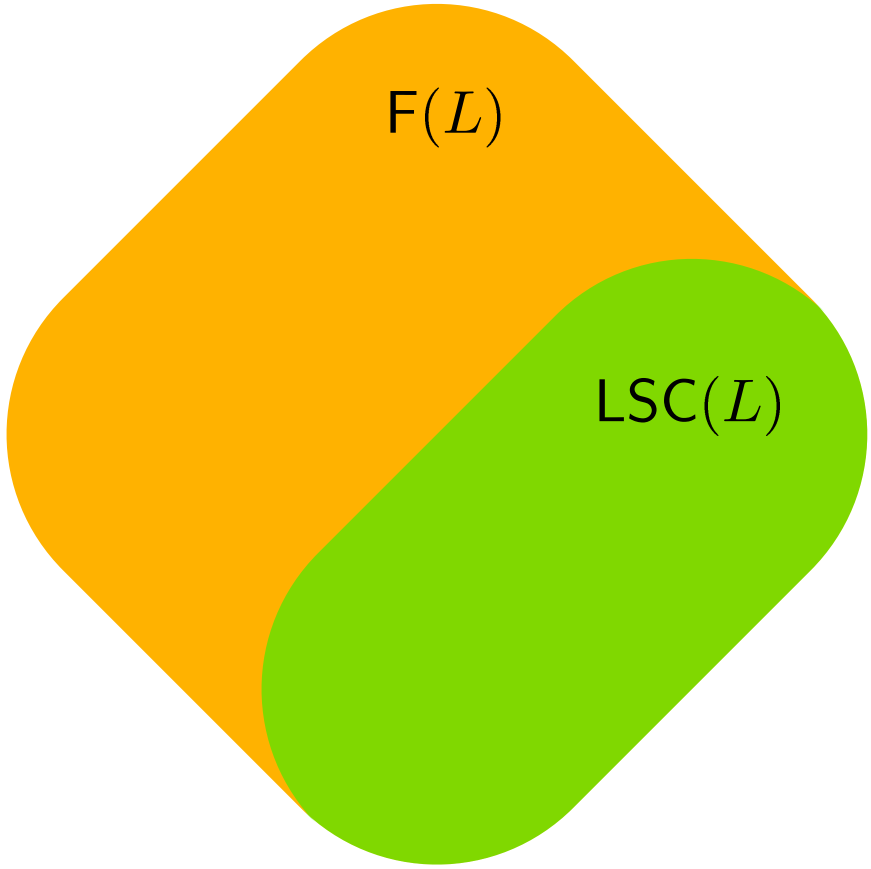

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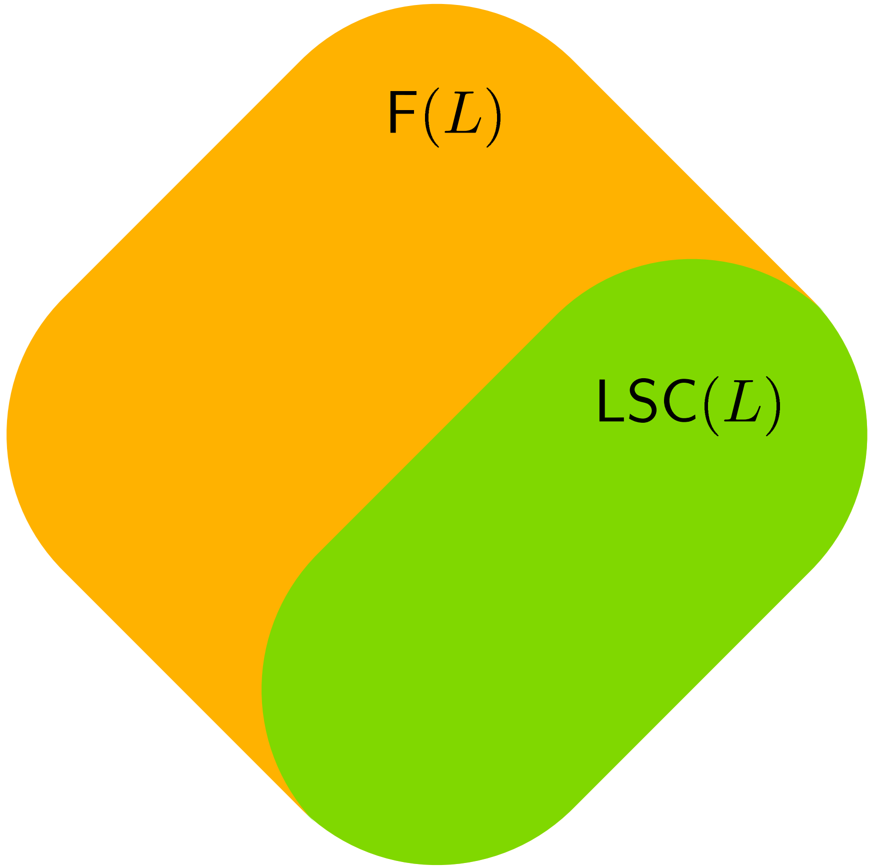
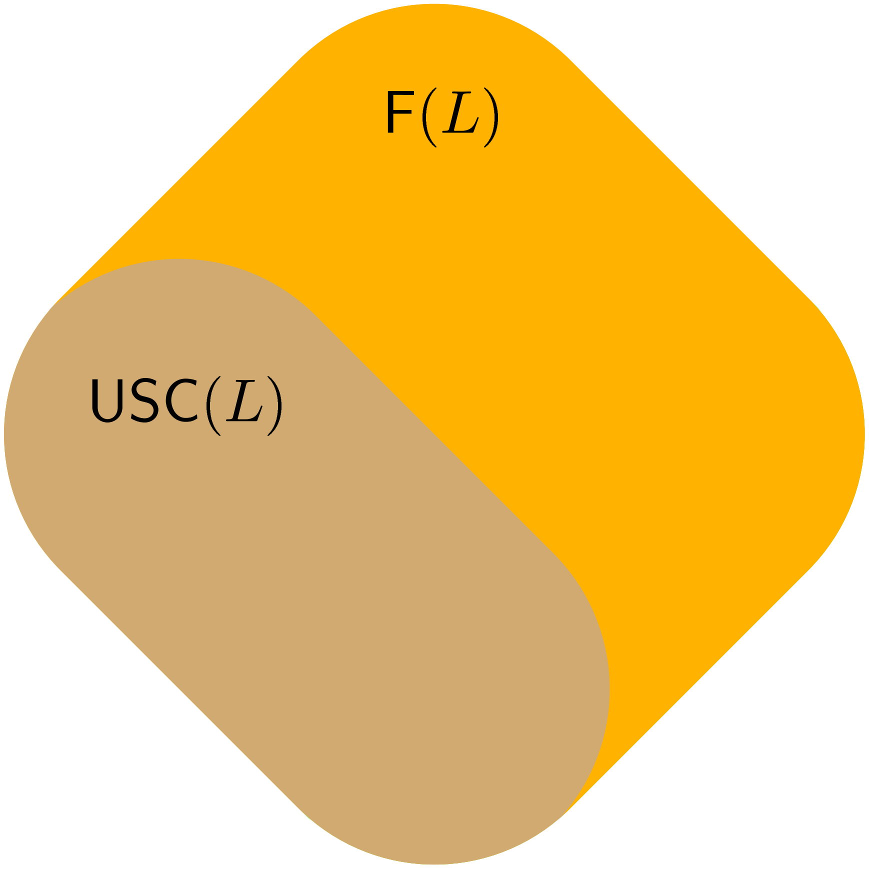


$f(p, —) \in \mathfrak{c}L$

SEMICONTINUITY AND CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$f(-, q) \in \mathfrak{c}L$



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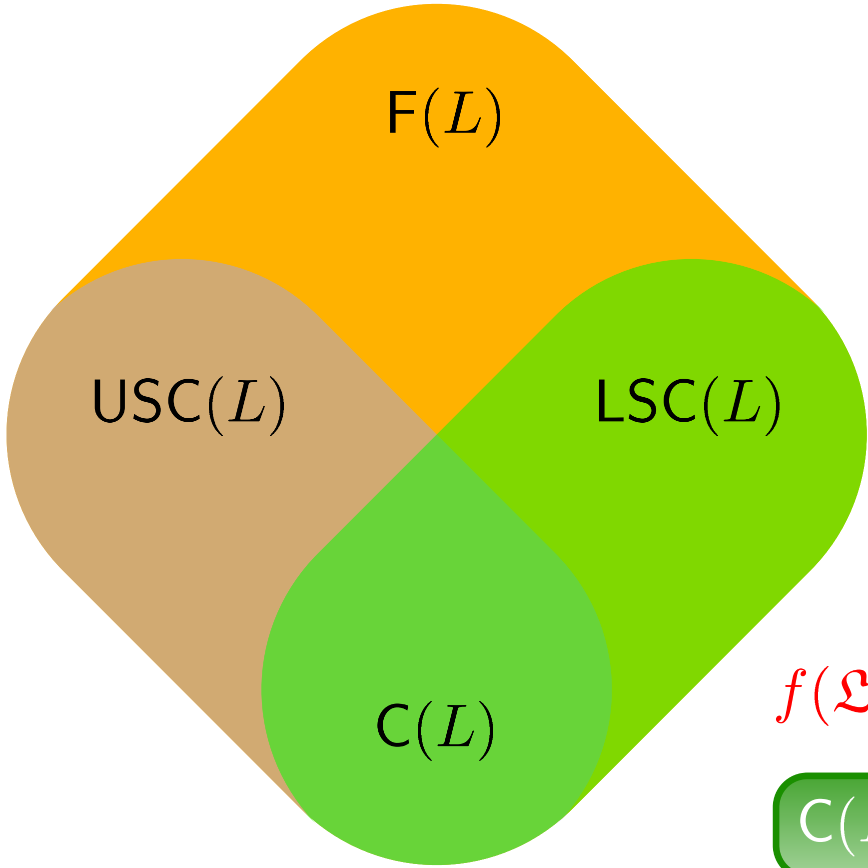

$f(p,-) \in \mathfrak{c}L$

$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

$\boxed{\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$

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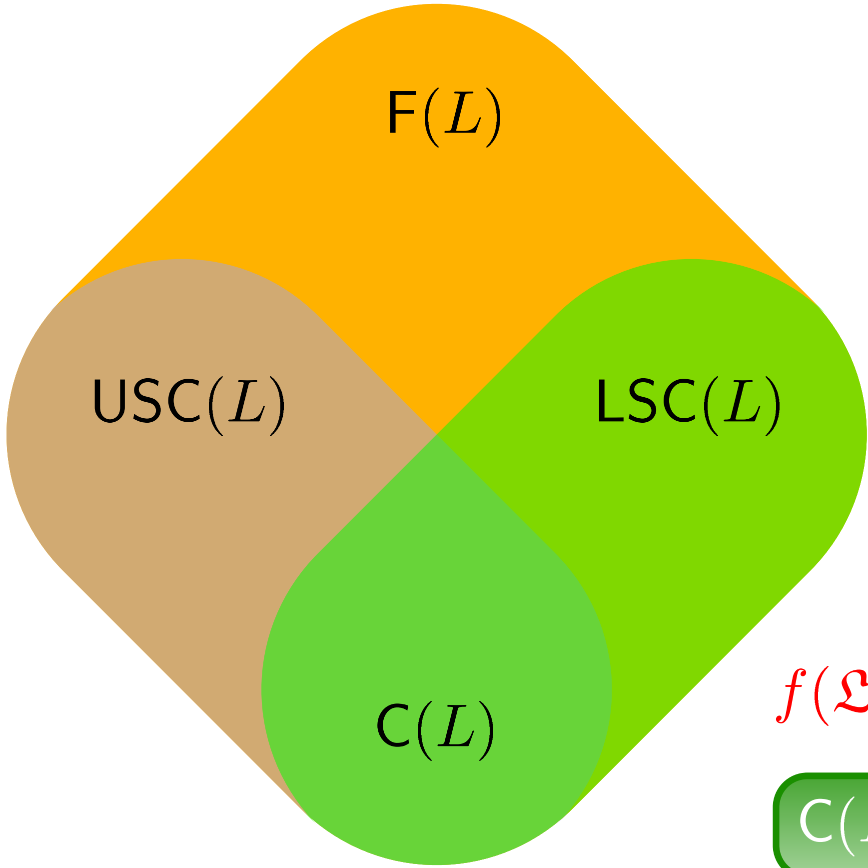

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$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

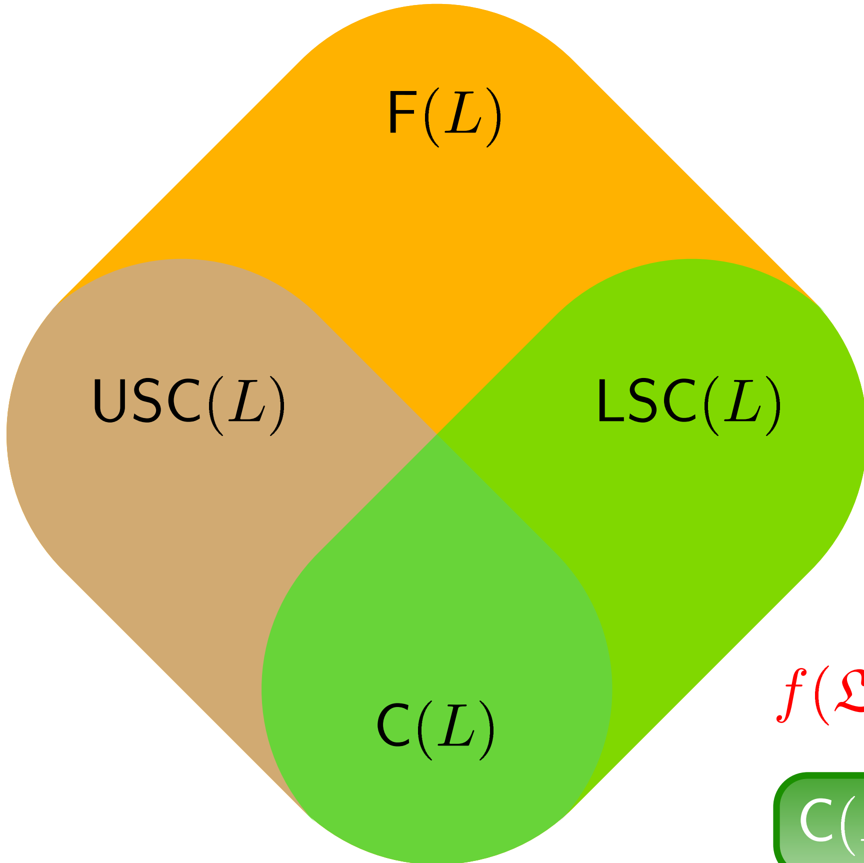
$\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$

$$\boxed{f \leqslant g \;\equiv\; f(p, -) \leqslant g(p, -),\ \forall p \in \mathbb{Q}}$$

SEMICONTINUITY AND CONTINUITY

$f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$f(-,q) \in \mathfrak{c}L$



$f(p,-) \in \mathfrak{c}L$

$f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$

$\mathsf{C}(L) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$

J. Gutiérrez García, T. Kubiak & J. P.
Localic real functions: a general setting, J. Pure Appl. Algebra *213 (2009)* .

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L : \;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \;:\;\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

["$\Leftarrow$": $f(-,q) = \bigvee_{r<q} f(-,r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(-,q)$.]

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \;:\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$[\,\text{“}\Leftarrow\text{”: } f(\text{—}, q) = \bigvee_{r<q} f(\text{—}, r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(\text{—}, q).\,]$$

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

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["$\Leftarrow$": $f(\text{—}, q) = \bigvee_{r<q} f(\text{—}, r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(\text{—}, q)$.]

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\; f(q, \text{—}) \leqslant F_{p,q} \leqslant f(p, \text{—}).$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \iff \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \,:\;\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

["$\Leftarrow$": $f(-,q) = \bigvee_{r<q} f(-,r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(-,q)$.]

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \,:\;\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \,:\;\; f(q,-) \leqslant F_{p,q} \leqslant f(p,-).$$

$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \; : \; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$[\; \text{“}\Leftarrow\text{”}: \; f(\text{—}, q) = \bigvee_{r<q} f(\text{—}, r) \leqslant \bigvee_{r<q} F_{r,q} \leqslant f(\text{—}, q). \;]$$

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \; : \; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \; : \; f(q, \text{—}) \leqslant F_{p,q} \leqslant f(p, \text{—}).$$

$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

Clearly: f is upper $\mathscr{A}$-semicont. iff it is lower $\mathscr{A}^{\mathsf{c}}$-semicont.

f is $\mathscr{A}^{\mathsf{c}}$-continuous iff it is $\mathscr{A}$-continuous.

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}$	**upper $\mathscr{A}$-sc**	**lower $\mathscr{A}$-sc**	**$\mathscr{A}$-continuous**
$\mathscr{A}_1 = \{\mathfrak{c}(a) : a \in L\}$	usc	lsc	continuous

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}$	upper $\mathscr{A}$-sc	lower $\mathscr{A}$-sc	$\mathscr{A}$-continuous
$\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}$	usc	lsc	continuous
$\mathscr{A}_2 = \{\mathfrak{c}(a^*) \colon a \in L\}$	normal usc	normal lsc	normal cont.

$$p < q \colon \; f(—, p) \leqslant \mathfrak{c}(a^*_{p,q}) \leqslant f(—, q)$$

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}$	**upper $\mathscr{A}$-sc**	**lower $\mathscr{A}$-sc**	**$\mathscr{A}$-continuous**
$\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}$	usc	lsc	continuous
$\mathscr{A}_2 = \{\mathfrak{c}(a^*) \colon a \in L\}$	normal usc	normal lsc	normal cont.

$$p < q\colon \ f(-,p) \leqslant \mathfrak{c}(a^*_{p,q}) \leqslant f(-,q)$$

$$\Leftrightarrow \ (f^\circ)^- = f$$

[Dilworth, 1950]

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}$	**upper $\mathscr{A}$-sc**	**lower $\mathscr{A}$-sc**	**$\mathscr{A}$-continuous**
$\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}$	usc	lsc	continuous
$\mathscr{A}_2 = \{\mathfrak{c}(a^*) \colon a \in L\}$	normal usc	normal lsc	normal cont.
$\mathscr{A}_3 = \{\mathfrak{c}(a) \colon a$ regular $G_\delta\}$	regular usc	regular lsc	regular cont.

[Lane, 1983]

$\mathscr{A}$-semicontinuity and $\mathscr{A}$-continuity: EXAMPLES

$\mathscr{A}$	upper $\mathscr{A}$-sc	lower $\mathscr{A}$-sc	$\mathscr{A}$-continuous
$\mathscr{A}_1 = \{\mathfrak{c}(a) \colon a \in L\}$	usc	lsc	continuous
$\mathscr{A}_2 = \{\mathfrak{c}(a^*) \colon a \in L\}$	normal usc	normal lsc	normal cont.
$\mathscr{A}_3 = \{\mathfrak{c}(a) \colon a \text{ regular } G_\delta\}$	regular usc	regular lsc	regular cont.
$\mathscr{A}_4 = \{\mathfrak{c}(\mathrm{coz} f) \colon f \in \mathsf{C}(L)\}$	zero usc	zero lsc	zero cont.

[Stone, 1949]

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

$S, T \in \mathcal{S}(L)$

$$\boxed{S \Subset_{\mathscr{A}} T \equiv \ \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}}\colon \ S \leqslant V \leqslant U \leqslant T}$$

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Katětov relation?

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Katětov relation?

(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

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Katětov relation?

(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

(K2) $S' \leqslant S \Subset_{\mathscr{A}} T \leqslant T' \Rightarrow S' \Subset_{\mathscr{A}} T'$.

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$S, T \in \mathcal{S}(L)$

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(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

(K2) $S' \leqslant S \Subset_{\mathscr{A}} T \leqslant T' \Rightarrow S' \Subset_{\mathscr{A}} T'$.

(K3) $S \Subset_{\mathscr{A}} T$ and $S' \Subset_{\mathscr{A}} T \Rightarrow (S \vee S') \Subset_{\mathscr{A}} T$.

(K4) $S \Subset_{\mathscr{A}} T$ and $S \Subset_{\mathscr{A}} T' \Rightarrow S \Subset_{\mathscr{A}} (T \wedge T')$.

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

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$$\boxed{S \Subset_{\mathscr{A}} T \equiv \ \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}}\colon\ S \leqslant V \leqslant U \leqslant T}$$

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$\mathscr{A}$ is a Katětov class

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

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(K4) $S \Subset_{\mathscr{A}} T$ and $S \Subset_{\mathscr{A}} T' \Rightarrow S \Subset_{\mathscr{A}} (T \wedge T')$.

$\mathscr{A}$ is a Katětov class

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice

$[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

$S, T \in \mathcal{S}(L)$

$\boxed{S \Subset_{\mathscr{A}} T \equiv \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}} : \; S \leqslant V \leqslant U \leqslant T}$

Katětov relation?

(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

(K2) $S' \leqslant S \Subset_{\mathscr{A}} T \leqslant T' \Rightarrow S' \Subset_{\mathscr{A}} T'$.

(K3) $S \Subset_{\mathscr{A}} T$ and $S' \Subset_{\mathscr{A}} T \Rightarrow (S \vee S') \Subset_{\mathscr{A}} T$.

(K4) $S \Subset_{\mathscr{A}} T$ and $S \Subset_{\mathscr{A}} T' \Rightarrow S \Subset_{\mathscr{A}} (T \wedge T')$.

$\mathscr{A}$ is a Katětov class

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice $[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$
- or closed under binary meets and $U_1, U_2 \in \mathscr{A}, U_1 \vee U_2 \leqslant V \in \mathscr{A}^{\mathsf{c}} \Rightarrow \exists\, U' \in \mathscr{A} : U_1 \vee U_2 \leqslant U' \leqslant V$ $[\mathscr{A}_2]$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

$S, T \in \mathcal{S}(L)$

$$\boxed{S \Subset_{\mathscr{A}} T \equiv \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}}\colon\ S \leqslant V \leqslant U \leqslant T}$$

Katětov relation?

(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

(K2) $S' \leqslant S \Subset_{\mathscr{A}} T \leqslant T' \Rightarrow S' \Subset_{\mathscr{A}} T'$.

(K3) $S \Subset_{\mathscr{A}} T$ and $S' \Subset_{\mathscr{A}} T \Rightarrow (S \vee S') \Subset_{\mathscr{A}} T$.

(K4) $S \Subset_{\mathscr{A}} T$ and $S \Subset_{\mathscr{A}} T' \Rightarrow S \Subset_{\mathscr{A}} (T \wedge T')$.

$\mathscr{A}$ is a Katětov class

(K5) $S \Subset_{\mathscr{A}} T \Rightarrow \exists\, U \in \mathcal{S}(L)\colon S \Subset_{\mathscr{A}} U \Subset_{\mathscr{A}} T$.

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice $[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$
- or closed under binary meets and $U_1, U_2 \in \mathscr{A}, U_1 \vee U_2 \leqslant V \in \mathscr{A}^{\mathsf{c}} \Rightarrow \exists\, U' \in \mathscr{A} : U_1 \vee U_2 \leqslant U' \leqslant V$ $[\mathscr{A}_2]$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

$S, T \in \mathcal{S}(L)$

$$\boxed{S \Subset_{\mathscr{A}} T \equiv \exists\, U \in \mathscr{A}, \exists\, V \in \mathscr{A}^{\mathsf{c}} \colon\ S \leqslant V \leqslant U \leqslant T}$$

Katětov relation?

(K1) $S \Subset_{\mathscr{A}} T \Rightarrow S \leqslant T$.

(K2) $S' \leqslant S \Subset_{\mathscr{A}} T \leqslant T' \Rightarrow S' \Subset_{\mathscr{A}} T'$.

(K3) $S \Subset_{\mathscr{A}} T$ and $S' \Subset_{\mathscr{A}} T \Rightarrow (S \vee S') \Subset_{\mathscr{A}} T$.

(K4) $S \Subset_{\mathscr{A}} T$ and $S \Subset_{\mathscr{A}} T' \Rightarrow S \Subset_{\mathscr{A}} (T \wedge T')$.

$\mathscr{A}$ is a Katětov class

(K5) $S \Subset_{\mathscr{A}} T \Rightarrow \exists\, U \in \mathcal{S}(L) \colon S \Subset_{\mathscr{A}} U \Subset_{\mathscr{A}} T$. $\Leftrightarrow$ L is $\mathscr{A}$-normal

LEMMA 2

LEMMA 1. $\mathscr{A}$ is a Katětov class if it is

- a sublattice $[\mathscr{A}_1, \mathscr{A}_3, \mathscr{A}_4]$
- or closed under binary meets and
  $U_1, U_2 \in \mathscr{A}, U_1 \vee U_2 \leqslant V \in \mathscr{A}^{\mathsf{c}} \Rightarrow \exists\, U' \in \mathscr{A} : U_1 \vee U_2 \leqslant U' \leqslant V$ $[\mathscr{A}_2]$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-normal.

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-normal.
2. $$\underbrace{f}_{\mathscr{A}-\mathsf{USC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A} - \mathsf{C}(L) \colon \; f \leqslant h \leqslant g\,.$$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

Then the dual result for extremal $\mathscr{A}$-disconnectedness follows just by

COMPLEMENTATION:

THEOREM. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}^{\mathsf{c}}$-normal.
2. $\underbrace{f}_{\mathscr{A}^{\mathsf{c}}-\mathsf{USC}} \leqslant \underbrace{g}_{\mathscr{A}^{\mathsf{c}}-\mathsf{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A}^{\mathsf{c}} - \mathsf{C}(L):\ f \leqslant h \leqslant g\,.$

MAIN RESULT: Katětov-Tong insertion $\equiv$ Stone insertion

Then the dual result for extremal $\mathscr{A}$-disconnectedness follows just by

COMPLEMENTATION:

COROLLARY. TFAE for any Katětov class $\mathscr{A} \subseteq B(\mathcal{S}(L))$:

1. L is $\mathscr{A}$-extremally disconnected.
2. $$\underbrace{f}_{\mathscr{A}-\mathsf{LSC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{USC}} \quad \Rightarrow \quad \exists\, h \in \mathscr{A} - \mathsf{C}(L) : \; f \leqslant h \leqslant g\,.$$

EXTENSION

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$\begin{array}{ccc} \mathcal{S}(L) & & A \\ \Big\downarrow \varphi_S & & \rightsquigarrow \\ \mathcal{S}(S) & & S \vee A \end{array}$$

EXTENSION

$S \in \mathcal{S}(L)$

Continuous Extension:

$$
\begin{tikzcd}
 & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\
\mathfrak{L}(\mathbb{R}) \arrow[r, "f \in \mathsf{C}(S)"'] & \mathcal{S}(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, rightsquigarrow] \\
S \vee A
\end{tikzcd}
$$

$S \in \mathcal{S}(L)$

Continuous Extension:

$$\begin{tikzcd}
 & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, dotted, "\exists\ \widetilde{f} \in \mathsf{C}(L)"] \arrow[r, "f \in \mathsf{C}(S)"'] & \mathcal{S}(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, rightsquigarrow] \\
S \vee A
\end{tikzcd}$$

CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, dotted, "\exists\, \tilde{f} \in \mathsf{C}(L)"] \arrow[r, "f \in \mathsf{C}(S)"'] & \mathcal{S}(S) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, squiggly] \\ S \vee A \end{tikzcd}$$

(RELATIVE) CONTINUOUS EXTENSION:

$$\mathscr{A} \subseteq B(\mathcal{S}(L))$$

$$\mathscr{A}_S = \{S \vee A \mid A \in \mathscr{A}\}$$

EXTENSION

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f \in \mathsf{C}(S)"'] \arrow[ur, dotted, "\exists\, \tilde{f} \in \mathsf{C}(L)"] & \mathcal{S}(S) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, squiggly] \\ S \vee A \end{tikzcd}$$

(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A} \subseteq B(\mathcal{S}(L))$

$\mathscr{A}_S = \{S \vee A \mid A \in \mathscr{A}\}$

$$\mathfrak{L}(\mathbb{R}) \xrightarrow[f]{} \mathcal{S}(S)$$

$\mathscr{A}_S$-continuous

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f \in \mathsf{C}(S)"'] \arrow[ur, dotted, "\exists\, \tilde{f} \in \mathsf{C}(L)"] & \mathcal{S}(S) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, rightsquigarrow] \\ S \vee A \end{tikzcd}$$

(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A} \subseteq B(\mathcal{S}(L))$

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$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f"'] & \mathcal{S}(S) \end{tikzcd}$$

$\mathscr{A}_S$-continuous

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, dotted, "\exists\, \tilde{f} \in \mathsf{C}(L)"] \arrow[r, "f \in \mathsf{C}(S)"'] & \mathcal{S}(S) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, squiggly] \\ S \vee A \end{tikzcd}$$

(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A} \subseteq B(\mathcal{S}(L))$

$\mathscr{A}_S = \{S \vee A \mid A \in \mathscr{A}\}$

$\mathscr{A}$-continuous

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, dotted, "\exists\, \tilde{f}"] \arrow[r, "f"'] & \mathcal{S}(S) \end{tikzcd}$$

$\mathscr{A}_S$-continuous

EXTENSION

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

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\begin{tikzcd}
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\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, squiggly] \\
S \vee A
\end{tikzcd}
$$

(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A} \subseteq B(\mathcal{S}(L))$

$\mathscr{A}_S = \{S \vee A \mid A \in \mathscr{A}\}$

$\mathscr{A}$-continuous

$$
\begin{tikzcd}
 & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\
\mathfrak{L}(\mathbb{R}) \arrow[ur, dotted, "\exists\, \tilde{f}"] \arrow[r, "f"'] & \mathcal{S}(S)
\end{tikzcd}
$$

$\mathscr{A}_S$-continuous

"$C_{\mathscr{A}}$-embedded"

$S \in \mathcal{S}(L)$

CONTINUOUS EXTENSION:

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f \in \mathrm{C}(S)"'] \arrow[ur, dotted, "\exists\, \tilde{f} \in \mathrm{C}(L)"] & \mathcal{S}(S) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, squiggly] \\ S \vee A \end{tikzcd}$$

(RELATIVE) CONTINUOUS EXTENSION:

$\mathscr{A} \subseteq B(\mathcal{S}(L))$

$\mathscr{A}_S = \{S \vee A \mid A \in \mathscr{A}\}$

$\mathscr{A}$-continuous

$$\begin{tikzcd} & \mathcal{S}(L) \arrow[d, "\varphi_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[r, "f"'] \arrow[ur, dotted, "\exists\, \tilde{f}"] & \mathcal{S}(S) \end{tikzcd}$$

$\mathscr{A}_S$-continuous

"$C_{\mathscr{A}}$-embedded"

"$C^*_{\mathscr{A}}$-embedded"

EXTENSION: Tietze-type $\equiv$ Stone-type

CONDITIONS ON $\mathscr{A}$:

EXTENSION: Tietze-type $\equiv$ Stone-type

CONDITIONS ON $\mathscr{A}$:

(1') $\mathscr{A}$ is closed under finite meets

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COROLLARY.
~~THEOREM.~~ Let $(\mathscr{A})^c$ be a Tietze class of L. TFAE:

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Homomorphic IMAGES (Hausdorff): see the preprint ...