Tutorial on Localic Topology

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OUTLINE

- AIM: cover the basics of point-free topology
- Slides give motivation, definitions and results, few proofs

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- Part I. Frames: the algebraic facet of spaces
- Part II. Categorical aspects of $\mathbf{Frm}$
- Part III. Locales: the geometric facet of frames
- Part IV. Doing topology in $\mathbf{Loc}$

WHAT IS POINT-FREE TOPOLOGY?

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- The techniques may hide some geometrical intuition, but often offers powerful algebraic tools and opens new perspectives.

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'lattice theory applied to topology'

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> «*The topological structure of a locale cannot live in its points: the points, if any, live on the open sets rather than the other way about.*»
>
> P. T. JOHNSTONE
>
> [The art of pointless thinking, *Category Theory at Work* (1991)]

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MORE: different categorical properties with advantage to the point-free side.

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- RAMIFICATIONS: category theory, topos theory, logic and computer science.

MAIN BASIC REFERENCES

P. T. Johnstone, *Stone Spaces*, CUP 1982.

S. Vickers, *Topology via Logic*, CUP 1989.

S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic - A first introduction to topos theory*, Springer 1992.

B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, vol. 12, Univ. Coimbra 1997.

R. N. Ball and J. Walters-Wayland, *C- and C*-quotients in pointfree topology*, Dissert. Math, vol. 412, 2002.

JP, A. Pultr and A. Tozzi, *Locales*, Chapter II in “Categorical Foundations”, CUP 2004.

JP and A. Pultr, Locales treated mostly in a covariant way, Textos de Matemática, vol. 41, Univ. Coimbra 2008.

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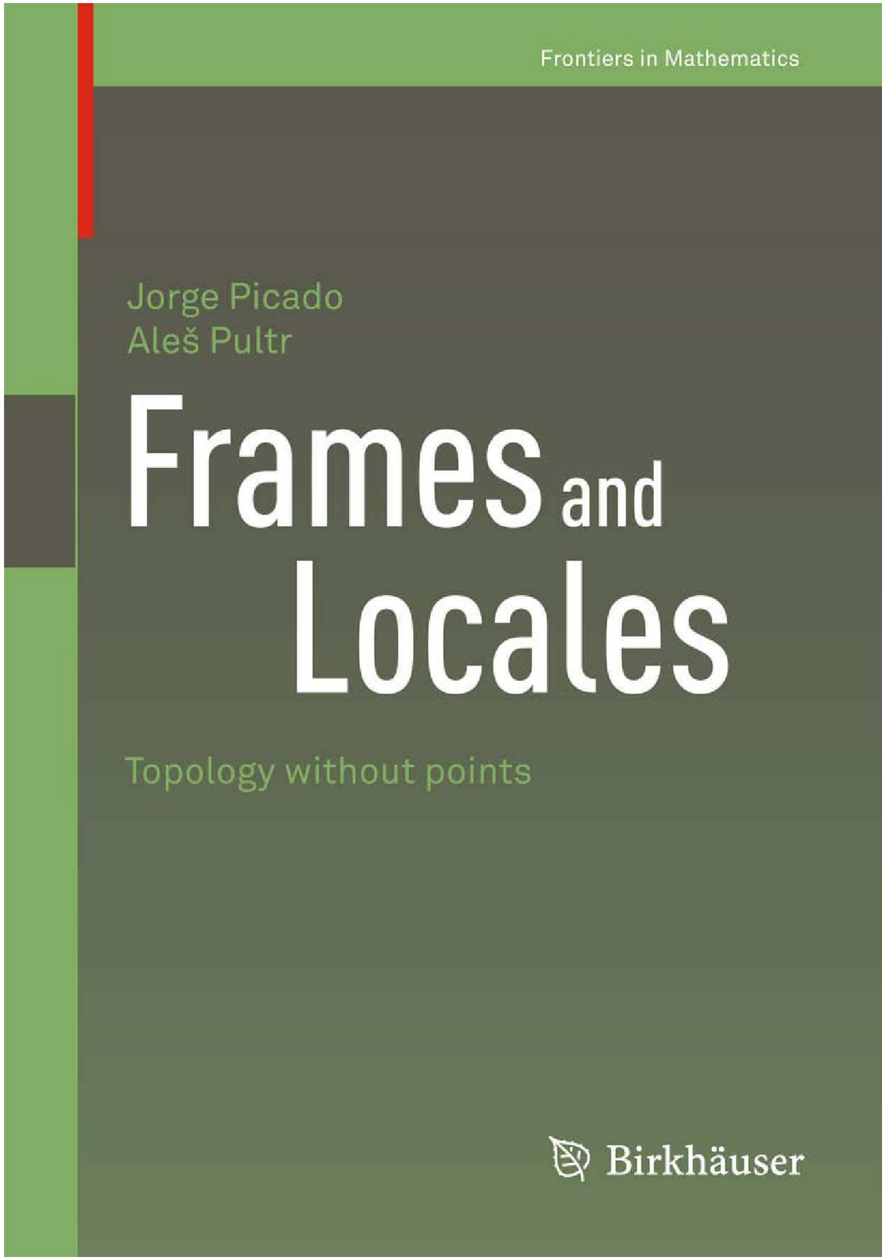
Frontiers in Mathematics

Jorge Picado
Aleš Pultr

Frames and Locales

Topology without points

Birkhäuser

PART I. ***Frames:***

the algebraic facet of spaces

FROM SPACES TO FRAMES

Top

$$(X, \mathcal{O}X)$$

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$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

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- complete lattice:

$$\bigvee U_i = \bigcup U_i, \quad 0 = \varnothing$$

$$U \wedge V = U \cap V, \quad 1 = X$$

$$\bigwedge U_i = \mathrm{int}(\bigcap U_i)$$

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more:

$$\boxed{U \wedge \bigvee_I V_i = \bigvee_I (U \wedge V_i)}$$

FROM SPACES TO FRAMES

Top

$$\begin{tikzcd}
(X,\mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X,\subseteq) \\
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\end{tikzcd}$$

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FROM SPACES TO FRAMES

Top

$$\begin{tikzcd}
(X,\mathcal{O}X) \arrow[r, squiggly, red] \arrow[d, "f"'] & \color{red}(\mathcal{O}X,\subseteq) \\
(Y,\mathcal{O}Y) \arrow[r, squiggly, red] & (\mathcal{O}Y,\subseteq) \arrow[u, "\boldsymbol{f^{-1}[-]}"']
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- $f^{-1}[-]$ preserves $\bigvee$ and $\wedge$

FROM SPACES TO FRAMES

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- complete lattice L

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frame:

$$\boxed{a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)}$$

- frame homomorphisms: $h\colon M \to L$ preserves $\bigvee$ and $\wedge$

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The algebraic nature of the objects of $\mathbf{Frm}$ is obvious.

More about that later on...

MORE EXAMPLES of frames

- Finite distributive lattices, complete Boolean algebras, complete chains.

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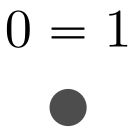
$\mathbb{1}$:



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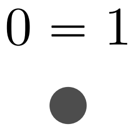

$\mathbb{2}$:


- subframe of a frame L: $S \subseteq L$ closed under arbitrary joins (in part. $0 \in S$) and finite meets (in part. $1 \in S$).

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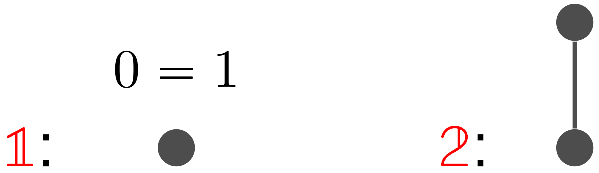

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- intervals of a frame L: $a, b \in L$, $a \leqslant b$
  $[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$, $\downarrow b = [0, b]$, $\uparrow a = [a, 1]$.

MORE EXAMPLES of frames

- For any $\wedge$-semilattice $(A, \wedge, 1)$, $\mathfrak{D}(A) = \{$down-sets of $A\}$ is a frame:

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\mathbf{SLat} \arrow[r, shift left, "\mathfrak{D}"] \arrow[r, phantom, "\perp"] & \mathbf{Frm} \arrow[l, shift left, "G"]
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(forgetful functor)

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$$\boxed{\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Frm}}(\mathfrak{D}(A), L) & \simeq & \mathrm{Hom}_{\mathbf{SLat}}(A, G(L)) \\ h & \longmapsto & (\widetilde{h}\colon a \mapsto h(\downarrow a)) \\ (\overline{g}\colon S \mapsto \bigvee g[S]) & \longleftarrow\!\shortmid & g \end{array}}$$

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(inclusion as a non-full subcategory)

EXAMPLES of frame homomorphisms

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initial object terminal object

-

$$\begin{array}{rcl} \bigvee : \mathfrak{I}(L) & \to & L \\ J & \mapsto & \bigvee J \end{array} \qquad \begin{array}{rcl} \bigvee : \mathfrak{D}(L) & \to & L \\ S & \mapsto & \bigvee S \end{array}$$

BACKGROUND: POSETS AS CATEGORIES

poset $(A, \leqslant)$

$(A, \leqslant)$ as a thin category

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MORPHISMS: $a \xrightarrow{\exists!} b$ whenever $a \leqslant b$

(there is at most one arrow between any pair of objects)

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$$\begin{tikzcd}
a \arrow[loop above, "1_a"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[loop above, "1_b"] \arrow[d, "g"] \\
 & c \arrow[loop below, "1_c"]
\end{tikzcd}$$

In fact, a preorder suffices:

(1) reflexivity: provides the identity morphisms 1_a.

(2) transitivity: provides the composition of morphisms $g \circ f$.

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$$\begin{tikzcd}
a \arrow[r, rightsquigarrow] \arrow[d, "\leqslant"'] & f(a) \arrow[d, "\leqslant"] \\
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order-preserving maps

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(binary) PRODUCTS:

$$\begin{tikzcd}
a & ? \arrow[l] \arrow[r] & b \\
& c \arrow[ul] \arrow[u, dotted] \arrow[ur] &
\end{tikzcd}$$

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order-preserving maps

(binary) PRODUCTS:

$$\begin{tikzcd}
a & a \wedge b \arrow[l] \arrow[r] & b \\
& c \arrow[ul] \arrow[u, dotted] \arrow[ur] &
\end{tikzcd}$$

meets

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order-preserving maps

(binary) COPRODUCTS:

$$\begin{tikzcd}
a \arrow[r] \arrow[dr] & a \vee b \arrow[d, dotted] & b \arrow[l] \arrow[dl] \\
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joins

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From this point of view: category theory is an extension of lattice th.

GALOIS ADJUNCTIONS

$$\begin{tikzcd}(A, \leqslant) \arrow[r, shift left, "f"] & (B, \leqslant) \arrow[l, shift left, "g"]\end{tikzcd}$$

$$\boxed{f(a) \leqslant b \ \text{iff} \ a \leqslant g(b)}$$

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$$\mathbf{Hom}_B(f(a), b) \cong \mathbf{Hom}_A(a, g(b))$$

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$$\Leftrightarrow \boxed{fg \leqslant \mathrm{id} \text{ and } \mathrm{id} \leqslant gf}$$

(“quasi-inverses”)

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("quasi-inverses")

Properties

① $fgf = f$ and $gfg = g$.

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1. $fgf = f$ and $gfg = g$.

2. $$\begin{tikzcd}(A, \leqslant) \arrow[r, shift left, "f"] & (B, \leqslant) \arrow[l, shift left, "g"] \\ g[B] \arrow[u, phantom, "\cup\!|"] \arrow[r, phantom, "\simeq"] & f[A] \arrow[u, phantom, "\cup\!|"]\end{tikzcd}$$

GALOIS ADJUNCTIONS

$$\mathbf{Hom}_B(f(a),b) \cong \mathbf{Hom}_A(a,g(b))$$

$$\begin{tikzcd} (A,\leqslant) \arrow[r, shift left=1.5ex, "f"] & (B,\leqslant) \arrow[l, shift left=1.5ex, "g"] \end{tikzcd}$$

$$f \dashv g \qquad \boxed{f(a) \leqslant b \text{ iff } a \leqslant g(b)}$$

$$\Leftrightarrow \boxed{fg \leqslant \mathrm{id} \text{ and } \mathrm{id} \leqslant gf}$$

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$$\begin{tikzcd} (A,\leqslant) \arrow[r, shift left=1.5ex, "f"] & (B,\leqslant) \arrow[l, shift left=1.5ex, "g"] \\ g[B] \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "\simeq"] & f[A] \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$$g[B] = \{a \in A \mid gf(a) = a\}$$

$$f[A] = \{b \in B \mid fg(b) = b\}$$

GALOIS ADJUNCTIONS

ADJOINT FUNCTOR THEOREM.

Let $f\colon A \to B$ be an order-preserving map between posets. Then:

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$$f(s) \leqslant b \ \ \forall\, s$$

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Let $f\colon A \to B$ be an order-preserving map between posets. Then:

(1) If f has a right adjoint, then f preserves all joins that exist in A.

(2) If A has all joins and f preserves them, then f has a right adjoint g, uniquely determined by f:

$$g(b) = \bigvee\{a \in A \mid f(a) \leqslant b\}.$$

PROOF: Let $f \dashv g$, $S \subseteq A$, $\bigvee S$ exists. $\quad \underbrace{f(\bigvee S)}_{\sim} \overset{?}{=} \bigvee\{f(s) \mid s \in S\}$

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FRAMES as HEYTING ALGEBRAS

Heyting algebra: lattice L with an extra $\rightarrow$ satisfying

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$\therefore$ frames = cHa. BUT different categories (morphisms).

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P3 $(\bigvee a_i)^* = \bigwedge(a_i)^*$. De Morgan law (Caution: not for $\bigwedge$)

PART II.

Categorical aspects of $\mathbf{Frm}$

ALGEBRAIC ASPECTS OF $\mathbf{Frm}$

① $\mathbf{Frm}$ is equationally presentable i.e.

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Objects are described by a (proper class of) operations and equations:

OPERATIONS

- 0-ary: $0, 1\colon L^0 \longrightarrow L$
- binary: $L^2 \longrightarrow L$, $(a, b) \mapsto a \wedge b$
- κ-ary (any cardinal κ): $L^\kappa \longrightarrow L$, $(a_i)_\kappa \mapsto \bigvee_\kappa a_i$

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EQUATIONS

- $(L, \wedge, 1)$ is an idempotent commutative monoid
- with a zero 0 sat. the absorption law $a \wedge 0 = 0 = 0 \wedge a \ \forall a$.
- $\bigvee_0 a_i = 0$, $a_j \wedge \bigvee_\kappa a_i = a_j$, $a \wedge \bigvee_\kappa a_i = \bigvee_\kappa (a \wedge a_i)$.

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COROLLARY.

$\mathbf{Frm}$ has all (small) limits (i.e., it is a COMPLETE category) and they are constructed exactly as in $\mathbf{Set}$ (i.e., the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$ preserves them).

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② $\mathbf{Frm}$ has free objects: there is a free functor $\mathbf{Set} \to \mathbf{Frm}$ (i.e., a left adjoint of the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$):

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CONSTRUCTION (in two steps):

$$\begin{tikzcd}
\mathbf{SLat} & & & \mathbf{Frm} \arrow[lll, "\text{forgets } \bigvee"']
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$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & F(X) \arrow[d, "F(f)"] \\
Y \arrow[r, mapsto] & F(Y)
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$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & F(X) = (\{S \subseteq X \mid S \text{ finite}\}, \supseteq) \arrow[d, "F(f)"] & S \arrow[d, mapsto] \\ Y \arrow[r, mapsto] & F(Y) & f[S] \end{tikzcd}$$

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THE UNIT:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[rrdd, bend right, "f"'] & & F(X) \arrow[dd, dotted, bend left, "\exists!\, \varphi \in \mathbf{SLat}"] \\
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$$x \rightsquigarrow \{x\}$$

$$\begin{tikzcd} \{x_1, x_2, \ldots, x_n\} \arrow[d, mapsto] \\ \bigwedge_{i=1}^{n} f(x_i) \end{tikzcd}$$

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A \arrow[r, mapsto] \arrow[d, "h"'] & \mathfrak{D}(A) \arrow[d, "\mathfrak{D}(h)"] \\
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THE UNIT:

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[ddrr, bend right, "f"'] & & \mathfrak{D}(A) \arrow[dd, dotted, bend left, "\exists!\ \varphi \in \mathbf{Frm}"] \\
& a \arrow[r, squiggly] & \downarrow a \\
& & L
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[d, mapsto] \\
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Then, by general results of category theory

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(4) $(RegEpi, Mono)$ is a factorization system.

(5) Quotients are described by congruences; there exist presentations by generators and relations.

EXAMPLE: PRESENTATIONS

PRESENTATIONS BY GENERATORS AND RELATIONS: ☺

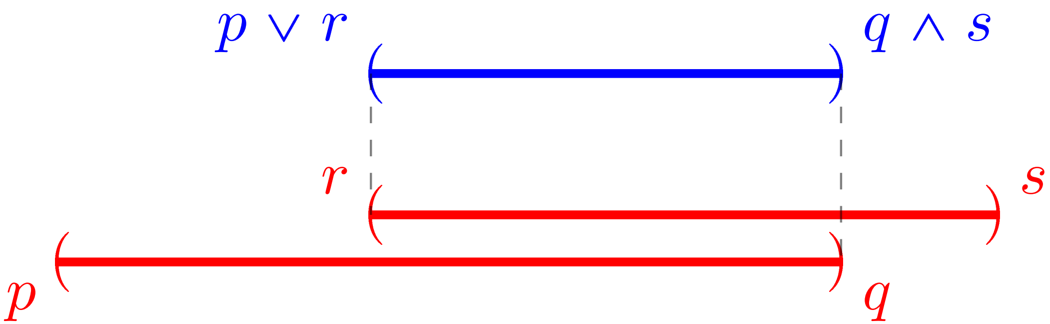
just take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,



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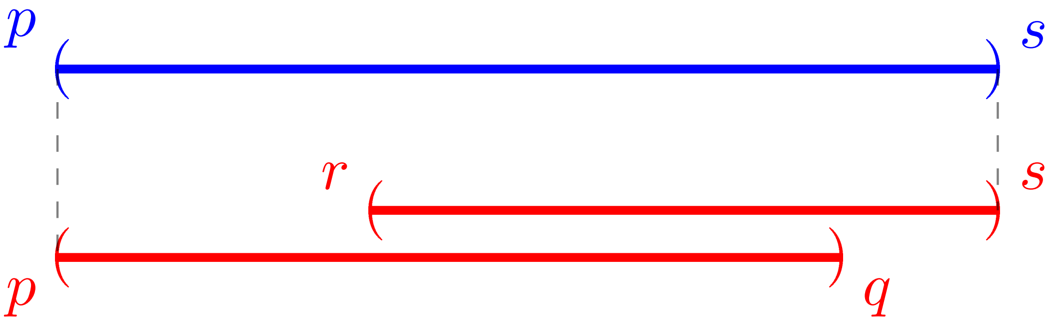

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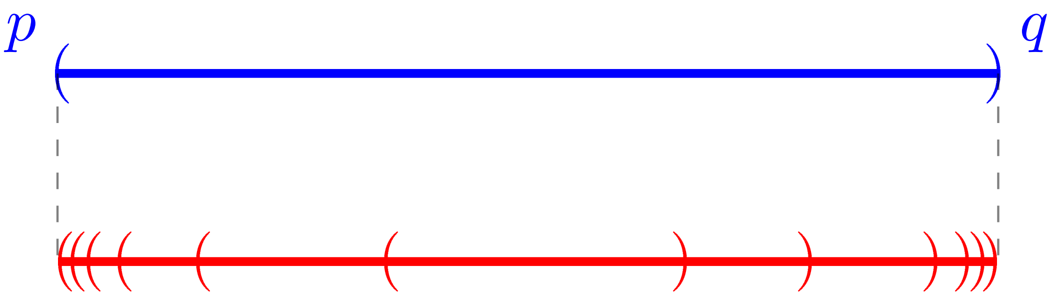

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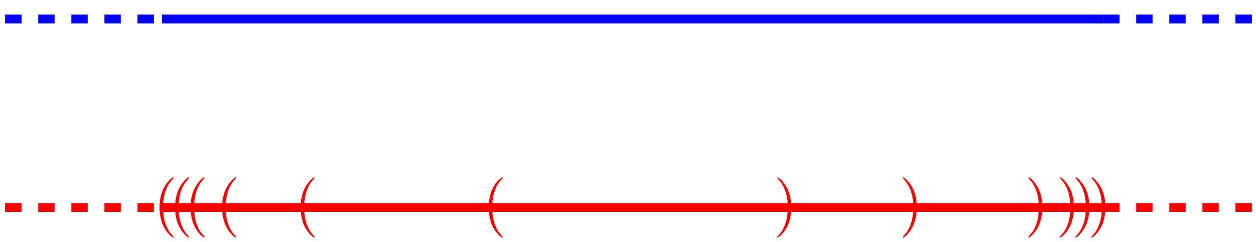
(R4) $\bigvee_{p,q \in \mathbb{Q}}(p, q) = 1$.



EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

Nice features: Continuous real functions,

semicontinuous real functions, ...

MORE, in next lectures.

Tutorial on Localic Topology

Jorge Picado

Department of Mathematics
University of Coimbra
PORTUGAL

cmuc
Centre for Mathematics
University of Coimbra

PART III. *Locales:*

the geometric facet of frames

MAKING THE PICTURE COVARIANT: the category of locales

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\mathcal{O}"] & \mathbf{Frm} \end{tikzcd}$$

Contravariant

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, squiggly] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, squiggly] & (\mathcal{O}Y, \subseteq) \arrow[u, "f^{-1}[-]"']
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preserves $\bigvee$ (incl. 0)

$\wedge$ (incl. 1)

$\mathbf{Loc}$ $\qquad$ $\mathbf{Frm}$

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frame homomorphisms taken backwards

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Loc **Frm**

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Each $h\colon M \to L$ in $\mathbf{Frm}$ has a UNIQUELY defined right adjoint

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that can be used as a representation of the h as a mapping going in the proper direction.

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LOCALIC MAP: a map $f\colon L \to M$ that has a left adjoint f^* in $\mathbf{Frm}$, i.e., preserving finite meets:

(1) $f^*(1) = 1$.

(2) $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$.

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PROPOSITION. Let $f : L \to M$ have a left adjoint f^*. Then:

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- OBJECTS: locales = frames (=cHa)

- MORPHISMS:

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$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "\mathsf{Lc}"] & \mathbf{Loc} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}X \\
Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"]
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MAKING THE PICTURE COVARIANT: the category of locales

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$$\begin{tikzcd}
U \arrow[d, squiggly] \\
Y \smallsetminus \overline{f[X \smallsetminus U]}
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\qquad
\begin{tikzcd}
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Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$

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Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, squiggly] \\
Y \smallsetminus \overline{f[X \smallsetminus U]}
\end{tikzcd}$$

Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathsf{c}}]^{\mathsf{c}}$)

MAKING THE PICTURE COVARIANT: the category of locales

$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$ is immediately modifiable to a functor

$$\mathbf{Top} \xrightarrow{\mathsf{Lc}} \mathbf{Loc}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}X \arrow[d, dotted, "(\mathcal{O}f)_*=\mathsf{Lc}(f)"] & U \arrow[d, squiggly] \\
Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"] & Y \smallsetminus \overline{f[X \smallsetminus U]}
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Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathrm{c}}]^{\mathrm{c}}$)

iff $V \subseteq \operatorname{int}(Y \smallsetminus f[X \smallsetminus U]) = Y \smallsetminus \overline{f[X \smallsetminus U]}$.

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map $\{*\} \longrightarrow X$

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map

Loc

$$\begin{tikzcd}
\{*\} \arrow[r, ""{name=U, below}] & X \\
\mathsf{Lc}(\{*\}) = \mathbb{2} \arrow[r, ""{name=D, above}] & \mathsf{Lc}(X)
\arrow[from=U, to=D, squiggly, "\mathsf{Lc}"]
\end{tikzcd}$$

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Top a point x of X is a continuous map $\{*\} \longrightarrow X$

Loc

$$\begin{tikzcd} \{*\} \longrightarrow X \arrow[d, squiggly, "\mathsf{Lc}"] \\ \mathsf{Lc}(\{*\}) = 2 \longrightarrow \mathsf{Lc}(X) \end{tikzcd}$$

Extension: a point of a *general* locale L is a localic map

$$p\colon 2 \;\to\; L$$

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Extension: a point of a *general* locale L is a localic map

$$\begin{array}{rcl} p\colon 2 & \rightarrow & L \\ 1 & \mapsto & 1 \end{array}$$

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Top a point x of X is a continuous map

Loc

$$\begin{tikzcd} \{*\} \arrow[rr] & {} \arrow[d, rightsquigarrow, "\mathsf{Lc}"] & X \\ \mathsf{Lc}(\{*\}) = 2 \arrow[rr] & {} & \mathsf{Lc}(X) \end{tikzcd}$$

Extension: a point of a *general* locale L is a localic map

$$\begin{array}{rcl} p\colon 2 & \to & L \\ 1 & \mapsto & 1 \\ 0 & \mapsto & a \neq 1. \end{array}$$

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Loc

$$\begin{tikzcd} \{*\} \arrow[r, ""{name=A}] & X \\ \mathsf{Lc}(\{*\}) = \mathbb{2} \arrow[r, ""{name=B}] & \mathsf{Lc}(X) \arrow[from=A, to=B, squiggly, "\mathsf{Lc}"] \end{tikzcd}$$

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$$\begin{array}{rcl} p\colon \mathbb{2} & \to & L \\ 1 & \mapsto & 1 \\ 0 & \mapsto & a \neq 1. \end{array}$$

$x \wedge y \leqslant a = p(0)$ iff $p^*(x) \wedge p^*(y) \leqslant 0$

THE SPECTRUM OF A LOCALE

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$$\begin{tikzcd} \{*\} \arrow[r] & X \arrow[d, rightsquigarrow, "\mathsf{Lc}"] \\ \mathsf{Lc}(\{*\}) = 2 \arrow[r] & \mathsf{Lc}(X) \end{tikzcd}$$

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$a \neq 1$

$\wedge$-IRREDUCIBLES

(PRIME ELEMENTS)

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$\wedge$-IRREDUCIBLES

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$\mathsf{Pt}(L)$

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$a \in L$, $\Sigma_a = \{p \in \mathsf{Pt}(L) \mid a \nleq p\}$.

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$$\begin{tikzcd}
\mathbf{Loc} \arrow[r, "\mathsf{Pt}"] & \mathbf{Top} \\
L \arrow[r, squiggly] & \mathsf{Pt}(L)
\end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
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Localic maps send points to points

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Localic maps send points to points

$$\mathsf{Pt}(f)^{-1}(\Sigma_b) = \{p \in \mathsf{Pt}(L) \mid b \not\leq f(p)\} = \{p \mid f^*(b) \not\leq p\} = \Sigma_{f^*(b)}.$$

SPACES AND LOCALES

$$\begin{tikzcd}\mathbf{Top} \arrow[r, shift left, "\mathsf{Lc}"] & \mathbf{Loc} \arrow[l, shift left, "\mathsf{Pt}"]\end{tikzcd}$$

SPACES AND LOCALES

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left=2, "\mathsf{Lc}"] & \mathbf{Loc} \arrow[l, shift left=2, "\mathsf{Pt}"] \end{tikzcd}$$

A frame is SPATIAL if it is isomorphic to some topology.

SPACES AND LOCALES

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$\mathsf{Lc}(X)$ is always spatial.

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A space X is SOBER if every meet-irreducible open is of the form

$$X \smallsetminus \overline{\{x\}}$$

for a unique $x \in X$.

$$\boxed{\mathbf{T_2} \subset \mathbf{Sob} \subset \mathbf{T_0}}$$

no relation with $\mathbf{T_1}$

SPACES AND LOCALES

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SPACES AND LOCALES

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left=2, "\mathsf{Lc}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, shift left=2, "\mathsf{Pt}"]
\end{tikzcd}$$

SPACES AND LOCALES

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UNIT:

$$\begin{aligned} \eta_X : X &\longrightarrow \mathsf{Pt}\,\mathsf{Lc}(X) \\ x &\longmapsto X \setminus \overline{\{x\}} \end{aligned}$$

SPACES AND LOCALES

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SPACES AND LOCALES

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SPACES AND LOCALES

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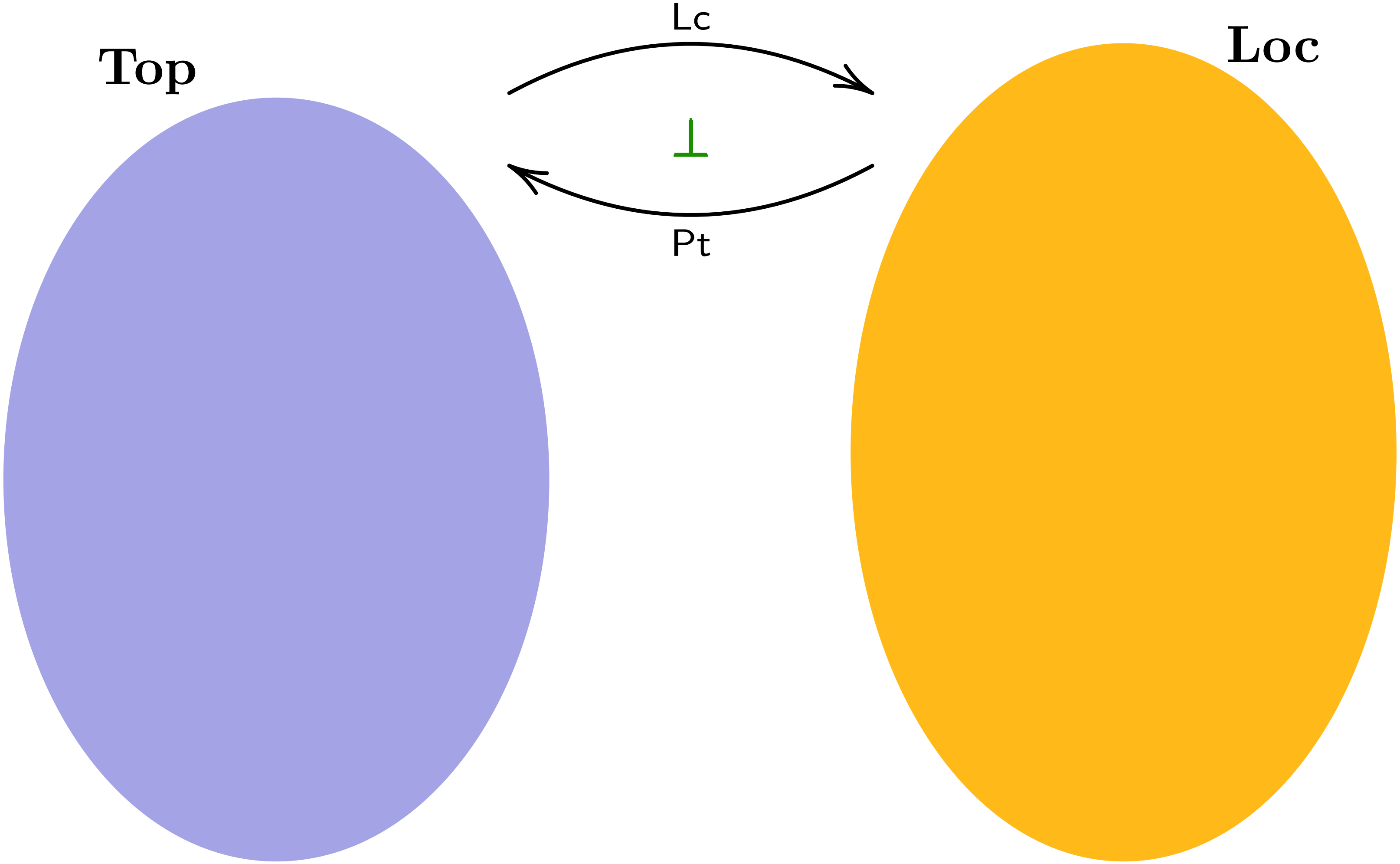
PROPOSITION: ε_L is an isomorphism iff L is spatial.

SPACES and LOCALES



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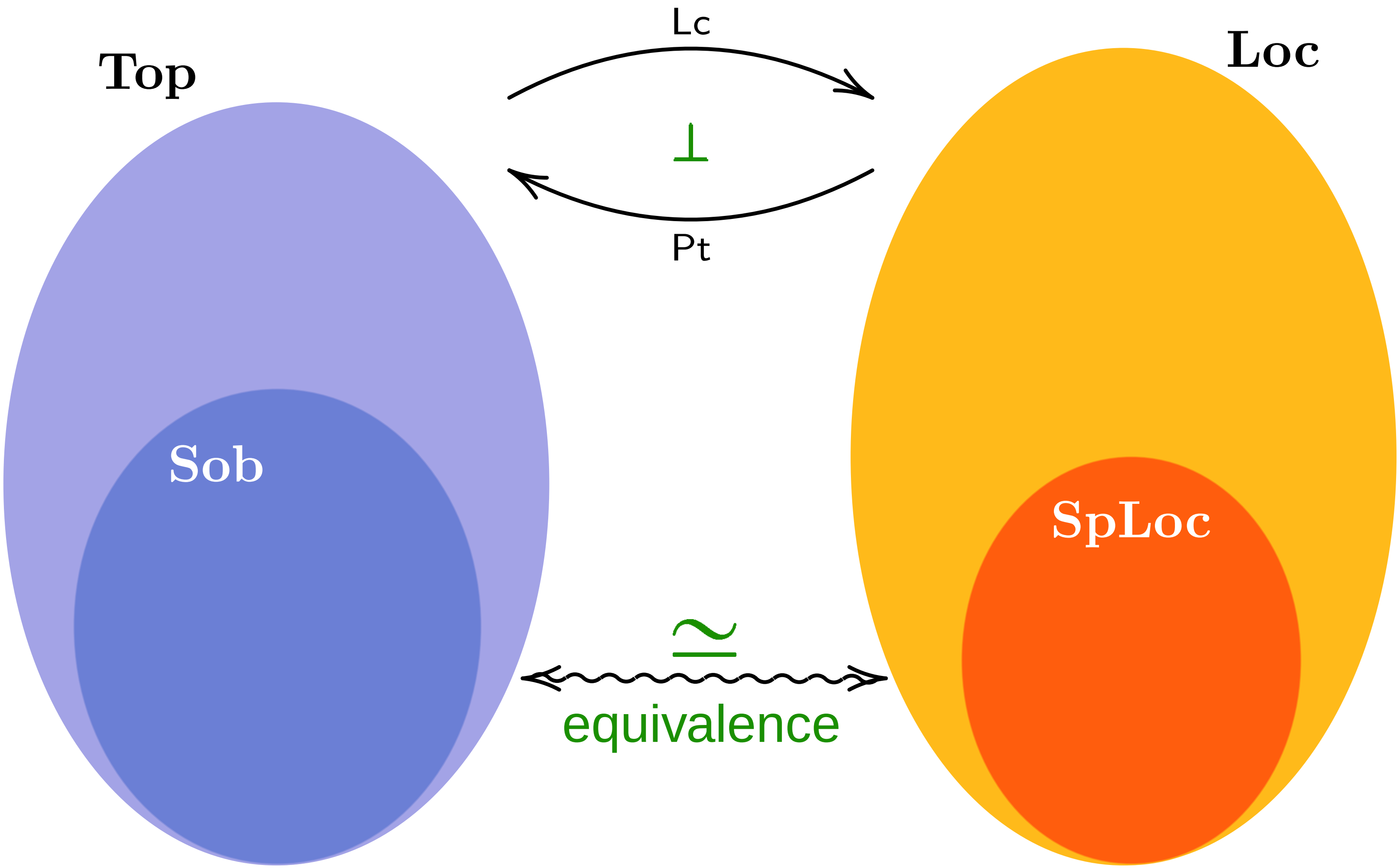
SPACES and LOCALES



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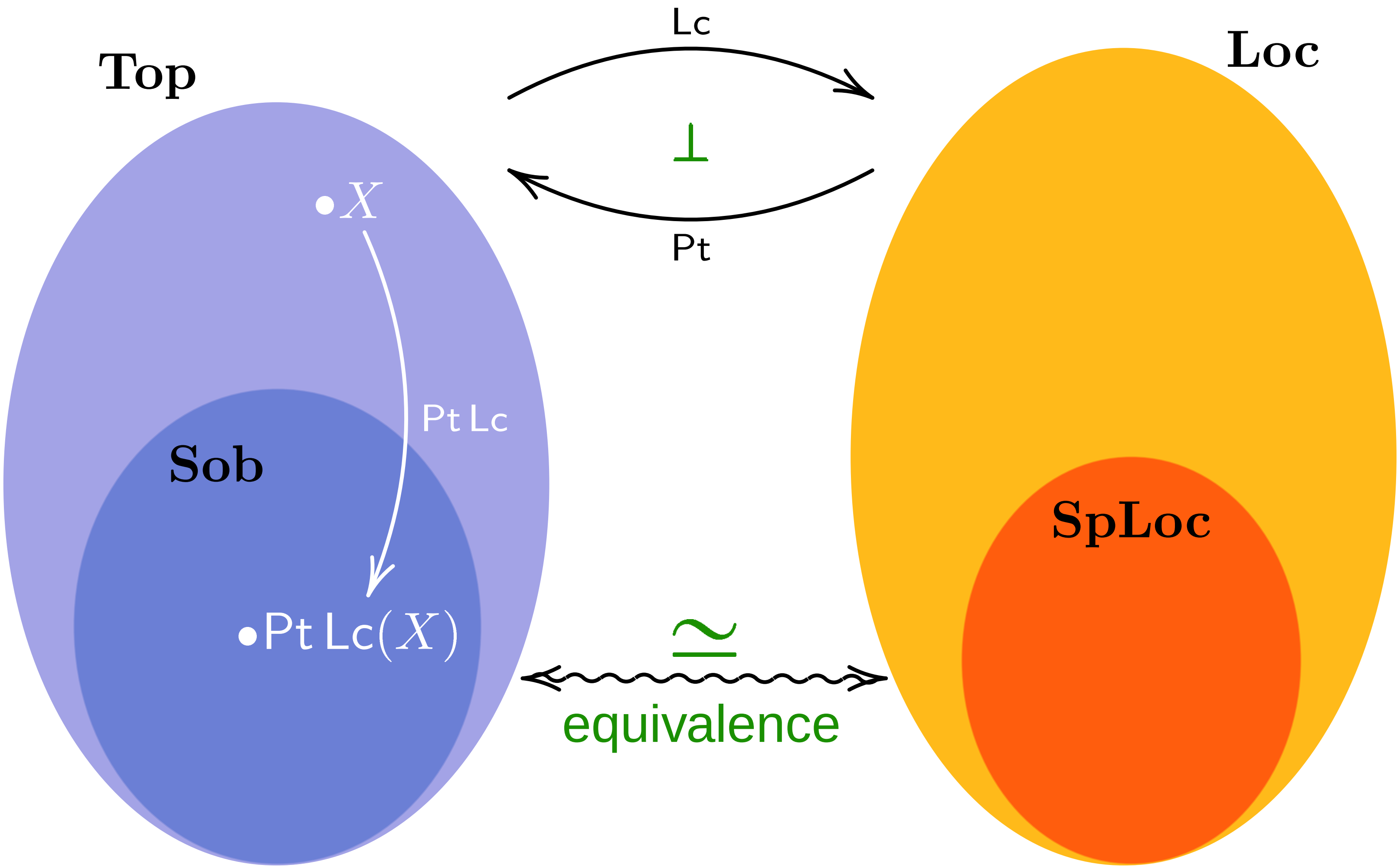
$$\begin{tikzcd}\mathbf{Sob} \arrow[r, leftrightsquigarrow, "\simeq"] & \mathbf{SpLoc}\end{tikzcd}$$

equivalence

Perception: **Sob** more representative of all of **Top** than **SpLoc** of **Loc**.

SPACES and LOCALES



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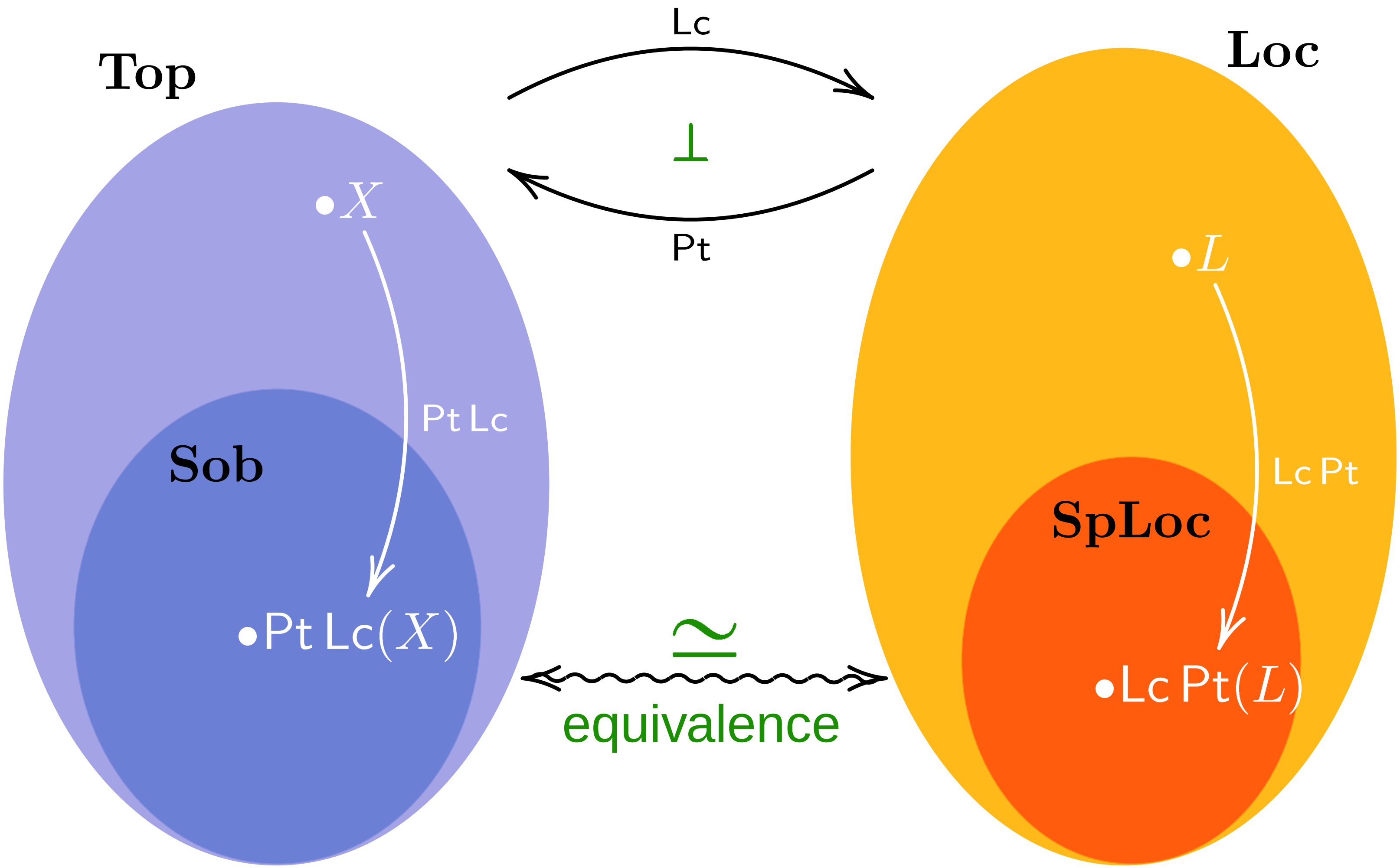
\mathbf{Sob} \arrow[r, leftrightsquigarrow, "\simeq"', "\text{equivalence}"] & \mathbf{SpLoc}
\end{tikzcd}$$

“soberification” of a space

SPACES and LOCALES



“soberification” of a space

“spatialization” of a locale

THE BOOLEAN CASE: non-spatial locales

PROPOSITION: L is spatial iff each $a \neq 1$ is a meet of points of L.

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By the Proposition,

B spatial $\Rightarrow$ each $a \neq 1$ in B is a meet of co-atoms

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By the Proposition,

B spatial $\Rightarrow$ each $a \neq 1$ in B is a meet of co-atoms

$\Leftrightarrow$ each $a \neq 1$ in B is a join of atoms (by complement.). ■

PART IV.

Doing topology in Loc

GENERALIZED SUBSPACES: SUBLOCALES

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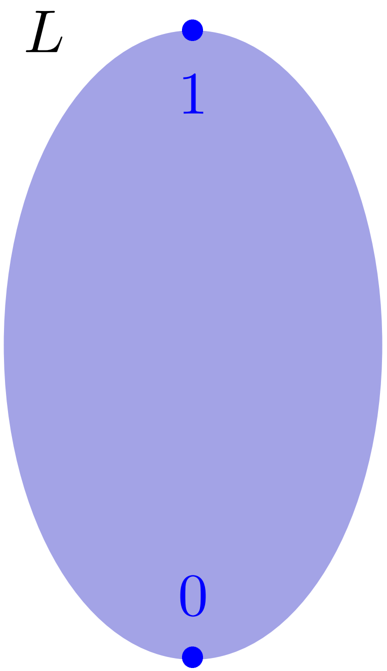

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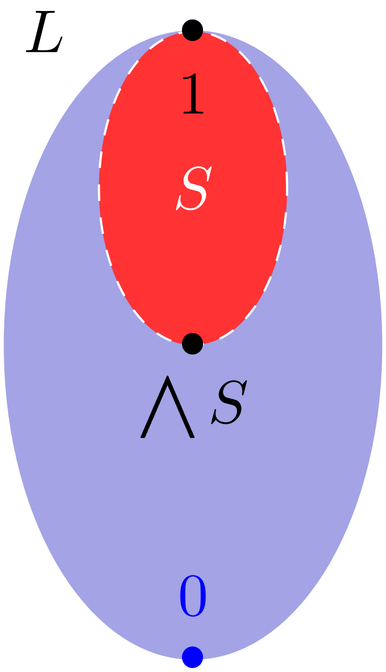
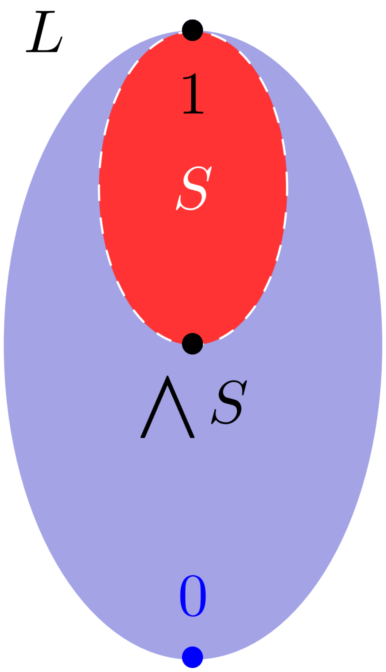


GENERALIZED SUBSPACES: SUBLOCALES

DEFINITION: $S \subseteq L$ is a SUBLOCALE of L if

(1) $\forall A \subseteq S, \bigwedge A \in S$,

(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.



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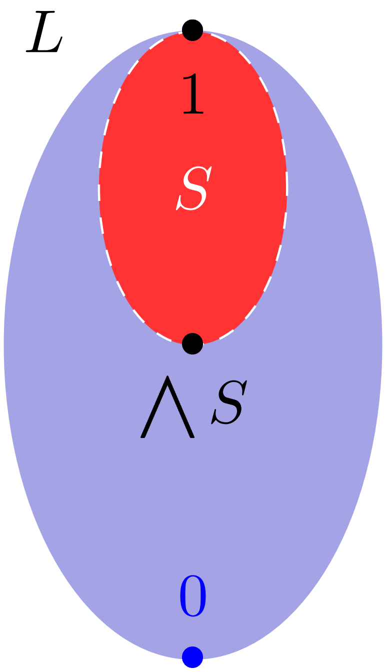
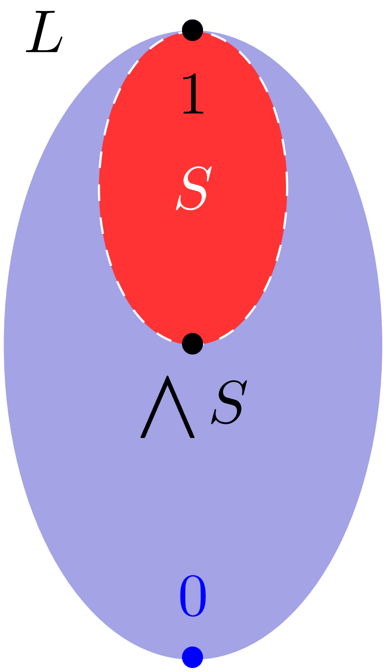


S is itself a locale: $\bigwedge_S = \bigwedge_L$, $\to_S = \to_L$

but $\bigsqcup s_i = \bigwedge\{s \in S \mid \bigvee s_i \leqslant s\}$.

GENERALIZED SUBSPACES: SUBLOCALES



DEFINITION: $S \subseteq L$ is a SUBLOCALE of L if

(1) $\forall A \subseteq S, \bigwedge A \in S$,

(2) $\forall a \in L, \forall s \in S,\ a \rightarrow s \in S$.

S is itself a locale: $\bigwedge_S = \bigwedge_L$, $\rightarrow_S = \rightarrow_L$

but $\bigsqcup s_i = \bigwedge\{s \in S \mid \bigvee s_i \leqslant s\}$.

Motivation for the definition:

PROP:

$S \subseteq L$ is a sublocale iff the embedding $j_S \colon S \subseteq L$ is a localic map.

THE SUBLOCALE LATTICE

$\mathcal{S}\ell(L)$: sublocales of L, ordered by $\subseteq$

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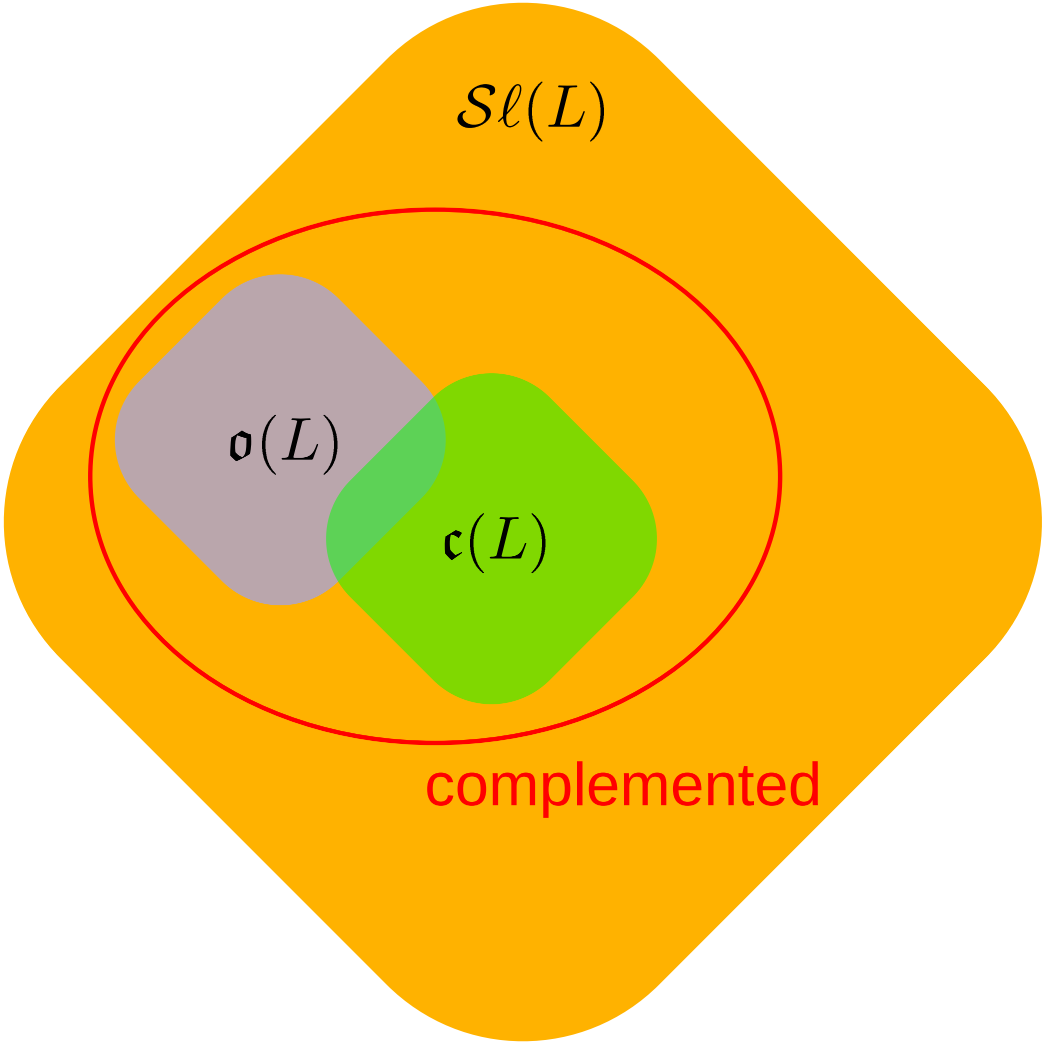

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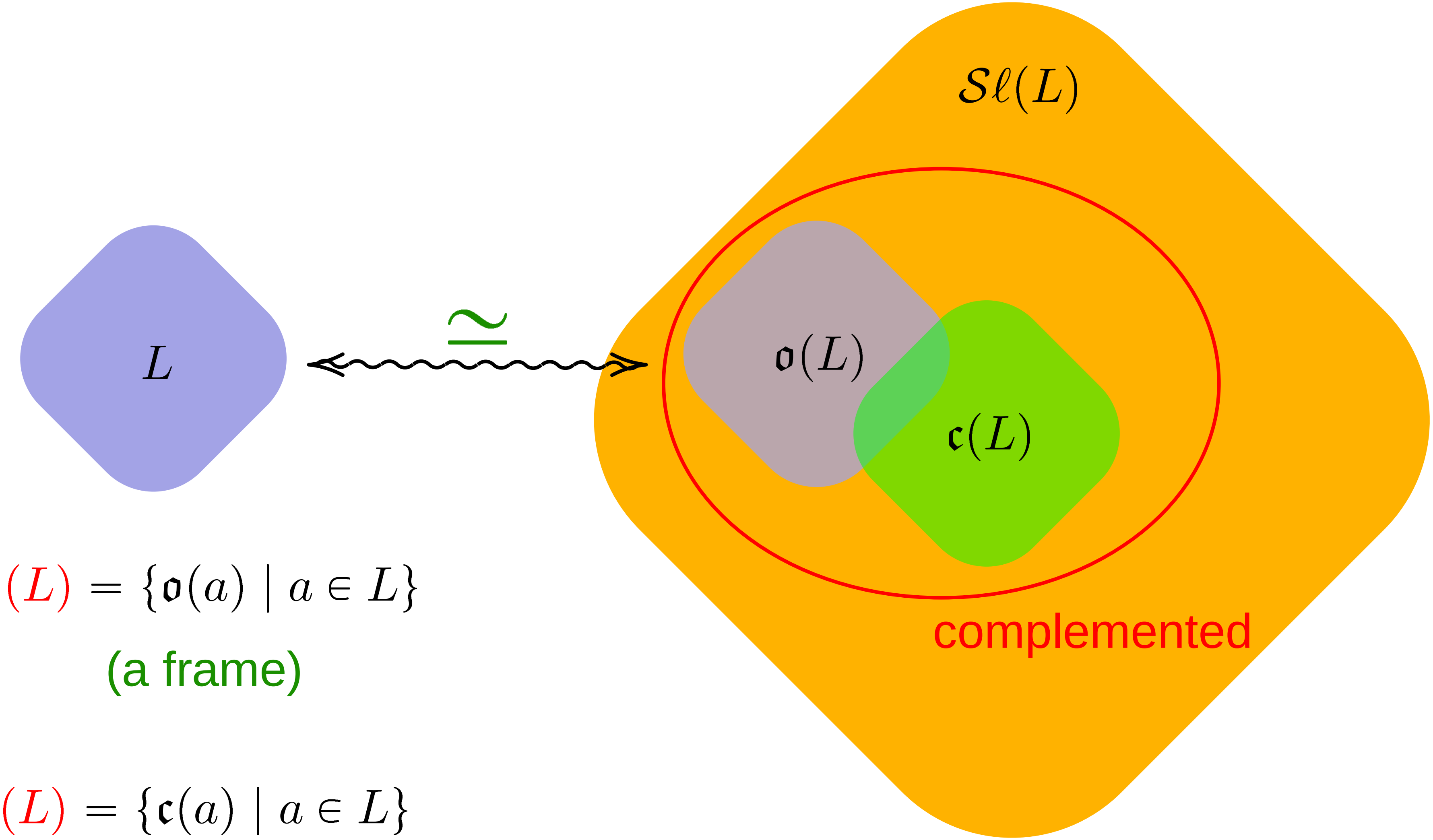

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By complementation, $\operatorname{int} \mathfrak{c}(b) = \mathfrak{o}(b^*)$.

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i.e., there exists the smallest dense sublocale of a locale!

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$$a \to x^* = a \to (x \to 0) = a \wedge x \to 0 = (a \wedge x)^*. \quad \blacksquare$$

Tutorial on Localic Topology

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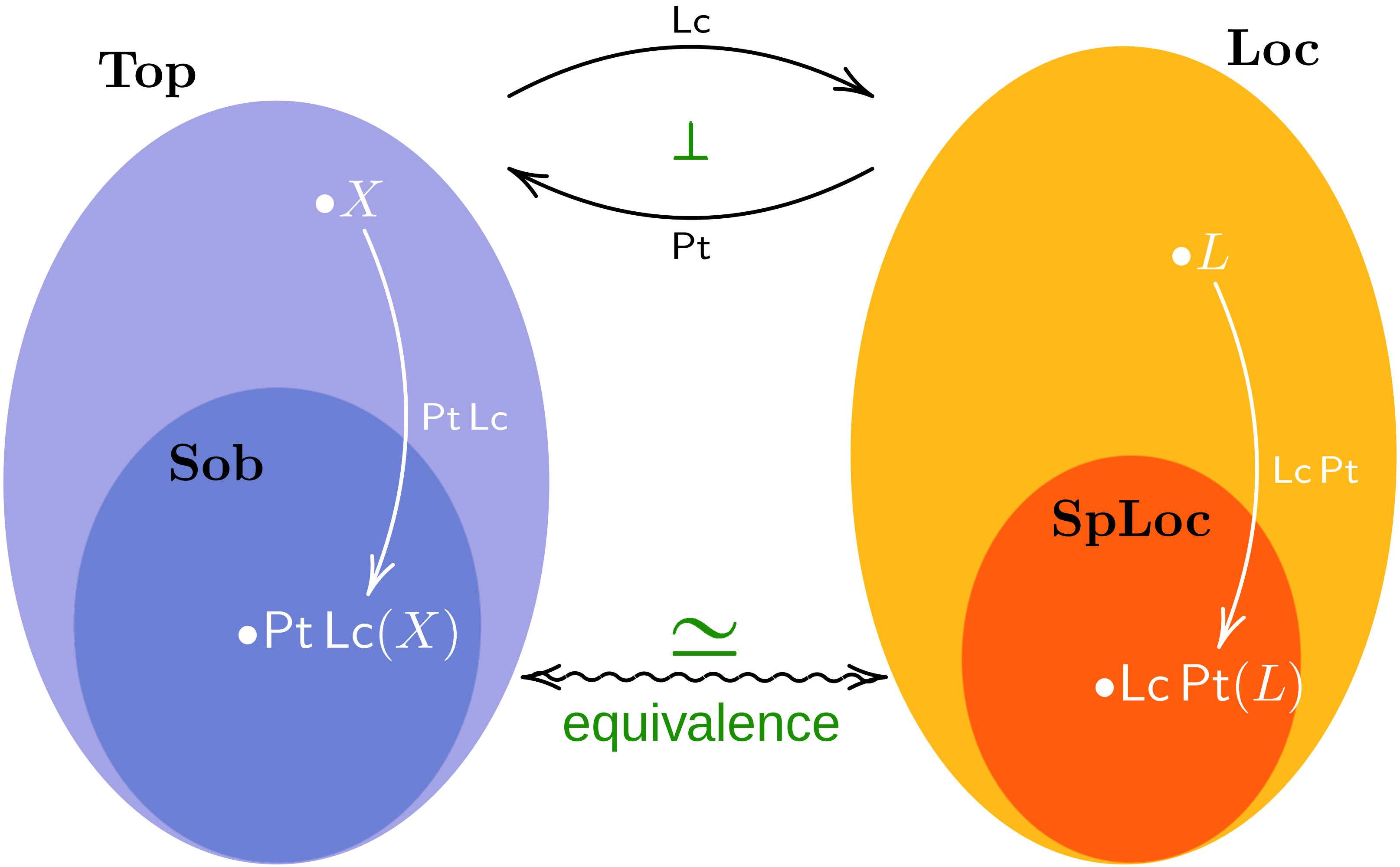
University of Coimbra

PORTUGAL

Centre for Mathematics
University of Coimbra

SPACES versus LOCALES



Top

Lc

$\perp$

Pt

Loc

$\bullet X$

$\mathsf{Pt}\,\mathsf{Lc}$

Sob

$\bullet\mathsf{Pt}\,\mathsf{Lc}(X)$

$\simeq$

equivalence

$\bullet L$

$\mathsf{Lc}\,\mathsf{Pt}$

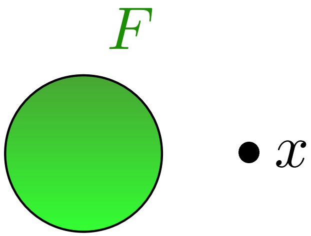
SpLoc

$\bullet\mathsf{Lc}\,\mathsf{Pt}(L)$

“soberification” of a space

“spatialization” of a locale

Regularity



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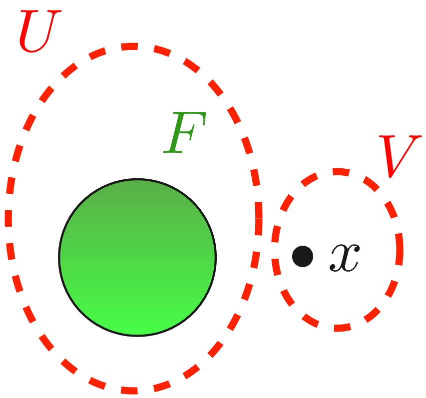
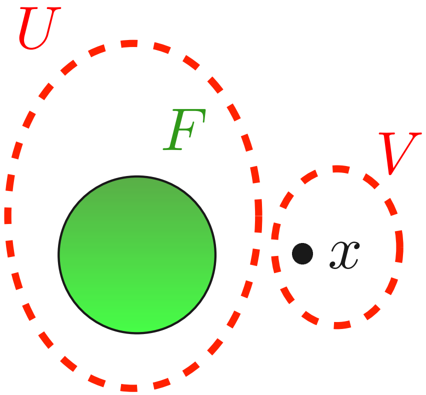


DOING TOPOLOGY IN Loc

Regularity

$$\forall U \in \mathcal{O}(X),\ \forall x \in U,\ \exists V \in \mathcal{O}(X):\ x \in V \subseteq \overline{V} \subseteq U.$$



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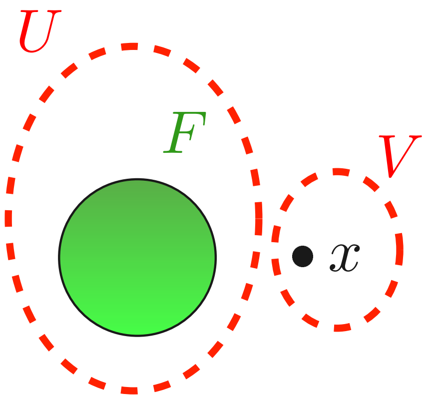

DOING TOPOLOGY IN Loc

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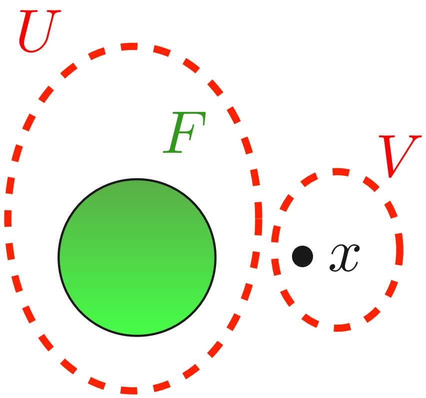

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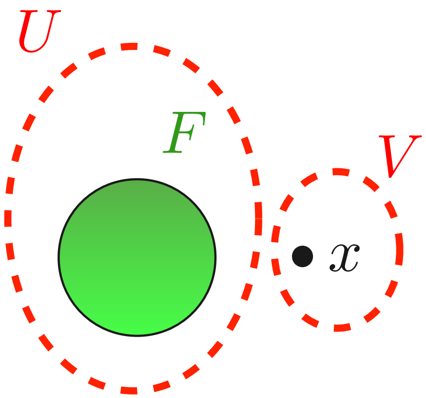

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$$V \prec U \Leftrightarrow X \smallsetminus \overline{V} \supseteq X \smallsetminus U \Leftrightarrow X \smallsetminus \overline{V} \cup U = X \Leftrightarrow V^* \cup U = X.$$

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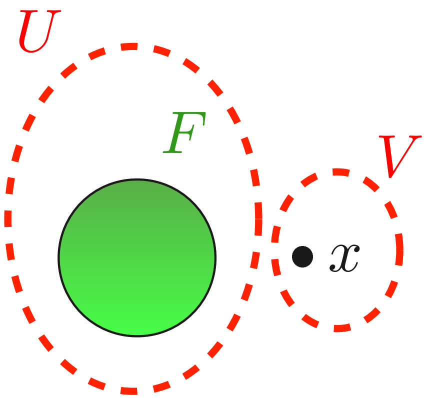

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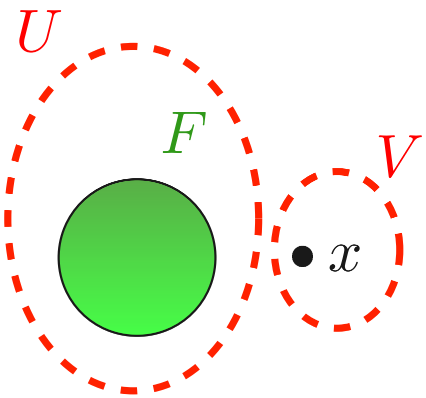
Regularity



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RECAP: SPECIAL SUBLOCALES

$a \in L$, $\mathfrak{c}(a) = {\uparrow}a$ CLOSED

$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN

$\Big\}$ complemented

Properties

(1) $a \leqslant b$ iff $\mathfrak{c}(a) \supseteq \mathfrak{c}(b)$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$.

(2) $\bigwedge \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee a_i)$.

(3) $\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$.

(4) $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.

(5) $\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.

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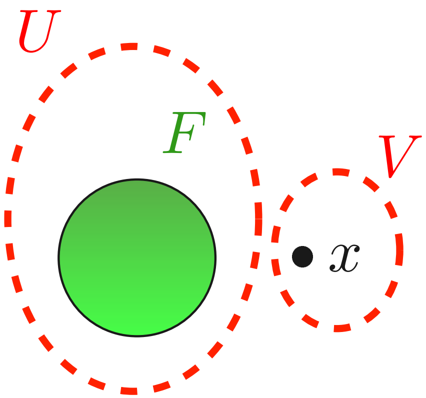

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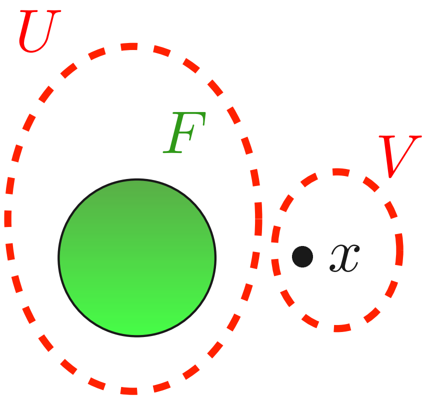

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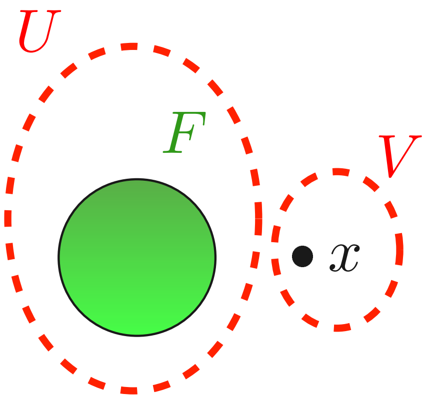

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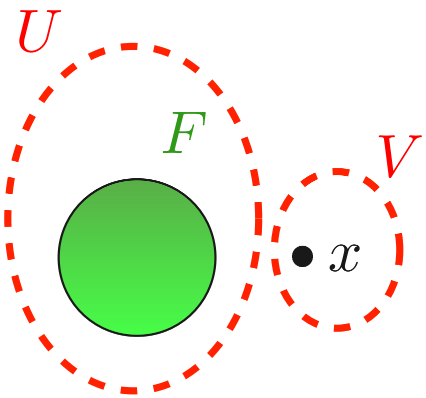

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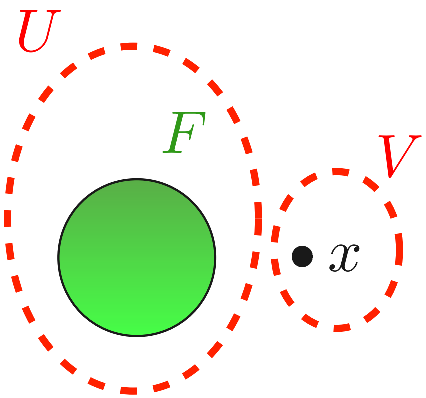

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(Conservative extension: X is regular iff the locale $\mathcal{O}(X)$ is regular.)

partial order $\prec$

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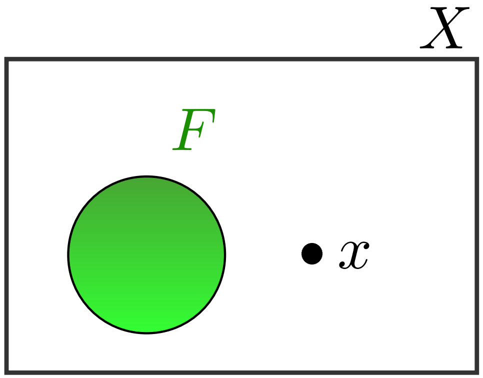
③ $$a_i \prec b_i \ (i = 1, 2) \ \Rightarrow \begin{cases} a_1 \vee a_2 \prec b_1 \vee b_2 \\ \\ a_1 \wedge a_2 \prec b_1 \wedge b_2 \end{cases}$$

Complete Regularity



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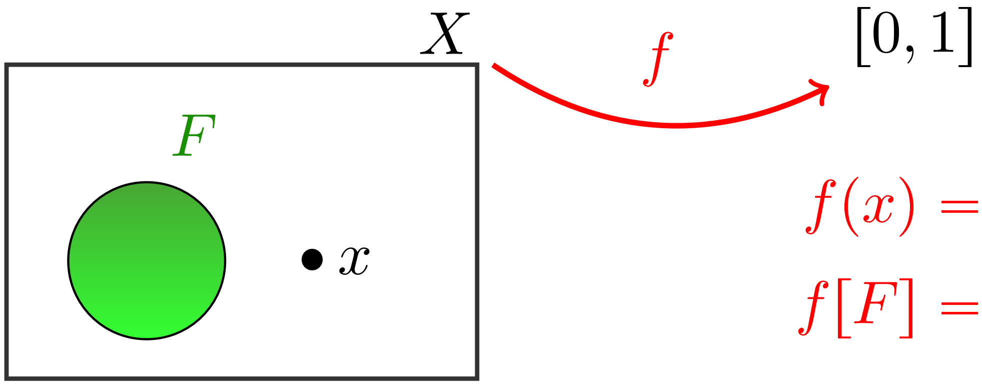

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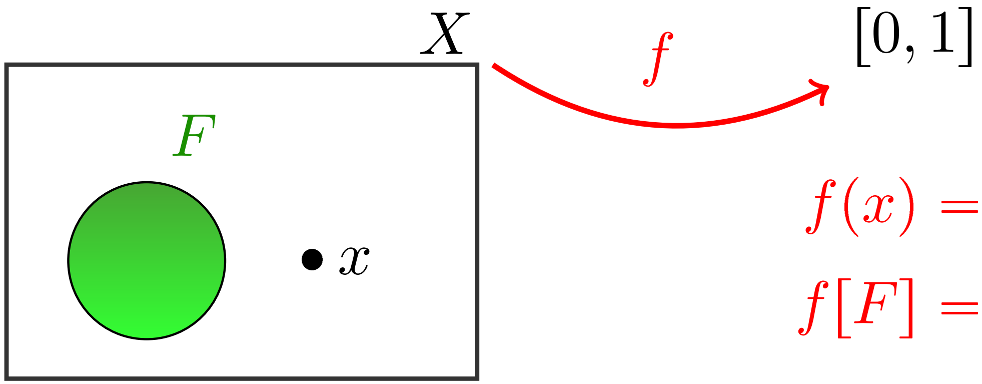

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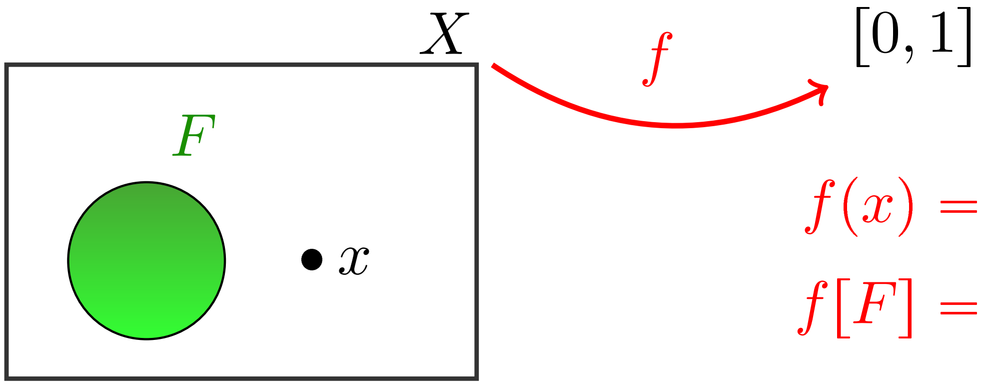

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$$V \lll U \equiv \exists\, (W_q)_{q \in \mathbb{Q} \cap [0,1]} :\ W_0 = V,\ W_1 = U,\ p < q \Rightarrow W_p \prec W_q.$$

[B. Banaschewski (1953)]

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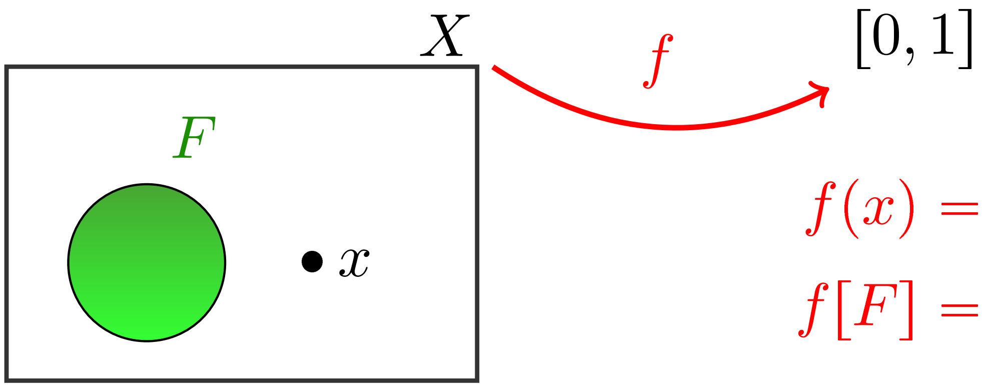

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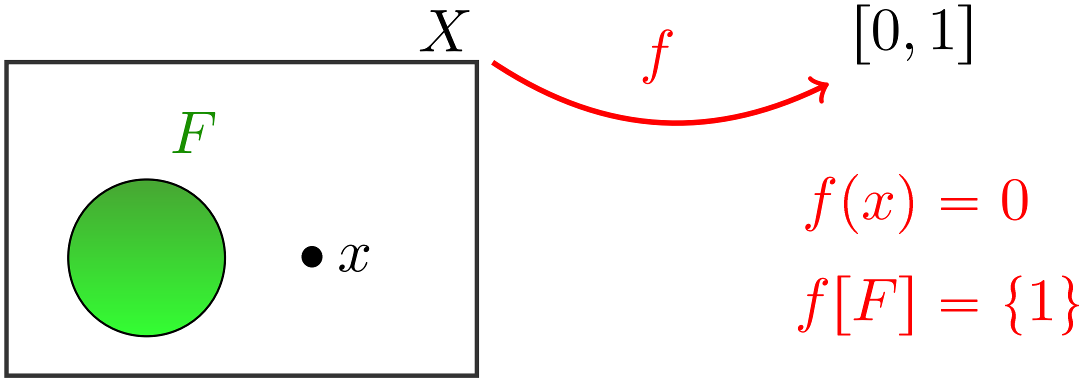

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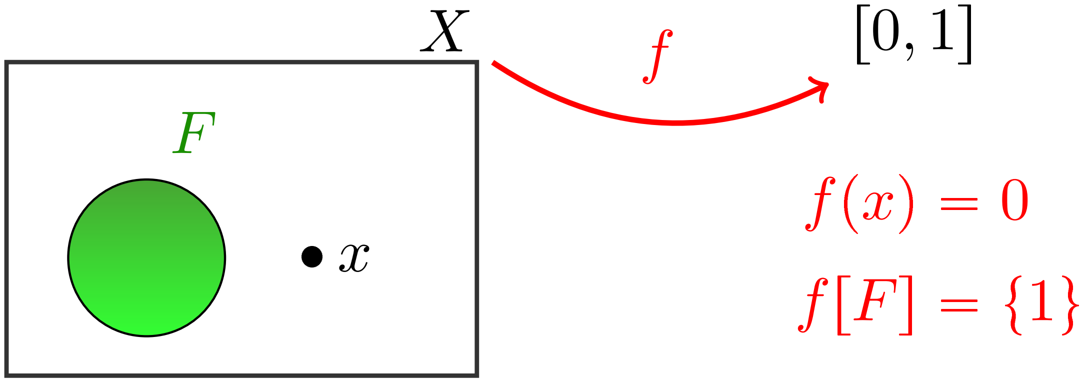

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Further

$$x_i \prec b \; (i = 1, \ldots, n) \;\Rightarrow\; c \prec b. \qquad \blacksquare$$

THE (constructive) STONE-Čech compactification

Ideals of L: $\mathfrak{I}(L)$ (I1) $b \leqslant a \in J \Rightarrow b \in J$, (I2) $a, b \in J \Rightarrow a \vee b \in J$

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$$x_j \in J_{i_j} \cap K \Rightarrow x \in \bigvee (J_i \cap K)$$

- $\bigvee J_i = L \ni 1 \Rightarrow 1 = x_1 \vee \cdots \vee x_n$ (some $x_j \in J_{i_j}$).

Then $1 \in \bigvee_{j=1}^{n} J_{i_j} \;\Rightarrow\; L = \bigvee_{j=1}^{n} J_{i_j}$. ■

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Regular ideal: (Ir) $\forall a \in J \; \exists b \in J : a \prec\!\!\prec b$. $\mathfrak{R}(L)$

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Now, suffices: $b \ll a$ in L $\Rightarrow$ $\sigma(b) \prec \sigma(a)$ in $\mathfrak{R}(L)$ which is easy!

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$$\boxed{h(a) = 0 \Rightarrow a = 0}$$

THE (constructive) STONE-Čech compactification

LEMMA 4. For each completely regular L,

$$\begin{array}{rcl}\beta_L : \mathfrak{R}(L) & \longrightarrow & L \\ J & \longmapsto & \bigvee J\end{array}$$

is a dense surjection.

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THEOREM. There is a functor $\mathfrak{R}\colon \mathbf{CRegFrm} \to \mathbf{CRegFrm}$

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(3) β_L is an isomorphism iff L is compact.

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SIDE NOTES

The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

Generators $(p, —), (—, q), \quad p, q \in \mathbb{Q}$

Relations

(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$

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Similarly, we have the extended continuous real functions:

$$\boxed{\overline{\mathrm{C}}(L)=\mathrm{Hom}_{\mathbf{Frm}}(\mathfrak{L}(\overline{\mathbb{R}}),L)}$$

B. Banaschewski, J. Gutiérrez García & J. P.
Extended real functions in pointfree topology, *J. Pure Appl. Algebra* **216 (2012)**

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I. MOZO CAROLLO, J. GUTIÉRREZ GARCÍA & J. P.
On the Dedekind completion of function rings, ***Forum Mathematicum*** **to appear**

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Natural extension:

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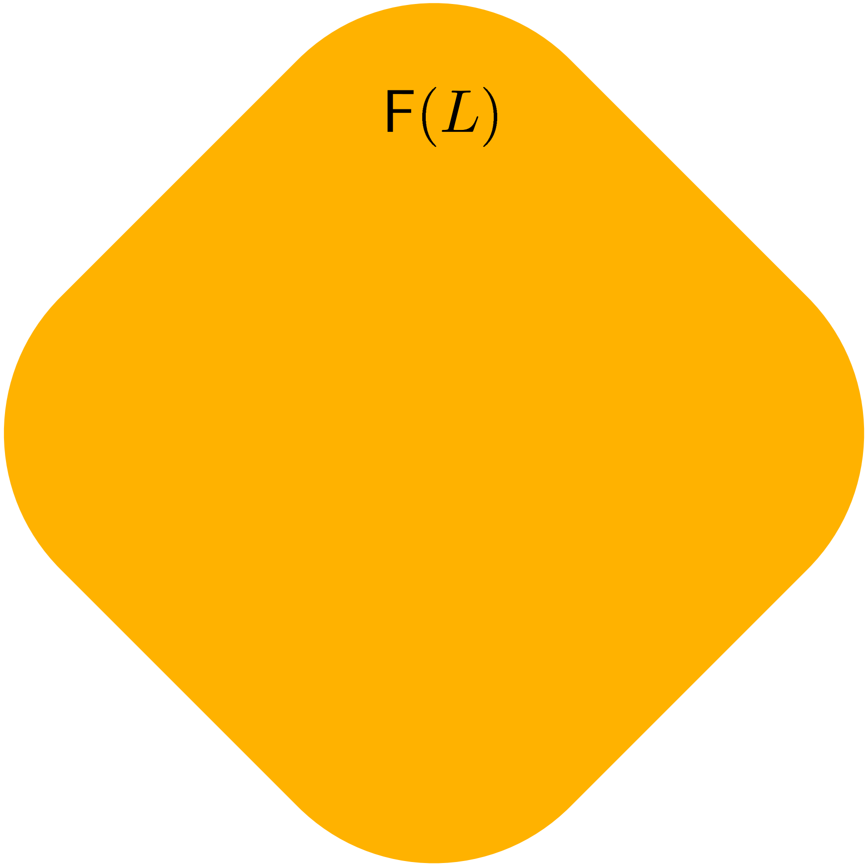

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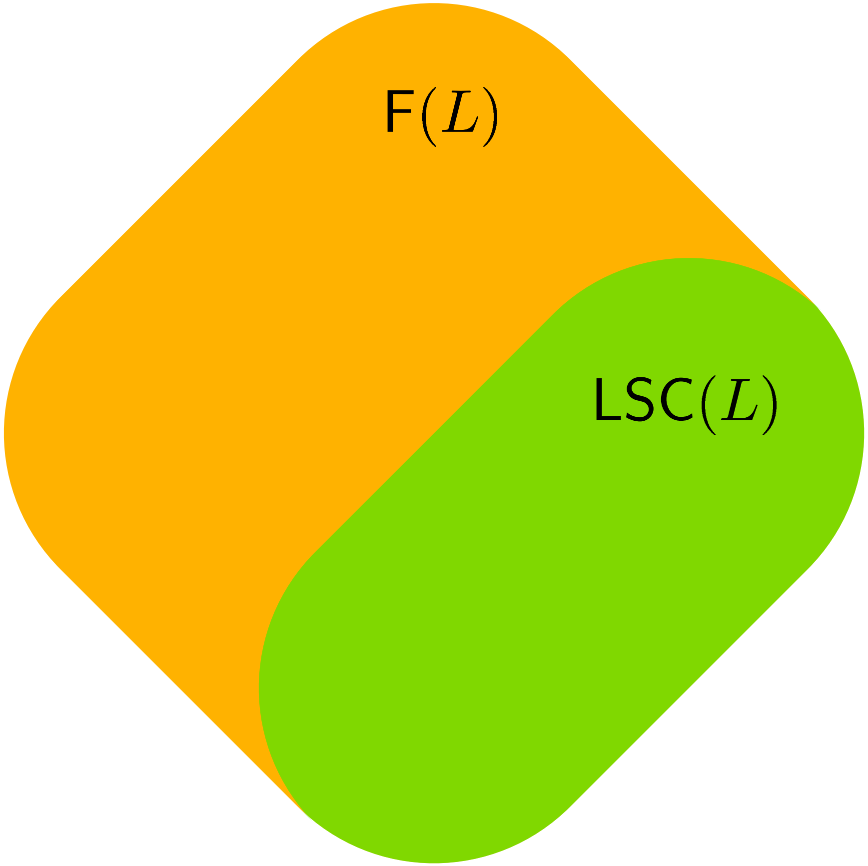

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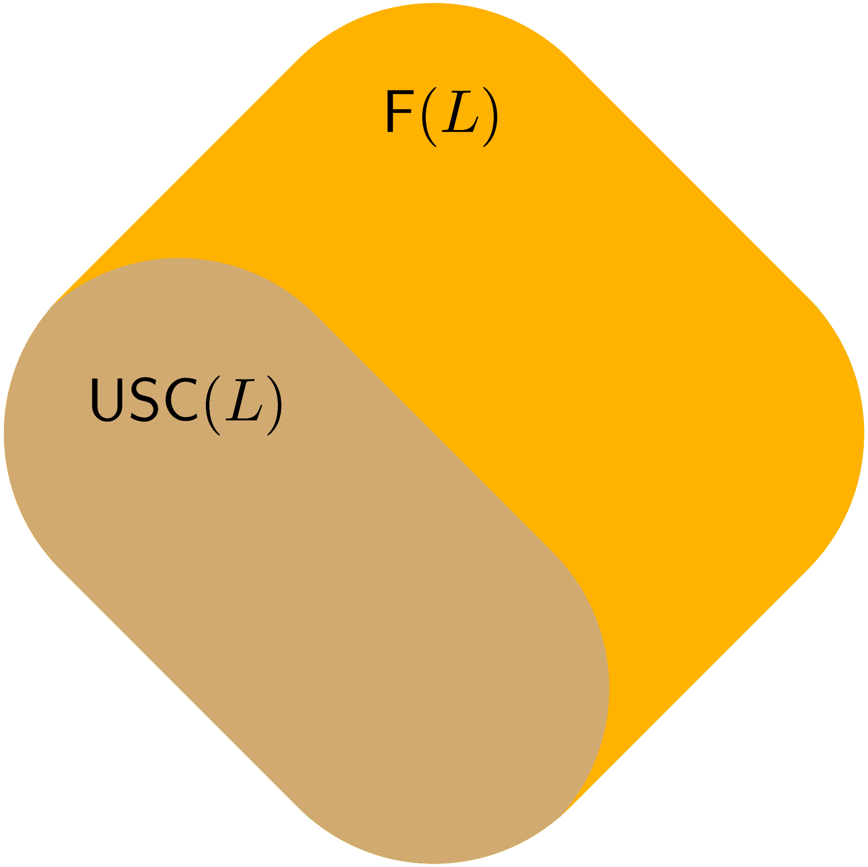

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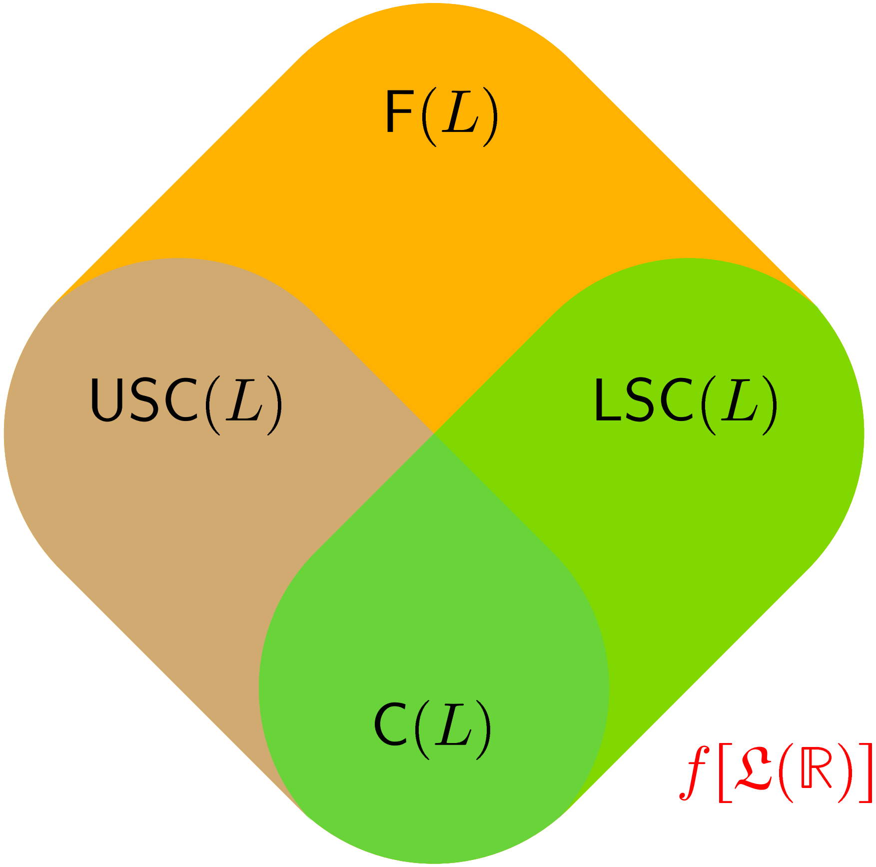

$f(-, q) \in \mathfrak{c}(L)$

$f(p, -) \in \mathfrak{c}(L)$

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J. GUTIÉRREZ GARCÍA, T. KUBIAK & J. P.

Localic real functions: a general setting, *J. Pure Appl. Algebra* **213 (2009) 1064-1074**

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- Dually: the upper regularization $f^- = -(-f)^\circ$

J. Gutiérrez García, T. Kubiak & J. P.
Lower and upper regularizations of frame semicontinuous real functions, *Alg. Univ.* (2009)

APPLICATIONS: insertion theorems

locale L

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(i) L is normal

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- Extension: $\mathfrak{L}(\mathbb{R}) \to \mathfrak{c}(a)$ [Tietze's Extension Theorem]

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TFAE:

(i) L is ~~normal~~ *extremally disconnected*

(ii) $\underbrace{f}_{\text{USC}} \geqslant \underbrace{g}_{\text{LSC}} \quad \Rightarrow \quad \exists\, h \in \mathsf{C}(L) : f \geqslant h \geqslant g$

- Classically: $L = \mathcal{O}X$ [Lane; Kubiak-de Prada Vicente insertion]
- Separation: $f = \chi_F,\ g = \chi_A$ [Gillman-Jerison]
- Extension: $\mathfrak{L}(\mathbb{R}) \to \mathfrak{o}(a)$ [Gillman-Jerison]

APPLICATIONS: insertion theorems

locale L

TFAE:

(i) L is ~~normal~~ *completely normal*

(ii) $\underbrace{f,g}_{\mathsf{F}(L)},\ f^{-}\leqslant g,\ f\leqslant g^{\circ} \quad\Rightarrow\quad \exists\, h\in \mathsf{LSC}(L): f\leqslant h\leqslant h^{-}\leqslant g$

- Classically: $L = \mathcal{O}X$

[General insertion: Kubiak]

APPLICATIONS: insertion theorems

More: monotone insertion [Kubiak],

strict insertion [Dowker],

bounded insertion [Michael], ...