Tutorial on Localic Topology

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OUTLINE

- AIM: cover the basics of point-free topology
- Slides give motivation, definitions and results, few proofs

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- Part I. Frames: the algebraic facet of spaces
- Part II. Categorical aspects of $\mathbf{Frm}$
- Part III. Locales: the geometric facet of frames
- Part IV. Doing topology in $\mathbf{Loc}$

WHAT IS POINT-FREE TOPOLOGY?

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- The techniques may hide some geometrical intuition, but often offers powerful algebraic tools and opens new perspectives.

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is developed in the categories

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‘topology itself’

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$$\mathbf{Frm} \left[\begin{array}{l} \text{frames} \\ \\ \text{frame homomorphisms} \end{array} \right.$$

$$\mathbf{Loc} = \mathbf{Frm}^{op} \left[\begin{array}{l} \text{locales} \\ \\ \text{localic maps} \end{array} \right.$$

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> «*The topological structure of a locale cannot live in its points: the points, if any, live on the open sets rather than the other way about.*»
>
> P. T. JOHNSTONE
>
> [The art of pointless thinking, *Category Theory at Work* (1991)]

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> «*(...) what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.*» R. BALL & J. WALTERS-WAYLAND
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> [C- and C*-quotients in pointfree topology, *Dissert. Math.* (2002)]

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MORE: different categorical properties with advantage to the point-free side.

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The idea of approaching topology via algebra (lattice theory) goes back to the ’30s-40’s:

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1st talk (H. Dowker, Prague Top. Symp. 1966)

groundbreaking paper (J. Isbell, Atomless parts of spaces, 1972)

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- RAMIFICATIONS: category theory, topos theory, logic and computer science.

MAIN BASIC REFERENCES

P. T. Johnstone, *Stone Spaces*, CUP 1982.

S. Vickers, *Topology via Logic*, CUP 1989.

S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic - A first introduction to topos theory*, Springer 1992.

B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, vol. 12, Univ. Coimbra 1997.

R. N. Ball and J. Walters-Wayland, *C- and C*-quotients in pointfree topology*, Dissert. Math, vol. 412, 2002.

JP, A. Pultr and A. Tozzi, *Locales*, Chapter II in “Categorical Foundations”, CUP 2004.

JP and A. Pultr, Locales treated mostly in a covariant way, Textos de Matemática, vol. 41, Univ. Coimbra 2008.

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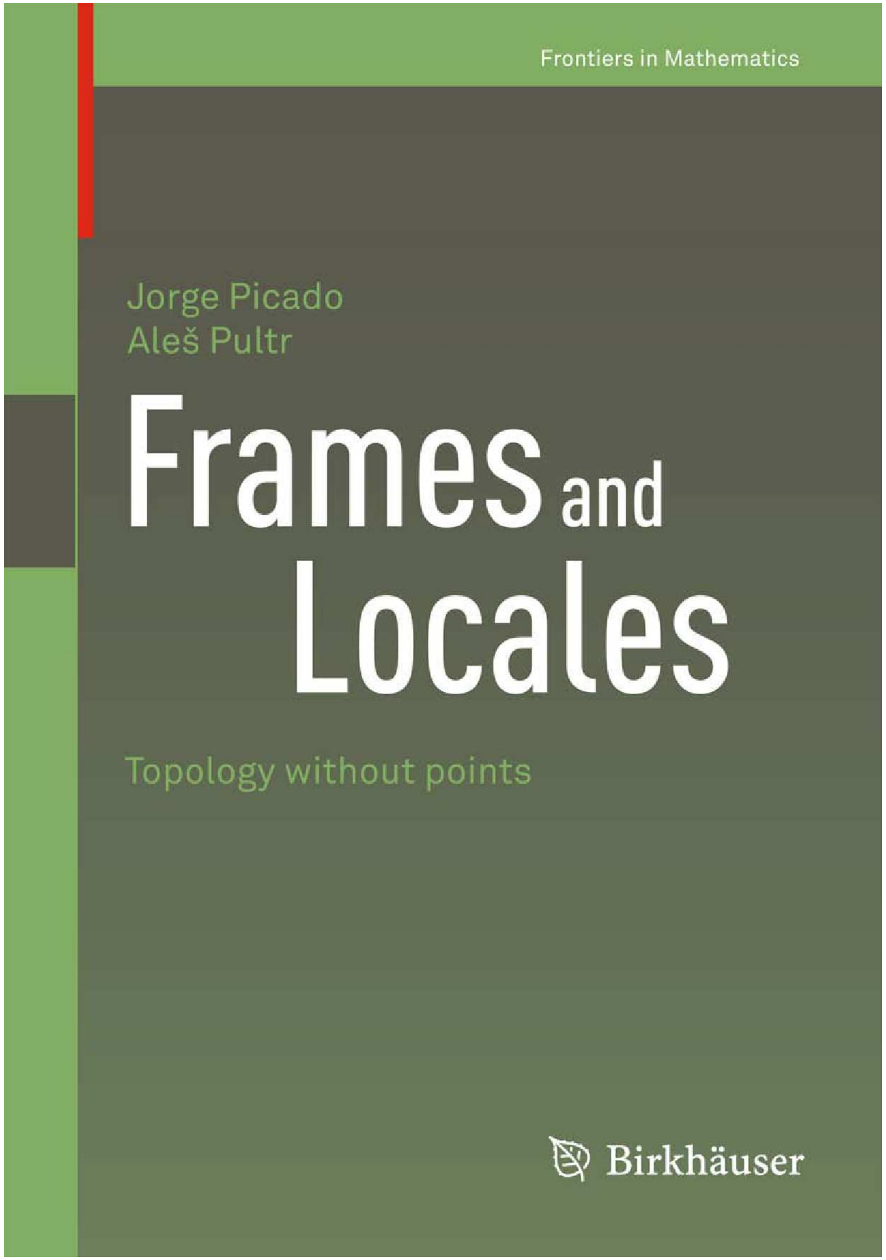

PART I. Frames:

the algebraic facet of spaces

FROM SPACES TO FRAMES

Top

$(X, \mathcal{O}X)$

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$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

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- complete lattice:

$$\bigvee U_i = \bigcup U_i, \quad 0 = \varnothing$$

$$U \wedge V = U \cap V, \quad 1 = X$$

$$\bigwedge U_i = \mathrm{int}(\bigcap U_i)$$

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more:

$$\boxed{U \wedge \bigvee_I V_i = \bigvee_I (U \wedge V_i)}$$

FROM SPACES TO FRAMES

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$$
\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y, \subseteq)
\end{tikzcd}
$$

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\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & \color{red}(\mathcal{O}X, \subseteq) \\
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- $f^{-1}[-]$ preserves $\bigvee$ and $\wedge$

FROM SPACES TO FRAMES

Top

• complete lattice L

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, squiggly] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
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\end{tikzcd}$$

frame:

$$\boxed{a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)}$$

• frame homomorphisms: $h \colon M \to L$ preserves $\bigvee$ and $\wedge$

$$\begin{tikzcd}[column sep=huge] \mathbf{Top} \arrow[r, "\mathcal{O}"] & \mathbf{Frm} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge] (X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\ (Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y, \subseteq) \arrow[u, "\boldsymbol{f^{-1}[-]}"'] \end{tikzcd}$$

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$$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$$

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The algebraic nature of the objects of $\mathbf{Frm}$ is obvious.

More about that later on...

MORE EXAMPLES of frames

- Finite distributive lattices, complete Boolean algebras, complete chains.

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$0 = 1$

$\mathbb{1}$: 

$\mathbb{2}$: 

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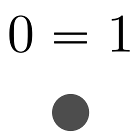

$\mathbb{2}$:


- subframe of a frame L: $S \subseteq L$ closed under arbitrary joins (in part. $0 \in S$) and finite meets (in part. $1 \in S$).

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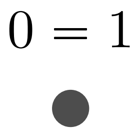
$\mathbb{1}$: 

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- intervals of a frame L: $a, b \in L$, $a \leqslant b$

  $[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$, $\downarrow b = [0, b]$, $\uparrow a = [a, 1]$.

MORE EXAMPLES of frames

- For any $\wedge$-semilattice $(A, \wedge, 1)$, $\mathfrak{D}(A) = \{$down-sets of $A\}$ is a frame:

$$\bigwedge = \bigcap, \quad \bigvee = \bigcup.$$

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$$\begin{tikzcd} \mathbf{SLat} \arrow[r, shift left, "\mathfrak{D}"] \arrow[r, phantom, "\perp"] & \mathbf{Frm} \arrow[l, shift left, "G"] \end{tikzcd}$$

(forgetful functor)

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$$\boxed{\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Frm}}(\mathfrak{D}(A), L) & \simeq & \mathrm{Hom}_{\mathbf{SLat}}(A, G(L)) \\ h & \longmapsto & (\widetilde{h} \colon a \mapsto h(\downarrow a)) \\ (\overline{g} \colon S \mapsto \bigvee g[S]) & \longleftarrow & g \end{array}}$$

MORE EXAMPLES of frames

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(inclusion as a non-full subcategory)

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- All homomorphisms of finite distributive lattices.

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initial object terminal object

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initial object terminal object

- $$\bigvee \colon \mathfrak{I}(L) \to L, \quad J \mapsto \bigvee J \qquad\qquad \bigvee \colon \mathfrak{D}(L) \to L, \quad S \mapsto \bigvee S$$

BACKGROUND: POSETS AS CATEGORIES

poset $(A, \leqslant)$

$(A, \leqslant)$ as a thin category $\left[\vphantom{\int}\right.$

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OBJECTS: $a \in A$

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poset $(A, \leqslant)$

$(A, \leqslant)$ as a thin category

- OBJECTS: $a \in A$
- MORPHISMS: $a \xrightarrow{\exists!} b$ whenever $a \leqslant b$

(there is at most one arrow between any pair of objects)

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$$\begin{tikzcd} a \arrow[loop above, "1_a"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[loop above, "1_b"] \arrow[d, "g"] \\ & c \arrow[loop below, "1_c"] \end{tikzcd}$$

In fact, a preorder suffices:

(1) reflexivity: provides the identity morphisms 1_a.

(2) transitivity: provides the composition of morphisms $g \circ f$.

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$$\begin{tikzcd}
a \arrow[r, squiggly] \arrow[d, "\leqslant"'] & f(a) \arrow[d, "\leqslant"] \\
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order-preserving maps

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$$\begin{tikzcd} a & ? \arrow[l] \arrow[r] & b \\ & c \arrow[ul] \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

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$$\begin{array}{ccc} a & \rightsquigarrow & f(a) \\ \leqslant \downarrow & & \downarrow \leqslant \\ a' & \rightsquigarrow & f(a') \end{array}$$

order-preserving maps

(binary) PRODUCTS:

$$\begin{array}{ccccc} a & \longleftarrow & a \wedge b & \longrightarrow & b \\ & \nwarrow & \uparrow & \nearrow & \\ & & c & & \end{array}$$

meets

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order-preserving maps

(binary) COPRODUCTS:

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & a \vee b \arrow[d, dotted] & b \arrow[l] \arrow[dl] \\ & c & \end{tikzcd}$$

joins

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From this point of view: category theory is an extension of lattice th.

GALOIS ADJUNCTIONS

$$\begin{tikzcd}(A,\leqslant) \arrow[r, shift left, "f"] & (B,\leqslant) \arrow[l, shift left, "g"]\end{tikzcd}$$

$$\boxed{f(a) \leqslant b \text{ iff } a \leqslant g(b)}$$

GALOIS ADJUNCTIONS

$$\textbf{Hom}_B(f(a), b) \cong \textbf{Hom}_A(a, g(b))$$

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$\Leftrightarrow \boxed{fg \leqslant \mathrm{id} \text{ and } \mathrm{id} \leqslant gf}$

("quasi-inverses")

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1 $\quad fgf = f$ and $gfg = g$.

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2.
$$\begin{array}{ccc} (A, \leqslant) & \underset{g}{\overset{f}{\rightleftarrows}} & (B, \leqslant) \\ \cup\!| & & \cup\!| \\ g[B] & \simeq & f[A] \end{array}$$

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$$\begin{array}{ccc} (A, \leqslant) & \underset{g}{\overset{f}{\rightleftarrows}} & (B, \leqslant) \\ \cup\!\mid & & \cup\!\mid \\ g[B] & \simeq & f[A] \end{array}$$

$$g[B] = \{a \in A \mid gf(a) = a\}$$

$$f[A] = \{b \in B \mid fg(b) = b\}$$

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ADJOINT FUNCTOR THEOREM.

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(2) If A has all joins and f preserves them, then f has a right adjoint g, uniquely determined by f:

$$g(b) = \bigvee\{a \in A \mid f(a) \leqslant b\}.$$

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Heyting algebra: lattice L with an extra $\rightarrow$ satisfying

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$\therefore$ frames = cHa. BUT different categories (morphisms).

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P3 $(\bigvee a_i)^* = \bigwedge (a_i)^*$. De Morgan law (Caution: not for $\bigwedge$)

PART II.

Categorical aspects of $\mathbb{F}\mathrm{rm}$

ALGEBRAIC ASPECTS OF $\mathbf{Frm}$

1. $\mathbf{Frm}$ is equationally presentable i.e.

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Objects are described by a (proper class of) operations and equations:

OPERATIONS

- 0-ary: $0, 1\colon L^0 \to L$
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EQUATIONS

- $(L, \wedge, 1)$ is an idempotent commutative monoid
- with a zero 0 sat. the absorption law $a \wedge 0 = 0 = 0 \wedge a \; \forall a$.
- $\bigvee_0 a_i = 0$, $a_j \wedge \bigvee_\kappa a_i = a_j$, $a \wedge \bigvee_\kappa a_i = \bigvee_\kappa (a \wedge a_i)$.

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COROLLARY.

$\mathbf{Frm}$ has all (small) limits (i.e., it is a COMPLETE category) and they are constructed exactly as in $\mathbf{Set}$ (i.e., the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$ preserves them).

ALGEBRAIC ASPECTS OF $\mathbf{Frm}$

2. $\mathbf{Frm}$ has free objects: there is a free functor $\mathbf{Set} \to \mathbf{Frm}$ (i.e., a left adjoint of the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$):

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& & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\{x_1, x_2, \dots, x_n\} \arrow[d, mapsto] \\
\bigwedge_{i=1}^{n} f(x_i)
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COROLLARY.

$\mathbf{Frm}$ is an ALGEBRAIC category. In particular:

(1) It has all (small) colimits (i.e., it is a COCOMPLETE category).

(2) Monomorphisms = injective.

(3) Epimorphisms need not be surjective;
Regular epis = surjective.

(4) $(RegEpi, Mono)$ is a factorization system.

(5) Quotients are described by congruences; there exist presentations by generators and relations.

EXAMPLE: PRESENTATIONS

PRESENTATIONS BY GENERATORS AND RELATIONS:

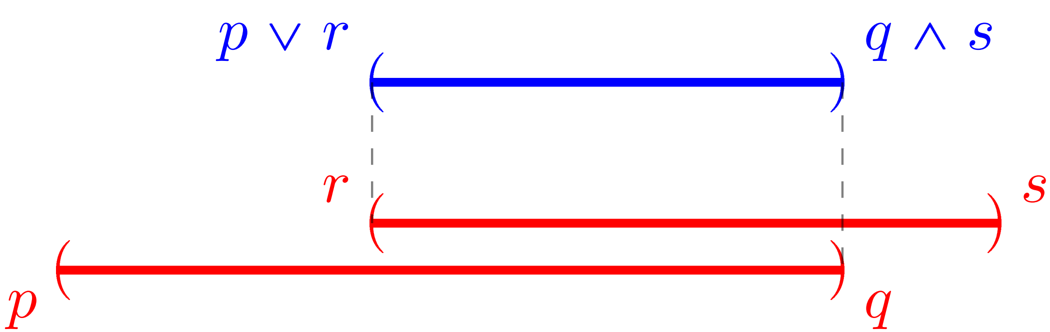
just take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p,q), $p,q \in \mathbb{Q}$, subject to the relations

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,



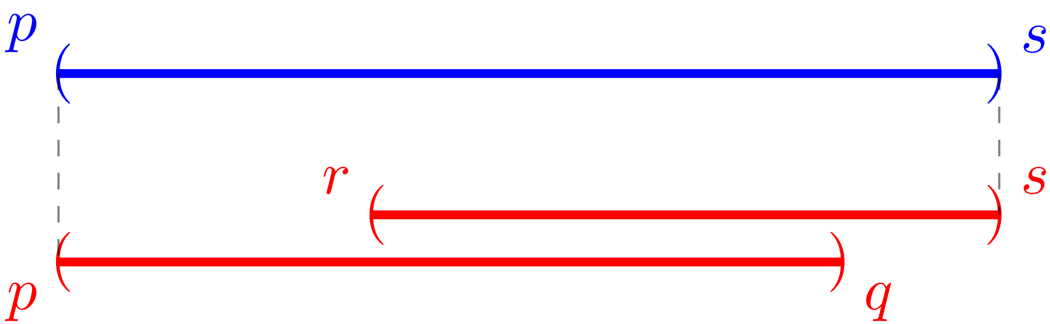
EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leqslant r < q \leqslant s$,



EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

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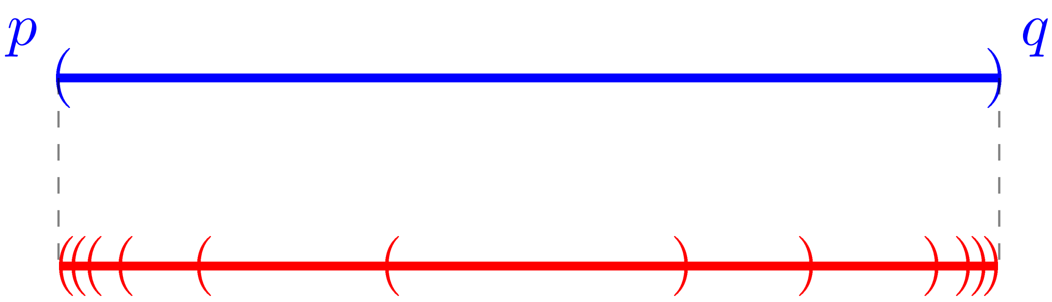
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,



Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

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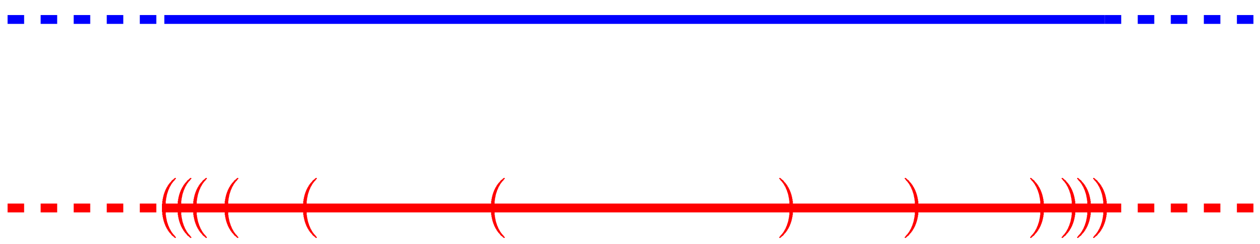
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.



EXAMPLE: PRESENTATIONS

Frame of reals $\mathfrak{L}(\mathbb{R})$

Nice features: Continuous real functions,

semicontinuous real functions, ...

MORE, in next lectures.