Tutorial on Localic Topology

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PART III. Locales:

the geometric facet of frames

MAKING THE PICTURE COVARIANT: the category of locales

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\mathcal{O}"] & \mathbf{Frm} \end{tikzcd}$$

Contravariant

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, rightsquigarrow] \arrow[d, "f"'] & (\mathcal{O}X, \subseteq) \\
(Y, \mathcal{O}Y) \arrow[r, rightsquigarrow] & (\mathcal{O}Y, \subseteq) \arrow[u, "{\boldsymbol{f^{-1}[-]}}"']
\end{tikzcd}$$

MAKING THE PICTURE COVARIANT: the category of locales

$$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$$

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$$\mathbf{Loc} = \mathbf{Frm}^{op}$$

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$$\begin{tikzcd}
L \arrow[d, "f"'] & L \\
M & M \arrow[u, "h"']
\end{tikzcd}$$

preserves $\bigvee$ (incl. 0)

$\wedge$ (incl. 1)

Loc **Frm**

MAKING THE PICTURE COVARIANT: the category of locales

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$$\begin{tikzcd} L \arrow[d, "f"'] \\ M \end{tikzcd}$$

frame homomorphisms taken backwards

$$\begin{tikzcd} L \\ M \arrow[u, "h"'] \end{tikzcd}$$

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Each $h\colon M \to L$ in $\mathbf{Frm}$ has a UNIQUELY defined right adjoint

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that can be used as a representation of the h as a mapping going in the proper direction.

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LOCALIC MAP: a map $f\colon L \to M$ that has a left adjoint f^* in $\mathbf{Frm}$, i.e., preserving finite meets:

(1) $f^*(1) = 1$.

(2) $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$.

MAKING THE PICTURE COVARIANT: the category of locales

PROPOSITION. Let $f : L \to M$ have a left adjoint f^*. Then:

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⇐: $f^*(a \wedge b) \leqslant x$ iff $a \leqslant b \to f(x) = f(f^*(b) \to x)$

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Loc

- OBJECTS: locales = frames (=cHa)

- MORPHISMS:

$$\begin{tikzcd} L \arrow[d, "f"'] \\ M \end{tikzcd}$$

- $f(\bigwedge S) = \bigwedge f[S]$

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$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "\mathsf{Lc}"] & \mathbf{Loc} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}X \\
Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"]
\end{tikzcd}$$

MAKING THE PICTURE COVARIANT: the category of locales

$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$ is immediately modifiable to a functor

$$\mathbf{Top} \xrightarrow{\mathsf{Lc}} \mathbf{Loc}$$

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X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}X \arrow[d, dotted, "(\mathcal{O}f)_*=\mathsf{Lc}(f)"] & U \arrow[d, squiggly] \\
Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"] & Y \smallsetminus \overline{f[X \smallsetminus U]}
\end{tikzcd}$$

MAKING THE PICTURE COVARIANT: the category of locales

$\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}$ is immediately modifiable to a functor

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Y \arrow[r, mapsto] & \mathcal{O}Y \arrow[u, bend left, "\mathcal{O}f"] & {\color{red}Y \smallsetminus \overline{f[X \smallsetminus U]}}
\end{tikzcd}$$

Why?

$$f^{-1}[V] \subseteq U \ \text{ iff } \ V \subseteq Y \smallsetminus f[X \smallsetminus U]$$

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$$\begin{tikzcd}
U \arrow[d, squiggly, red] \\
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Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathsf{c}}]^{\mathsf{c}}$)

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Why?

$f^{-1}[V] \subseteq U$ iff $V \subseteq Y \smallsetminus f[X \smallsetminus U]$ (since $f^{-1}[-] \dashv f[-^{\mathsf{c}}]^{\mathsf{c}}$)

iff $V \subseteq \operatorname{int}(Y \smallsetminus f[X \smallsetminus U]) = Y \smallsetminus \overline{f[X \smallsetminus U]}$.

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map $\{*\} \longrightarrow X$

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Top a point x of X is a continuous map

$$\begin{tikzcd} \{*\} \longrightarrow X \arrow[d, squiggly, "\mathsf{Lc}"] \\ \mathsf{Lc}(\{*\}) = \mathbb{2} \longrightarrow \mathsf{Lc}(X) \end{tikzcd}$$

Loc

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map

Loc

$$\begin{tikzcd} \{*\} \arrow[r, ""{name=U}] & X \\ \mathsf{Lc}(\{*\}) = \mathbb{2} \arrow[r, ""{name=D}] & \mathsf{Lc}(X) \arrow[from=U, to=D, squiggly, "\mathsf{Lc}"] \end{tikzcd}$$

Extension: a point of a *general* locale L is a localic map

$$p\colon \mathbb{2} \quad \to \quad L$$

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map $\{*\} \longrightarrow X$

Loc

$$\begin{tikzcd} \{*\} \arrow[r, ""{name=U, below}] & X \\ \mathsf{Lc}(\{*\}) = 2 \arrow[r, ""{name=D, above}] & \mathsf{Lc}(X) \arrow[from=U, to=D, squiggly, "\mathsf{Lc}"] \end{tikzcd}$$

Extension: a point of a *general* locale L is a localic map

$$\begin{aligned} p\colon 2 &\to L \\ 1 &\mapsto 1 \end{aligned}$$

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map

$$\begin{tikzcd} \{*\} \arrow[r] \arrow[d, squiggly, "\mathsf{Lc}"] & X \\ \mathsf{Lc}(\{*\}) = 2 \arrow[r] & \mathsf{Lc}(X) \end{tikzcd}$$

Loc

Extension: a point of a *general* locale L is a localic map

$$\begin{array}{rcl} p\colon 2 & \to & L \\ 1 & \mapsto & 1 \\ 0 & \mapsto & a \neq 1. \end{array}$$

THE SPECTRUM OF A LOCALE

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$$\begin{tikzcd} \{*\} \longrightarrow X \arrow[d, squiggly, "\mathsf{Lc}"] \\ \mathsf{Lc}(\{*\}) = 2 \longrightarrow \mathsf{Lc}(X) \end{tikzcd}$$

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$$\begin{array}{rcl} p\colon 2 & \to & L \\ 1 & \mapsto & 1 \\ 0 & \mapsto & a \neq 1. \end{array}$$

$x \wedge y \leqslant a = p(0)$ iff $p^*(x) \wedge p^*(y) \leqslant 0$

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Top a point x of X is a continuous map $\{*\} \longrightarrow X$

$$\begin{tikzcd} {} \arrow[d, squiggly, "\mathsf{Lc}"] \\ {} \end{tikzcd}$$

Loc $\mathsf{Lc}(\{*\}) = 2 \longrightarrow \mathsf{Lc}(X)$

Extension: a point of a *general* locale L is a localic map

$$\begin{array}{rcl} p\colon 2 & \to & L \\ 1 & \mapsto & 1 \\ 0 & \mapsto & a \neq 1. \end{array}$$

$$x \wedge y \leqslant a = p(0) \quad \text{iff} \quad p^*(x) \wedge p^*(y) \leqslant 0$$

$$\Rightarrow \quad p^*(x) = 0 \;\; \text{or} \;\; p^*(y) = 0$$

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Top a point x of X is a continuous map $\{*\} \longrightarrow X$

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$a \neq 1$

$\wedge$-IRREDUCIBLES

(PRIME ELEMENTS)

THE SPECTRUM OF A LOCALE

Top a point x of X is a continuous map $\{*\} \longrightarrow X$

Loc

$$\begin{tikzcd} \{*\} \arrow[r, ""{name=A, below}] & X \\ \mathsf{Lc}(\{*\}) = \mathbb{2} \arrow[r, ""{name=B}] & \mathsf{Lc}(X) \arrow[from=A, to=B, squiggly, "\mathsf{Lc}"] \end{tikzcd}$$

Extension: a point of a *general* locale L is a localic map

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$$\left.\begin{array}{l} x \wedge y \leqslant a = p(0) \ \text{ iff } \ p^*(x) \wedge p^*(y) \leqslant 0 \\ \qquad \Rightarrow \ p^*(x) = 0 \ \text{ or } \ p^*(y) = 0 \\ \qquad \text{iff } \ x \leqslant p(0) = a \ \text{ or } \ y \leqslant p(0) = a. \end{array}\right]$$

$a \neq 1$

$\wedge$-IRREDUCIBLES

(PRIME ELEMENTS)

$\mathsf{Pt}(L)$

THE SPECTRUM OF A LOCALE

$a \in L$, $\Sigma_a = \{p \in \mathsf{Pt}(L) \mid a \not\leq p\}$.

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Localic maps send points to points

$$\mathsf{Pt}(f)^{-1}(\Sigma_b) = \{p \in \mathsf{Pt}(L) \mid b \nleq f(p)\} = \{p \mid f^*(b) \nleq p\} = \Sigma_{f^*(b)}.$$

SPACES AND LOCALES

$$\begin{tikzcd}
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$$X \setminus \overline{\{x\}}$$

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SPACES AND LOCALES

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SPACES AND LOCALES

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$$\begin{aligned} \eta_X : X &\longrightarrow \mathsf{Pt}\,\mathsf{Lc}(X) \\ x &\longmapsto X \setminus \overline{\{x\}} \end{aligned}$$

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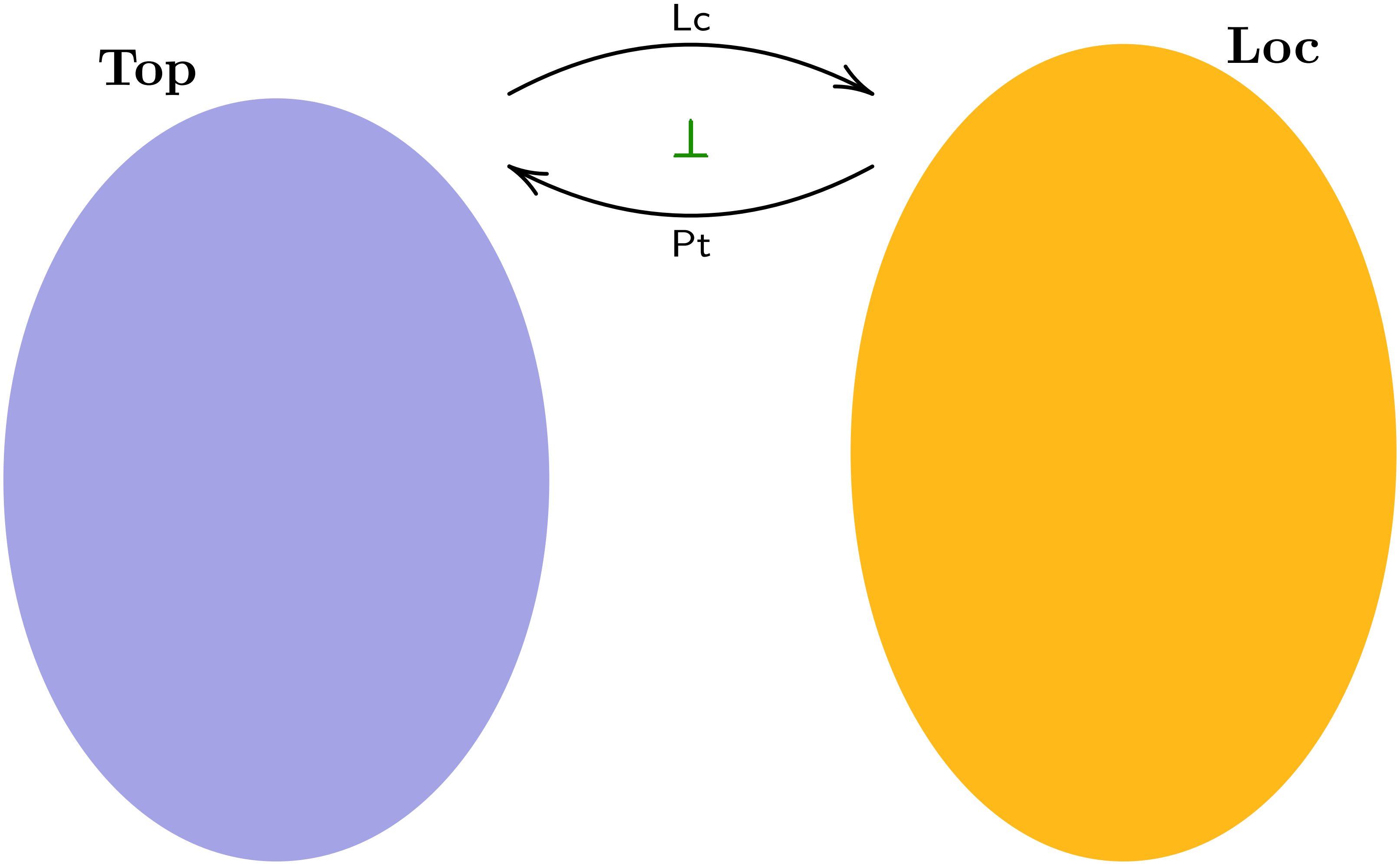
PROPOSITION: ε_L is an isomorphism iff L is spatial.

SPACES and LOCALES



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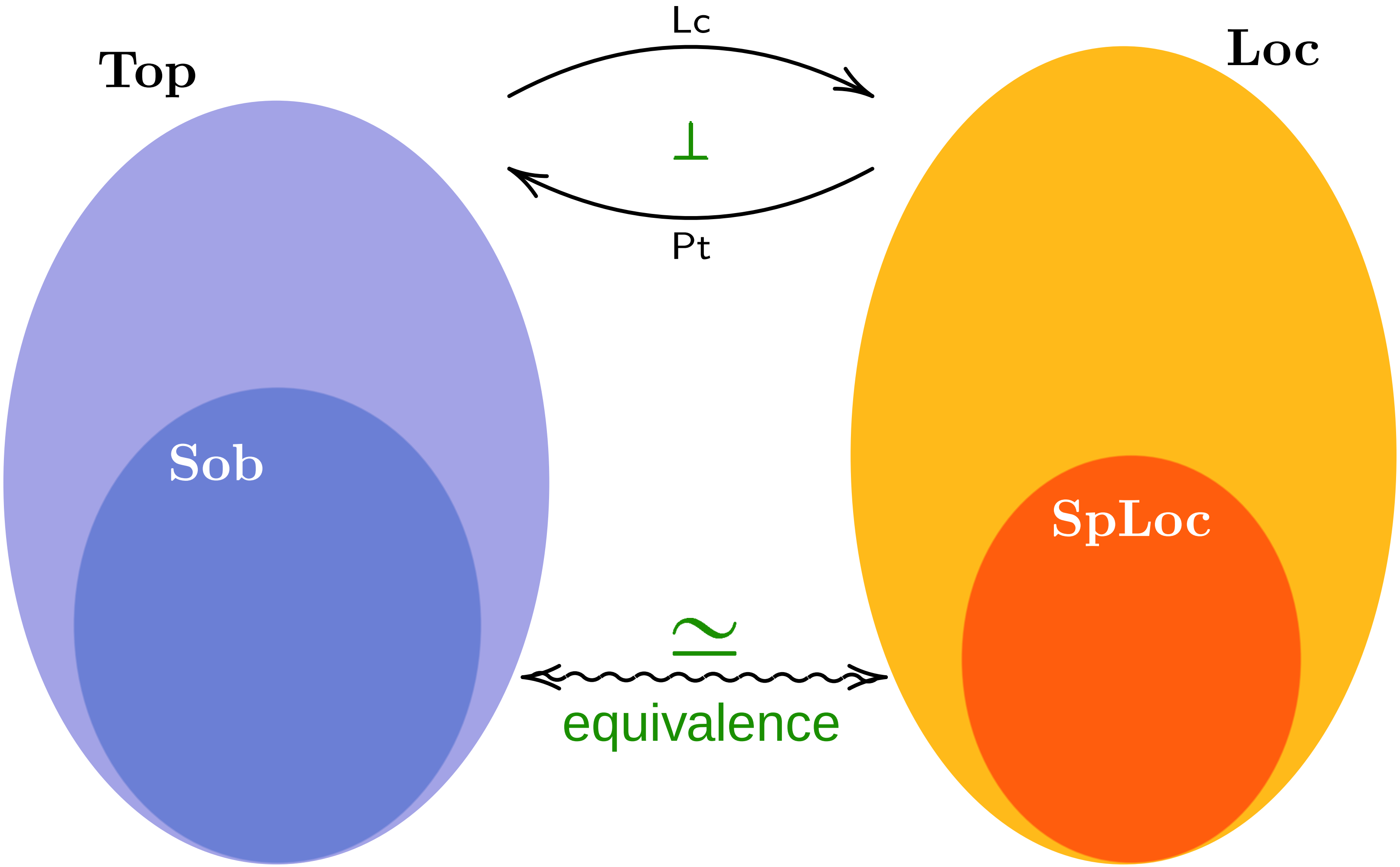
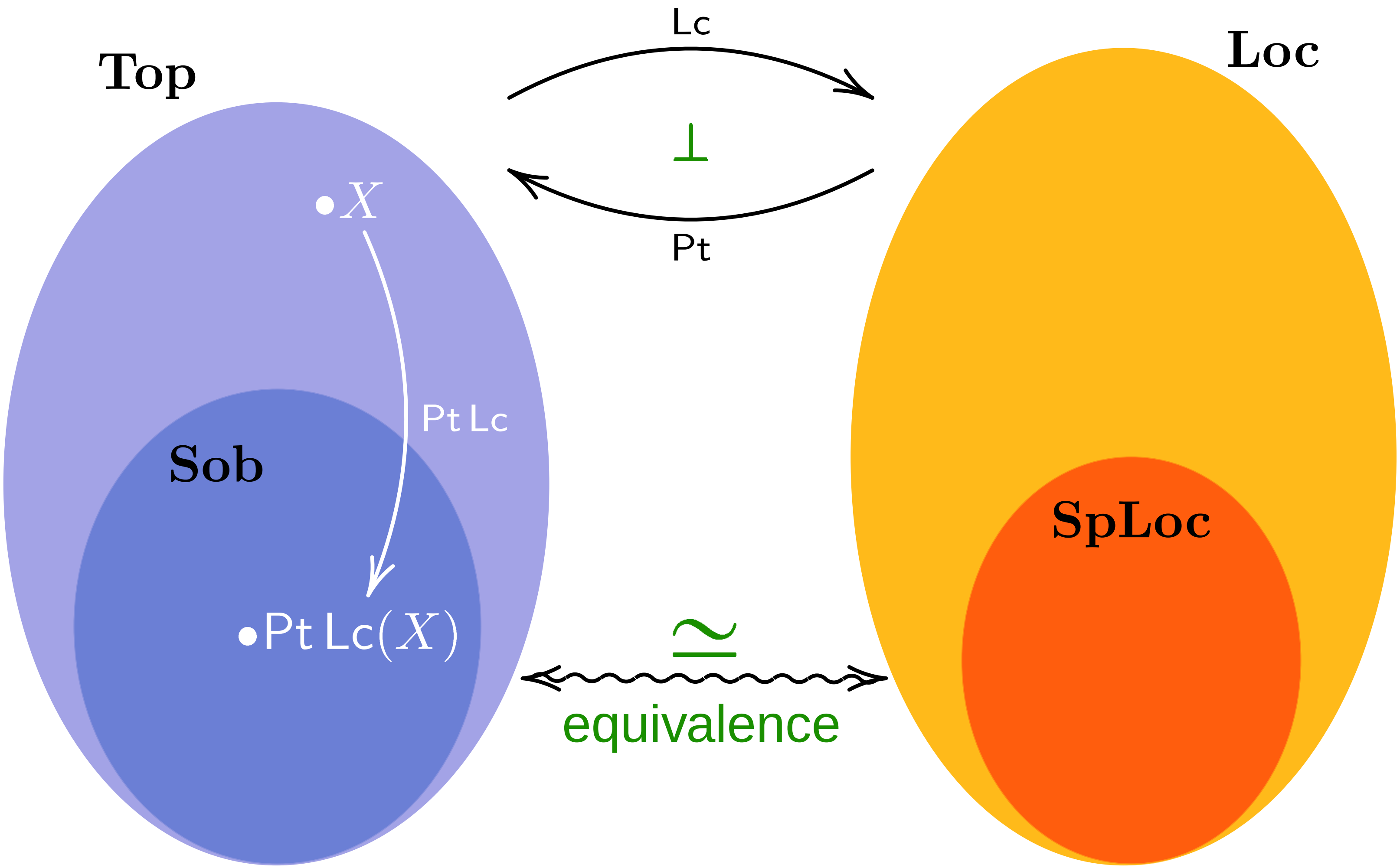


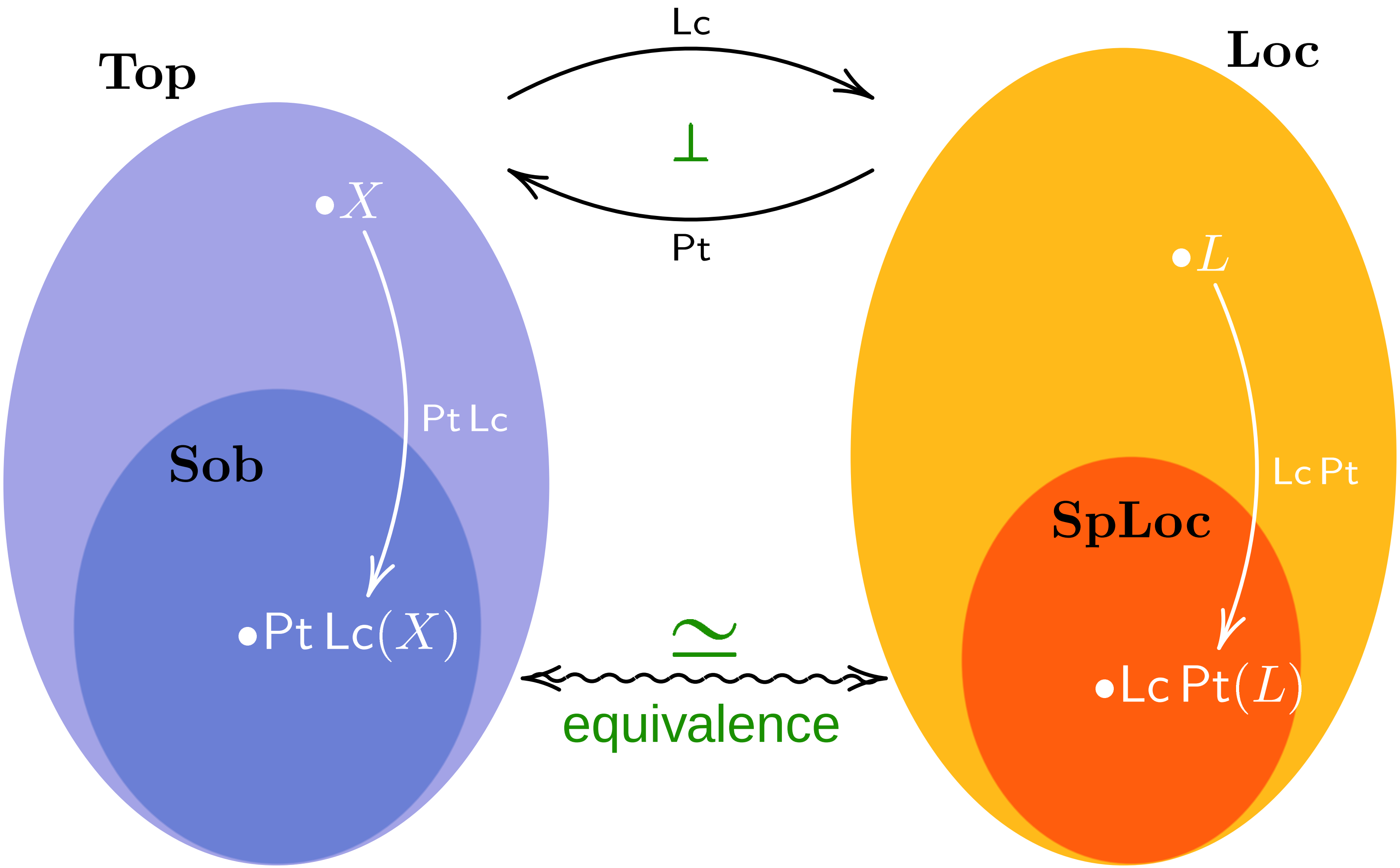
Perception: **Sob** more representative of all of **Top** than **SpLoc** of **Loc**.

SPACES and LOCALES



“soberification” of a space

SPACES and LOCALES



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Top

Sob

$\bullet X \xrightarrow{\mathsf{Pt}\,\mathsf{Lc}} \bullet \mathsf{Pt}\,\mathsf{Lc}(X)$

Loc

SpLoc

$\bullet L \xrightarrow{\mathsf{Lc}\,\mathsf{Pt}} \bullet \mathsf{Lc}\,\mathsf{Pt}(L)$

$\mathbf{Sob} \overset{\simeq}{\longleftrightarrow} \mathbf{SpLoc}$

equivalence

“soberification” of a space

“spatialization” of a locale

THE BOOLEAN CASE: non-spatial locales

PROPOSITION: L is spatial iff each $a \neq 1$ is a meet of points of L.

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By the Proposition,

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$\Leftrightarrow$ each $a \neq 1$ in B is a join of atoms (by complement.). ■

PART IV.

Doing topology in Loc

GENERALIZED SUBSPACES: SUBLOCALES

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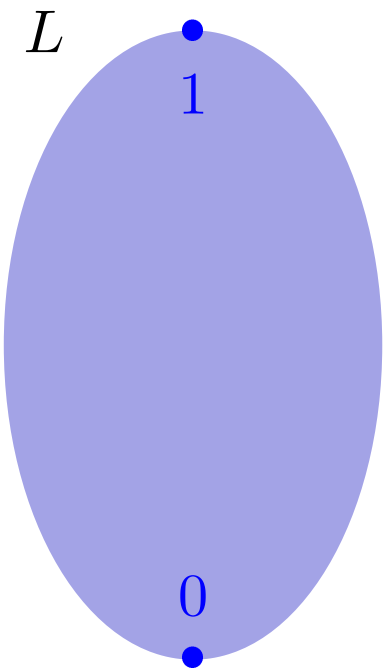

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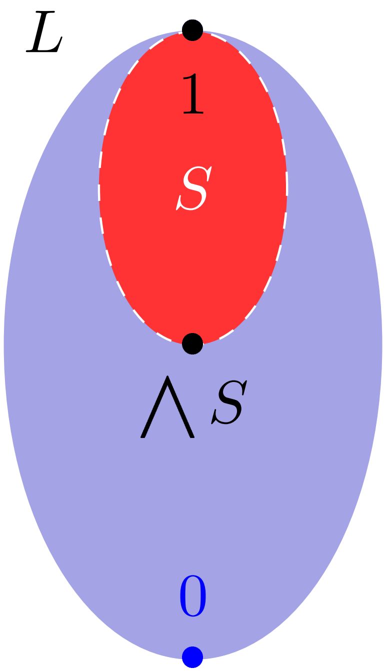

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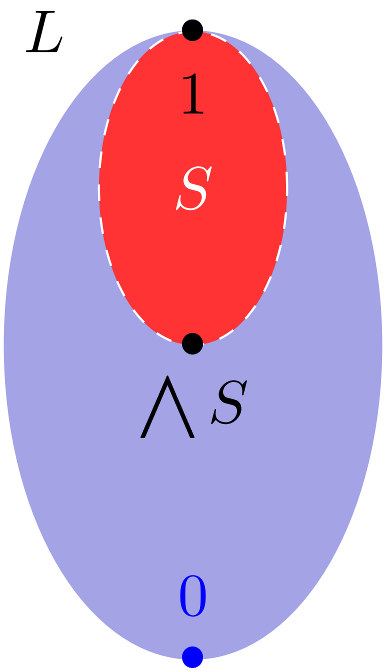

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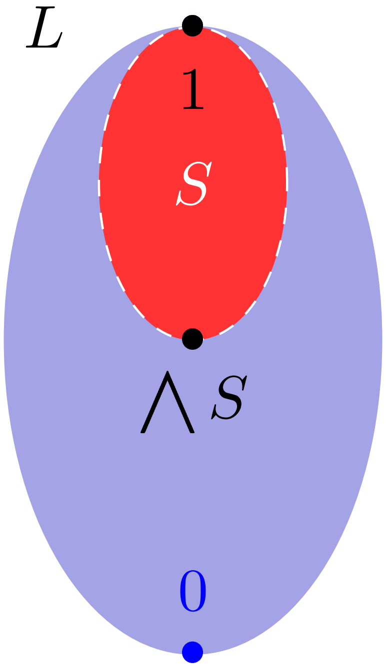

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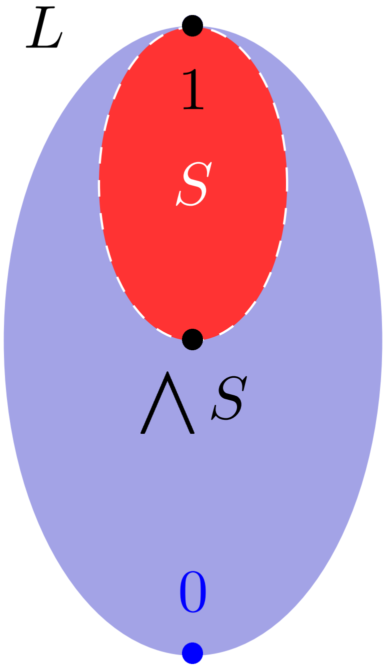

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Motivation for the definition:

PROP:

$S \subseteq L$ is a sublocale iff the embedding $j_S \colon S \subseteq L$ is a localic map.

THE SUBLOCALE LATTICE

$\mathcal{S}\ell(L)$: sublocales of L, ordered by $\subseteq$

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$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \boxed{\bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}}$$

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$$\bigcap(A \vee B_i) \overset{?}{\subseteq} A \vee (\bigcap B_i)$$

PROOF:

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Then $x = a \wedge b_i,\ \forall i \overset{\text{(H)}}{\Rightarrow} \underbrace{a \to b_i}_{b \in \bigcap B_i}$ does not depend on i.

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$$\overset{\text{(H)}}{=} a \wedge (a \to b_i) = a \wedge b \in A \vee (\bigcap B_i).$$

■

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$a \in L,\qquad \mathfrak{c}(a) = {\uparrow}a$ CLOSED

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SPECIAL SUBLOCALES

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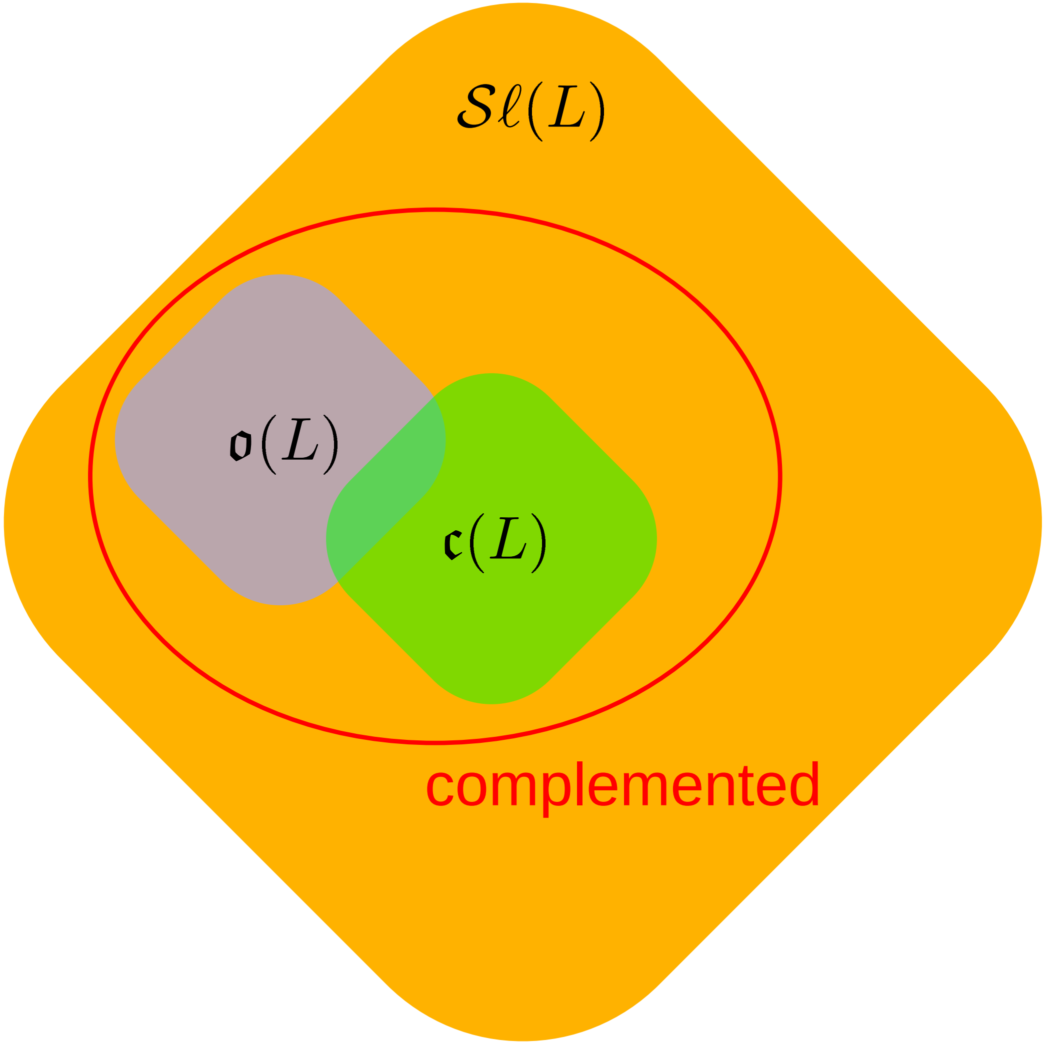

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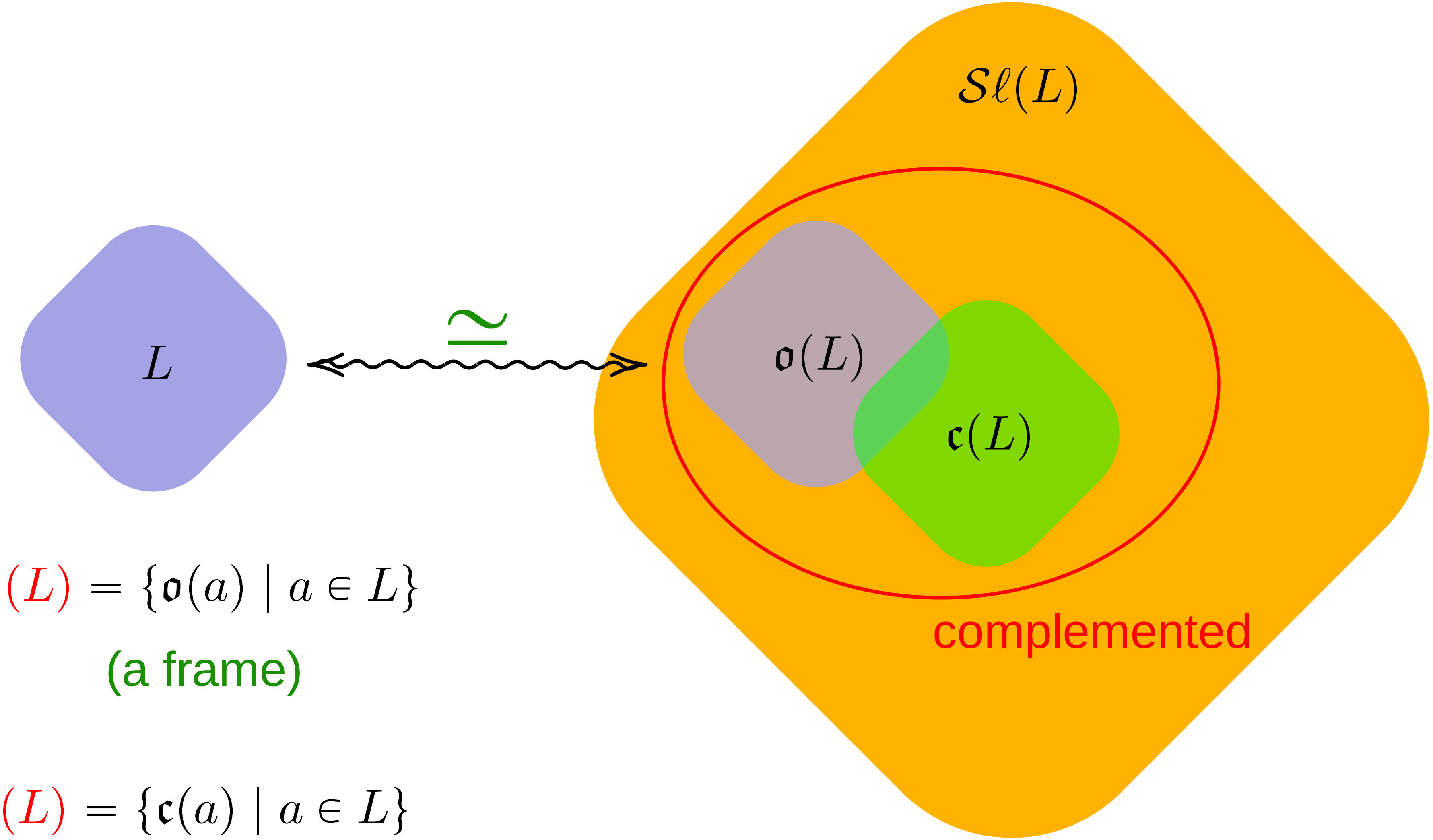

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i.e., there exists the smallest dense sublocale of a locale! ☺

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