# *Tutorial on Localic Topology*

Jorge Picado

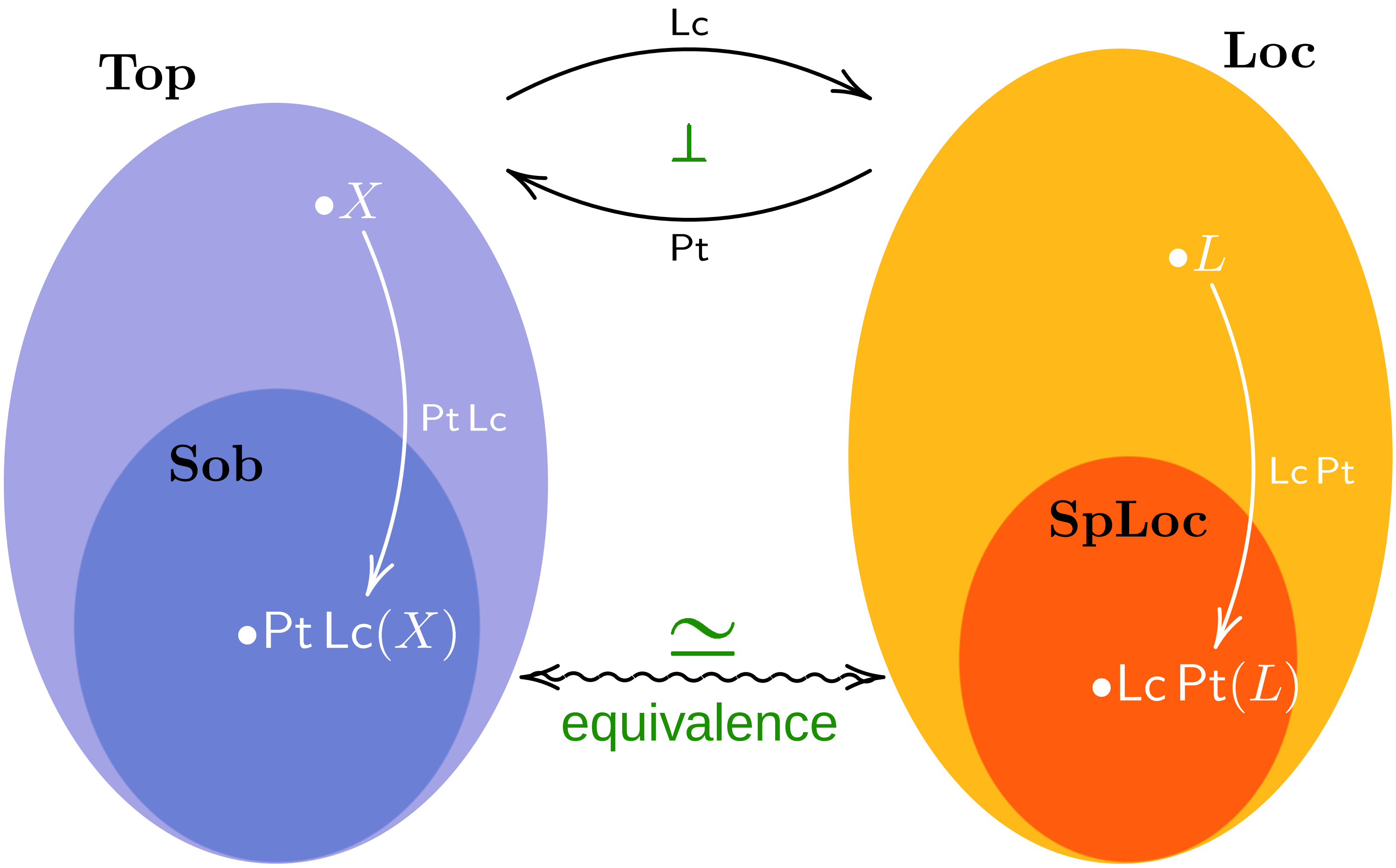
Department of Mathematics

University of Coimbra

PORTUGAL

Centre for Mathematics
University of Coimbra

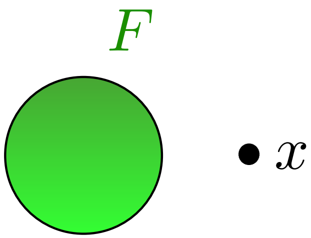
# SPACES versus LOCALES




“soberification” of a space

“spatialization” of a locale

# *Regularity*



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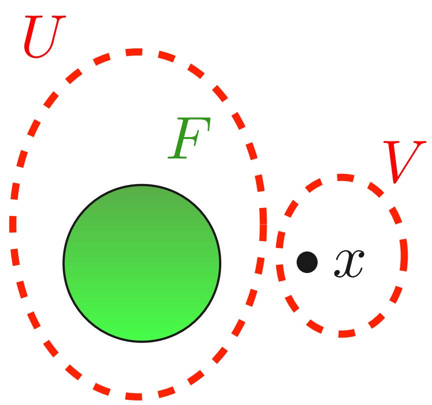

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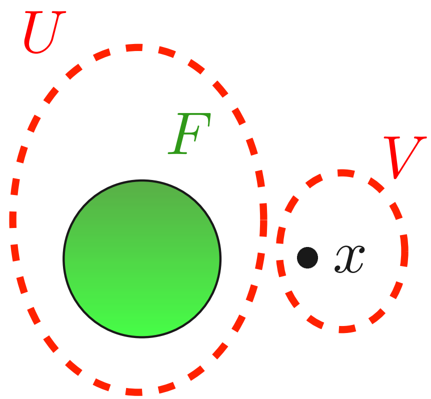

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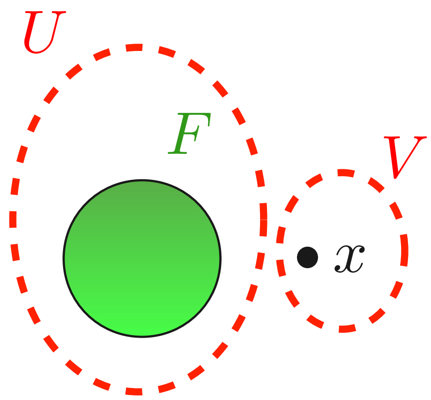


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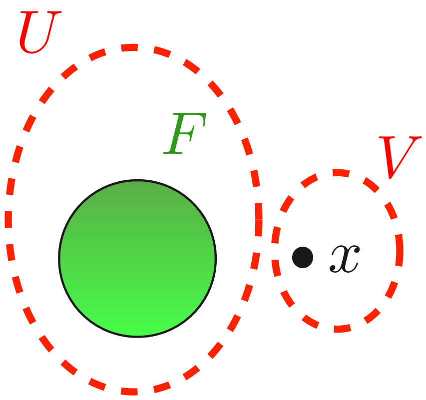


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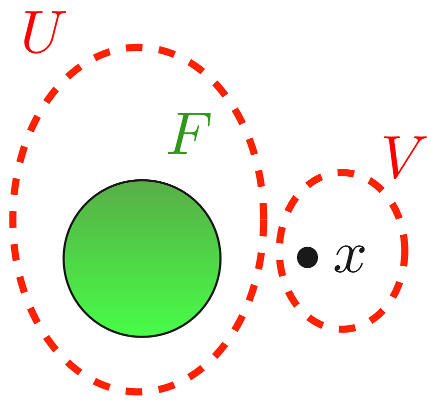


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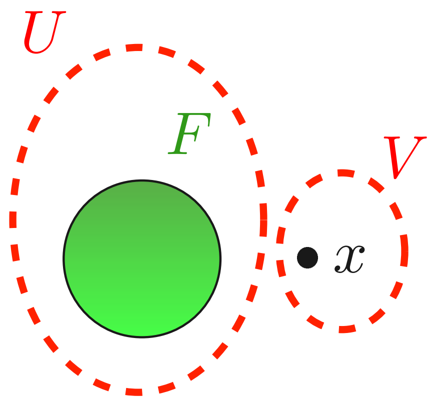


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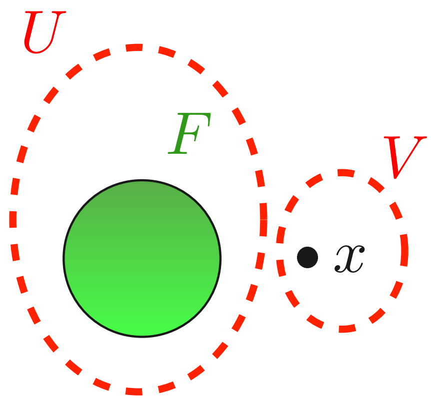


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## RECAP: SPECIAL SUBLOCALES

$a \in L$, $\mathfrak{c}(a) = \uparrow a$ CLOSED

$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN

} complemented

**Properties**

(1) $a \leqslant b$ iff $\mathfrak{c}(a) \supseteq \mathfrak{c}(b)$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$.

(2) $\bigwedge \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee a_i)$.

(3) $\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$.

(4) $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.

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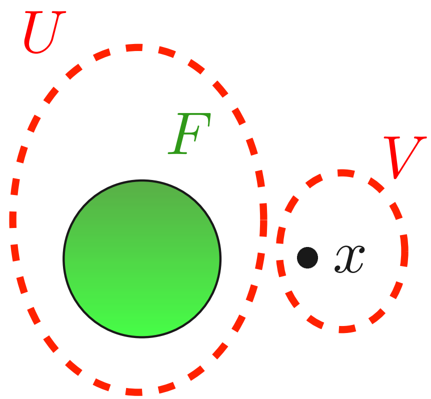


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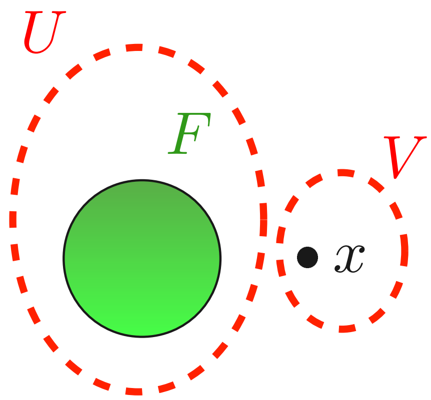


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$$\mathfrak{c}(b^*) \wedge \mathfrak{c}(a) = 0$$

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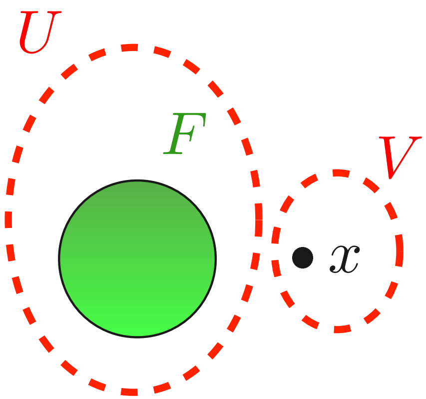


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$$\mathfrak{c}(b^* \vee a) = \mathfrak{c}(1)$$

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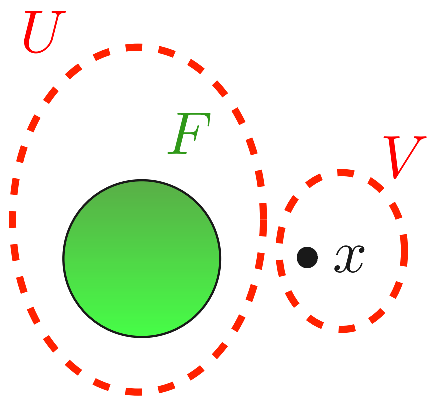


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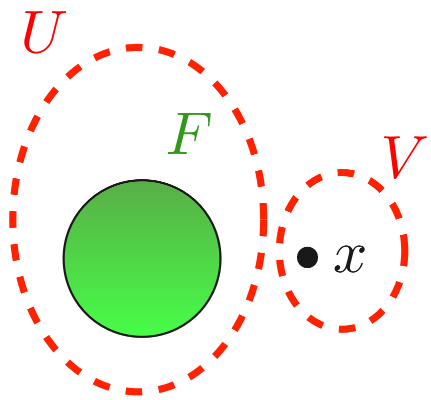


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(Conservative extension: $X$ is regular iff the locale $\mathcal{O}(X)$ is regular.)

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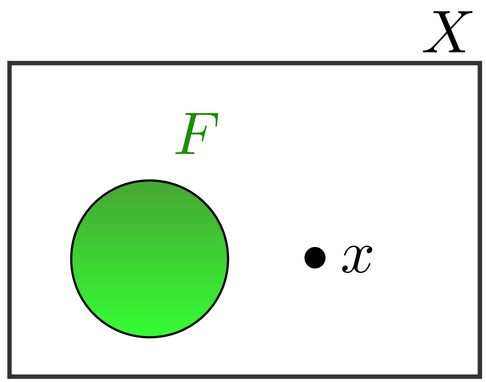
3. $$a_i \prec b_i \ (i = 1, 2) \;\Rightarrow\; \begin{cases} a_1 \vee a_2 \prec b_1 \vee b_2 \\ a_1 \wedge a_2 \prec b_1 \wedge b_2 \end{cases}$$

# Complete Regularity



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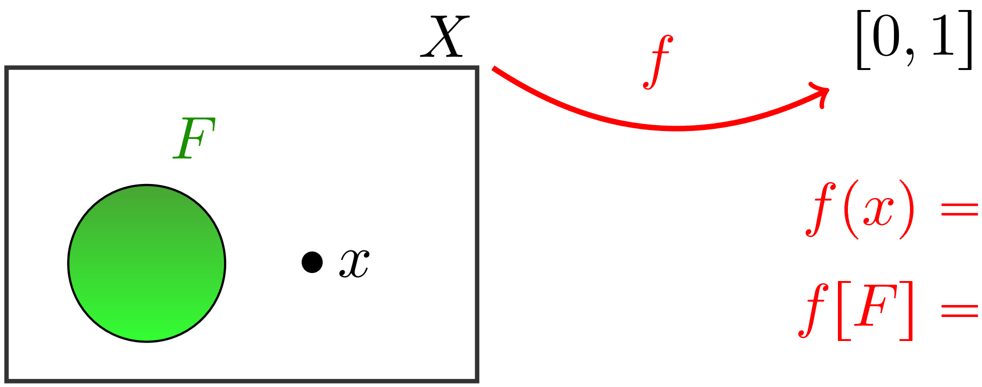


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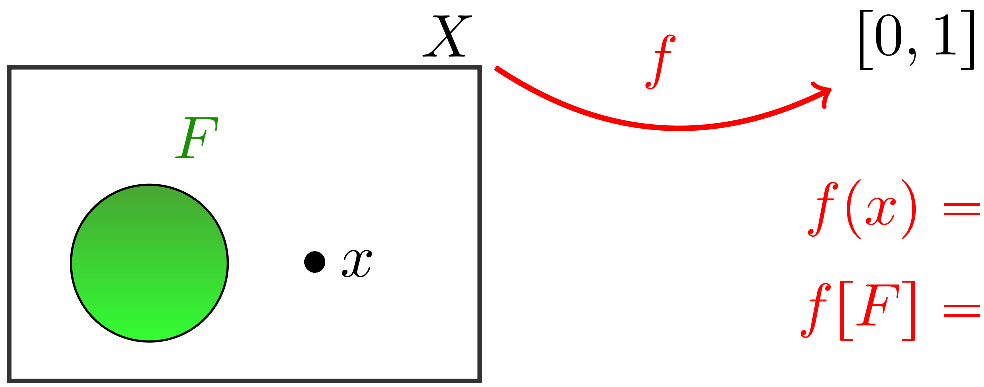


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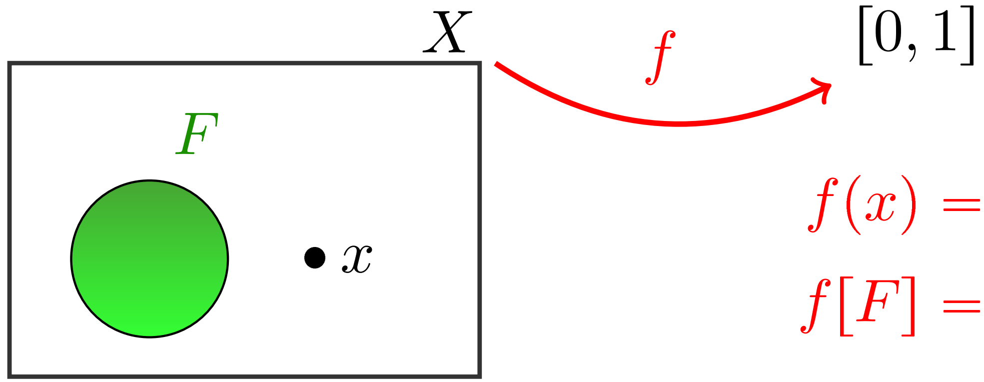


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$$V \lll U \equiv \exists\, (W_q)_{q \in \mathbb{Q} \cap [0,1]} \colon\ W_0 = V,\ W_1 = U,\ p < q \Rightarrow W_p \prec W_q.$$

[B. Banaschewski (1953)]

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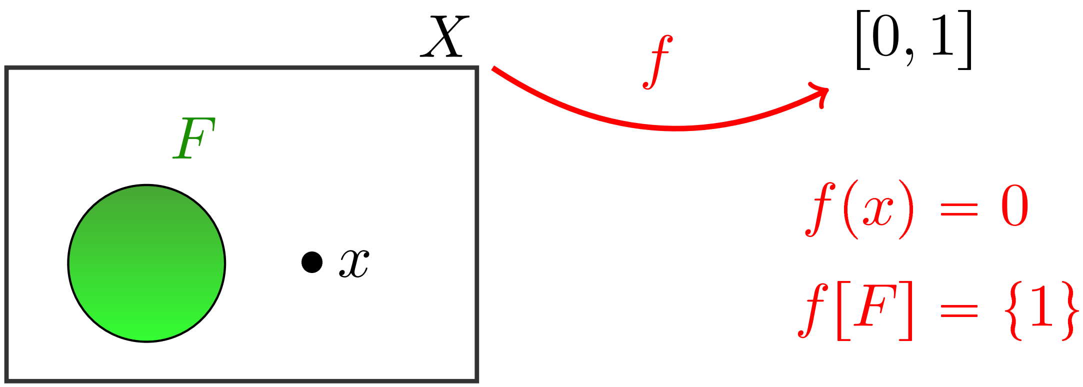


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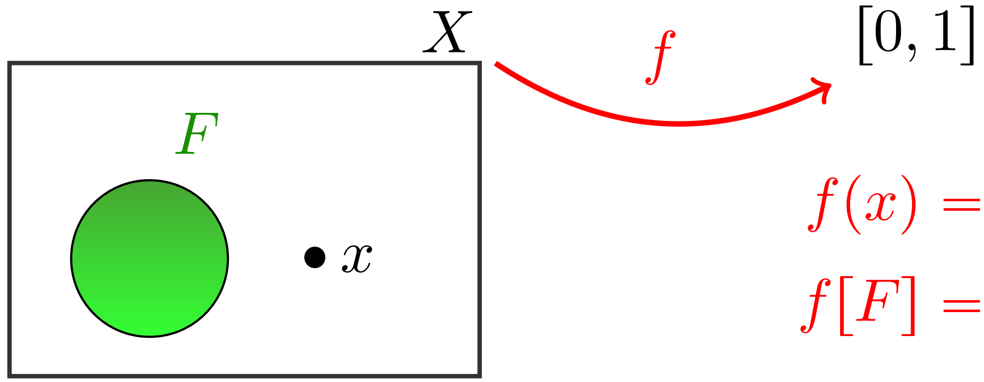


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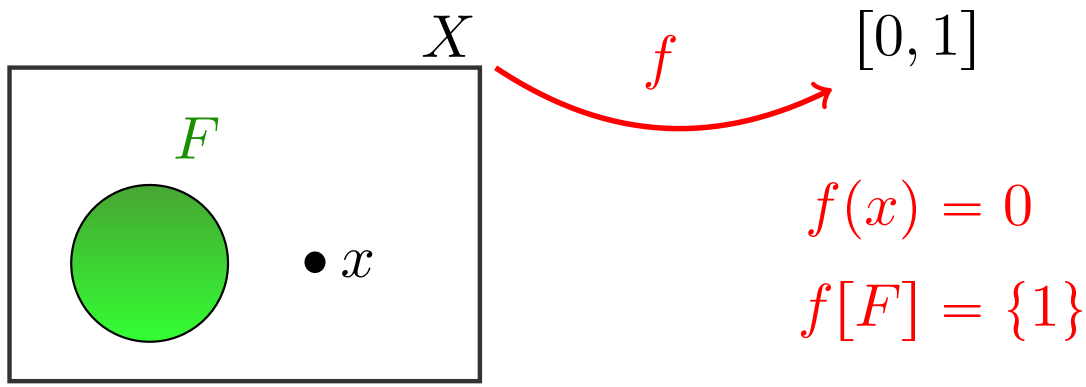


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(Conservative extension: $X$ is c. reg. iff the locale $\mathcal{O}(X)$ is c. reg.)

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where $c = x_1 \vee \cdots \vee x_n$.

Further

$$x_i \prec b \; (i = 1, \ldots, n) \;\Rightarrow\; c \prec b. \quad \blacksquare$$

# THE (constructive) STONE-Čech compactification

Ideals of $L$: $\mathfrak{I}(L)$ (I1) $b \leqslant a \in J \Rightarrow b \in J$, (I2) $a, b \in J \Rightarrow a \vee b \in J$

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$$\text{(I1)} \Big\Downarrow {\scriptstyle x_j \leqslant x \in K}$$

$$x_j \in J_{i_j} \cap K \Rightarrow x \in \bigvee (J_i \cap K)$$

## THE (constructive) STONE-Čech compactification

Ideals of $L$: $\mathfrak{I}(L)$ (I1) $b \leqslant a \in J \Rightarrow b \in J$, (I2) $a, b \in J \Rightarrow a \vee b \in J$

**LEMMA 1.** $\mathfrak{I}(L)$ is a compact locale.

PROOF: • $\bigwedge = \bigcap$ (any intersection of ideals is an ideal).

• $\bigvee J_i = \{\bigvee F \mid F$ finite, $F \subseteq \bigcup J_i\}$.

• $(\bigvee J_i) \cap K = \bigvee (J_i \cap K)$.

$$\subseteq:\ x = x_1 \vee \cdots \vee x_n \in (\bigvee J_i) \cap K \quad (x_j \in J_{i_j})$$

$$\text{(I1)} \Big\Downarrow \ x_j \leqslant x \in K$$

$$x_j \in J_{i_j} \cap K \Rightarrow x \in \bigvee (J_i \cap K)$$

• $\bigvee J_i = L \ni 1 \Rightarrow 1 = x_1 \vee \cdots \vee x_n$ (some $x_j \in J_{i_j}$).

Then $1 \in \bigvee_{j=1}^{n} J_{i_j} \ \Rightarrow \ L = \bigvee_{j=1}^{n} J_{i_j}$. ■

# THE (constructive) STONE-Čech compactification

Regular ideal: (Ir) $\forall a \in J\ \exists b \in J: a \prec\!\prec b$. $\mathfrak{R}(L)$

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Regular ideal: (Ir) $\forall a \in J\ \exists b \in J : a \ll b$. $\mathfrak{R}(L)$

**LEMMA 2.** $\mathfrak{R}(L)$ is a subframe of $\mathfrak{I}(L)$, hence compact.

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LEMMA 2. $\mathfrak{R}(L)$ is a subframe of $\mathfrak{I}(L)$, hence compact.

$a \in L$, $\sigma(a) = \{x \in L \mid x \lll a\}$.

By interpolation property of $\lll$, each $\sigma(a)$ is a regular ideal of $L$.

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Now, suffices: $b \lll a$ in $L \Rightarrow \sigma(b) \prec \sigma(a)$ in $\mathfrak{R}(L)$ which is easy!

# INTERMEZZO: DENSE MAPS

Dense localic map: $f\colon L \to M$ such that $f[L]$ is dense in $M$

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$$h(a) = 0 \Rightarrow a = 0$$

# THE (constructive) STONE-Čech compactification

LEMMA 4. For each completely regular $L$,

$$\begin{aligned} \beta_L : \mathfrak{R}(L) &\to L \\ J &\mapsto \bigvee J \end{aligned}$$

is a dense surjection.

# THE (constructive) STONE-Čech compactification

THEOREM. There is a functor $\mathfrak{R}\colon \mathbf{CRegFrm} \to \mathbf{CRegFrm}$

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(2) Each $\beta_L$ is a dense surjection.

(3) $\beta_L$ is an isomorphism iff $L$ is compact.

# REAL NUMBERS POINTFREELY

The frame of reals:

$$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\big\langle\, (p,\text{—}), (\text{—}, q) (p, q \in \mathbb{Q}) \ \big|$$

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# CONTINUOUS REAL FUNCTIONS POINTFREELY

$f: \mathfrak{L}(\mathbb{R}) \to L$

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- From the adjunction

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# The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

| | |
|---|---|
| Generators | $(p, —), (—, q), \quad p, q \in \mathbb{Q}$ |
| Relations | (R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$ |
| | (R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$ |
| | (R3) $(p, —) = \bigvee_{r>p}(r, —)$ and $(—, q) = \bigvee_{s<q}(—, s)$ |
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SIDE NOTES

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Similarly, we have the extended continuous real functions:

$$\overline{\mathsf{C}}(L)=\mathrm{Hom}_{\mathbf{Frm}}(\mathfrak{L}(\overline{\mathbb{R}}),L)$$

B. BANASCHEWSKI, J. GUTIÉRREZ GARCÍA & J. P.
**Extended real functions in pointfree topology,** *J. Pure Appl. Algebra* **216 (2012)**

## SIDE NOTES

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SIDE NOTES

## The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

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$\mathrm{Pt}(\mathfrak{L}(\mathbb{IR}))$ is the partial real line.

## *The frame of partial reals* $\mathfrak{L}(\mathbb{IR})$

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Similarly, we have the partial continuous real functions:

$$\mathsf{IC}(L) = \mathsf{Hom}_{\mathbf{Frm}}(\mathfrak{L}(\mathbb{IR}), L)$$

I. Mozo Carollo, J. Gutiérrez García & J. P.
**On the Dedekind completion of function rings,** *Forum Mathematicum* **to appear**

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## *Motivation*

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

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$$\mathsf{Hom}_{\mathbf{Frm}}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{S}(L)})$$ dual lattice of sublocales of $L$

Natural extension:

$$\boxed{\mathsf{F}(L) = \mathsf{Hom}_{\mathbf{Frm}}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))}$$

# (GENERAL) REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:

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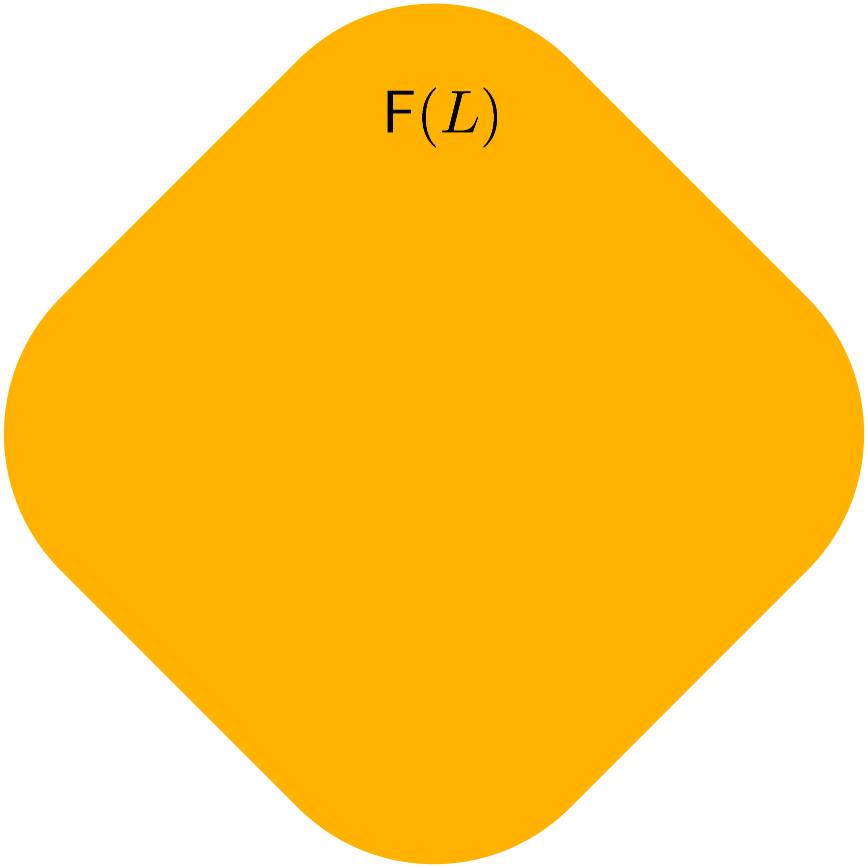

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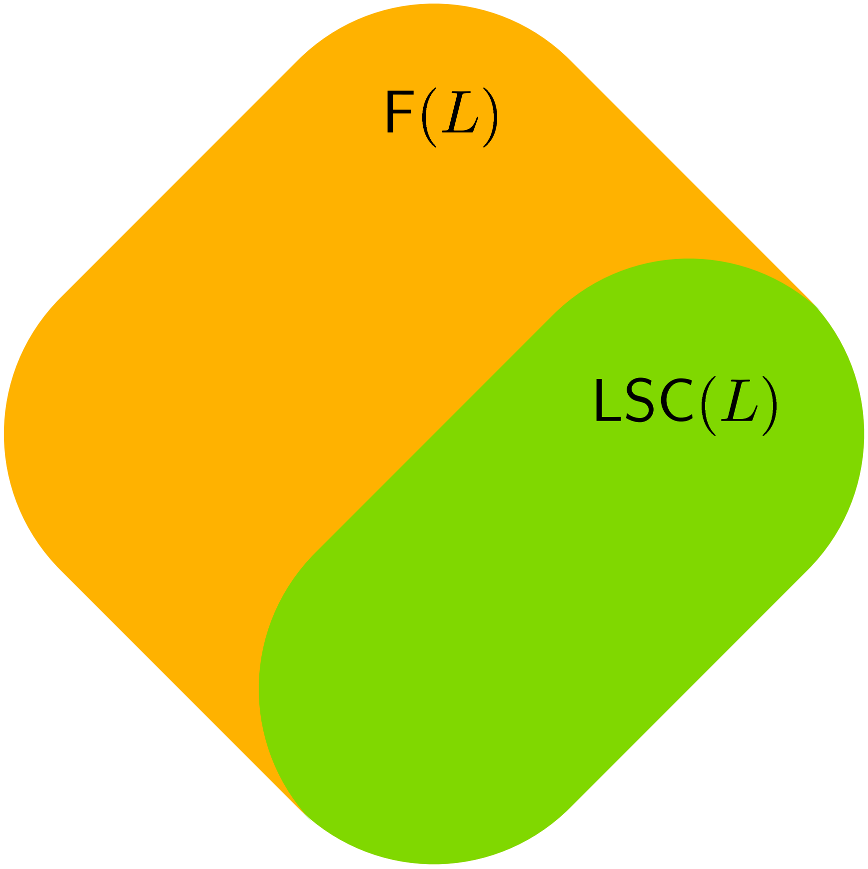


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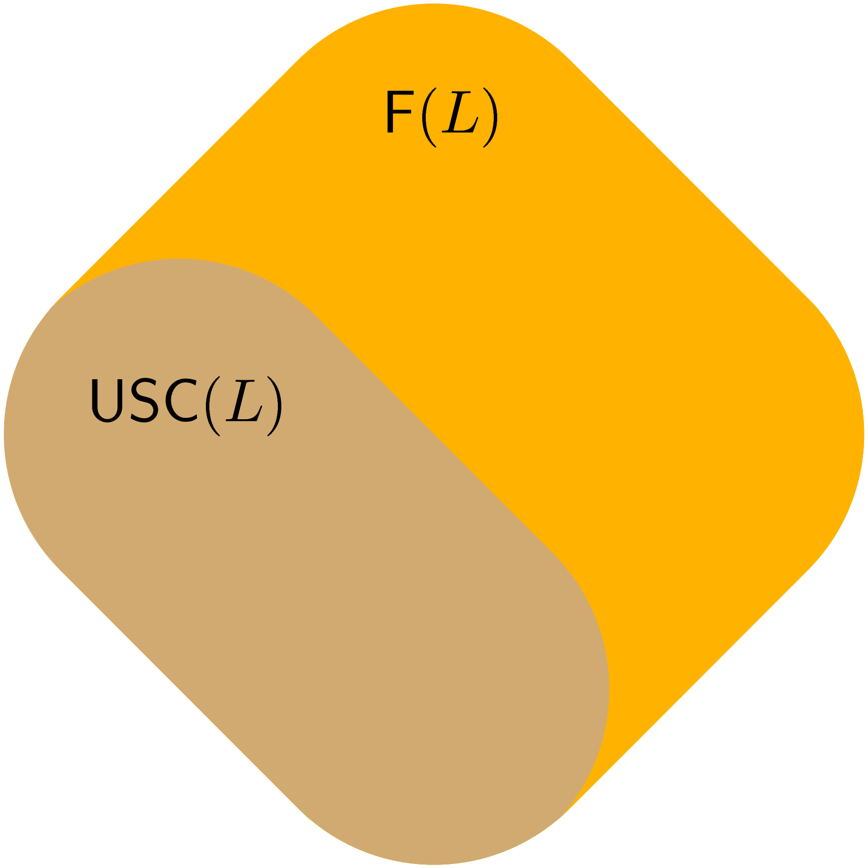


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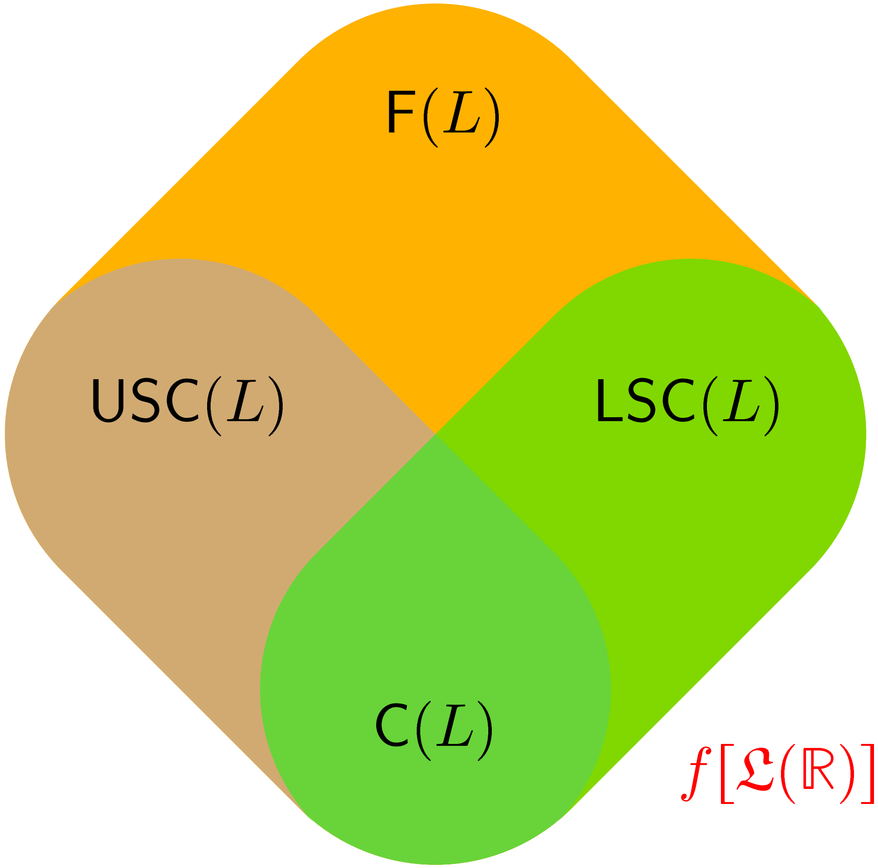


$f(p, -) \in \mathfrak{c}(L)$

$f[\mathfrak{L}(\mathbb{R})] \subseteq \mathfrak{c}(L) \simeq L$

J. Gutiérrez García, T. Kubiak & J. P.
**Localic real functions: a general setting,** *J. Pure Appl. Algebra* **213 (2009) 1064-1074**

# REGULARIZATIONS OF A REAL FUNCTION

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If $\{g \in \mathsf{LSC}(L) \mid g \leqslant f\} \neq \emptyset$ then $f^\circ \in \mathsf{LSC}(L)$.

## REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization $f^\circ$

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- Dually: the upper regularization $f^- = -(-f)^\circ$

J. GUTIÉRREZ GARCÍA, T. KUBIAK & J. P.
Lower and upper regularizations of frame semicontinuous real functions, *Alg. Univ.* (2009)

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- Extension: $\mathfrak{L}(\mathbb{R}) \to \mathfrak{c}(a)$ [Tietze's Extension Theorem]

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- Separation: $f = \chi_F,\ g = \chi_A$ [Gillman-Jerison]
- Extension: $\mathfrak{L}(\mathbb{R}) \to \mathfrak{o}(a)$ [Gillman-Jerison]

# APPLICATIONS: insertion theorems

locale $L$

TFAE:

(i) $L$ is ~~normal~~ *completely normal*

(ii) $\underbrace{f, g}_{\mathsf{F}(L)}, \ f^{-} \leqslant g, \ f \leqslant g^{\circ} \quad \Rightarrow \quad \exists\, h \in \mathsf{LSC}(L) : f \leqslant h \leqslant h^{-} \leqslant g$

- Classically: $L = \mathcal{O}X$

[General insertion: Kubiak]

## APPLICATIONS: insertion theorems

More: monotone insertion [Kubiak],

strict insertion [Dowker],

bounded insertion [Michael], ...