Extended real functions in pointfree topology

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— joint work with B. Banaschewski and J. Gutiérrez García

THE SETTING: POINTFREE SPACES

LOCALES   Loc$\equiv$Frm^{op}

categorical topology, topos theory, logic

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preserves $\bigvee$ (incl. the bottom 0)

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$$L \xrightarrow{f} M \qquad\qquad L \longleftarrow M$$

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"BEHAVES LIKE" Top

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POINTFREE SPACES: examples

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  (spatial locales)

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- finite distributive lattices

  ⋮

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≪(...) what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.≫

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$\mathcal{O}\mathbb{R}$ (usual topology) OR

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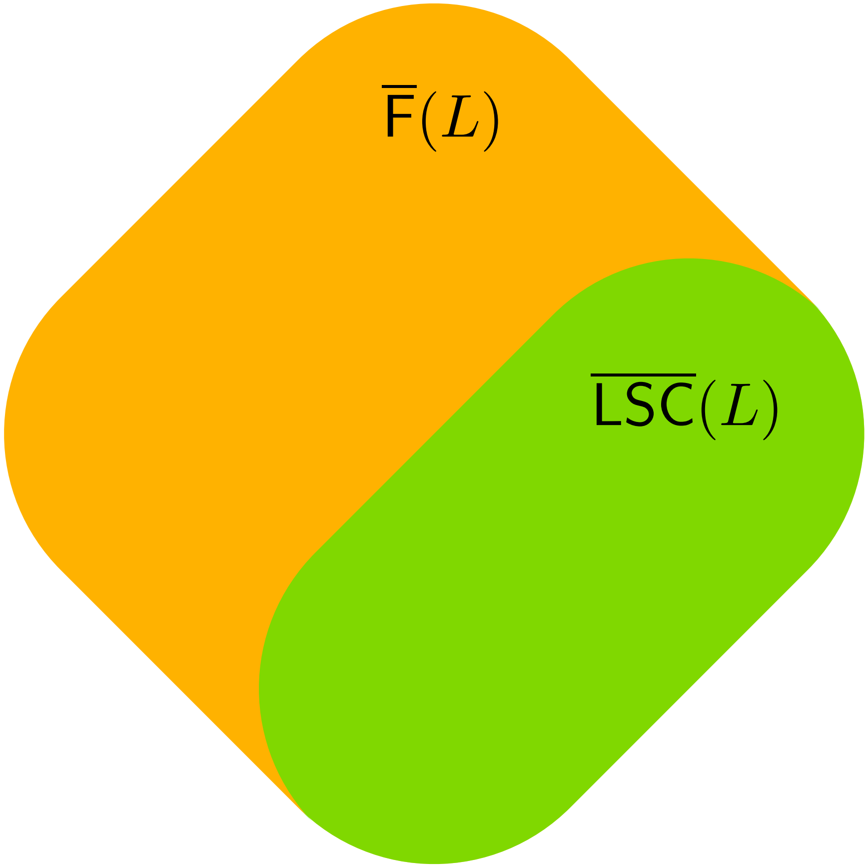
MOTIVATES: $\boxed{\overline{\mathsf{F}}(L) := \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{S}(L))}$

SEMICONTINUITY AND CONTINUITY

$f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)$



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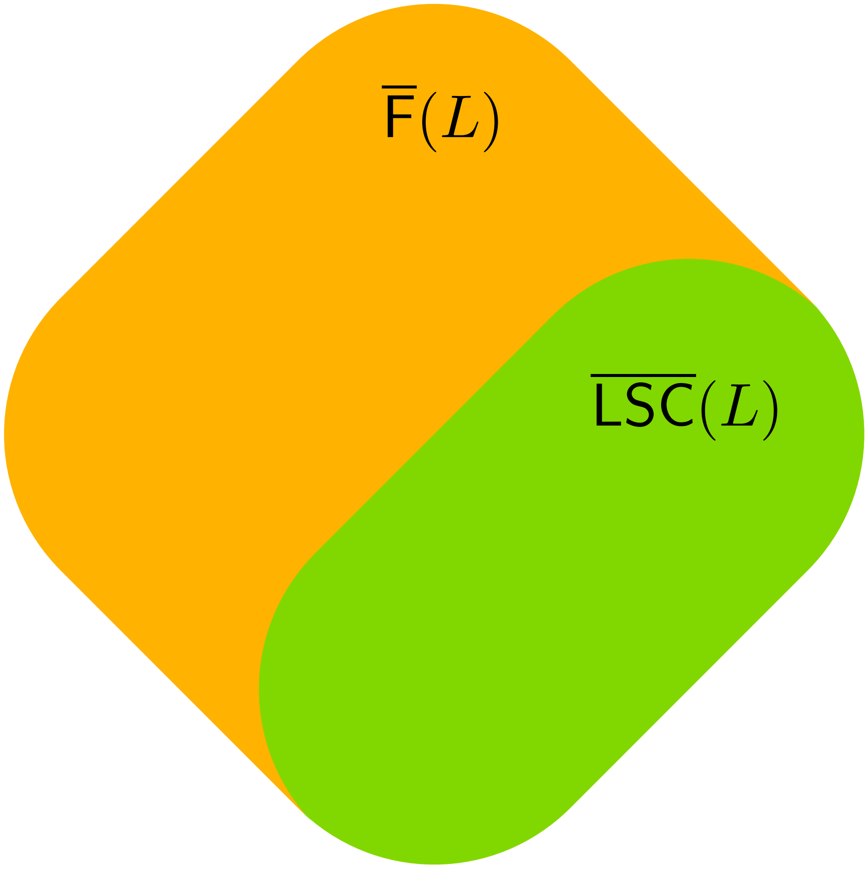

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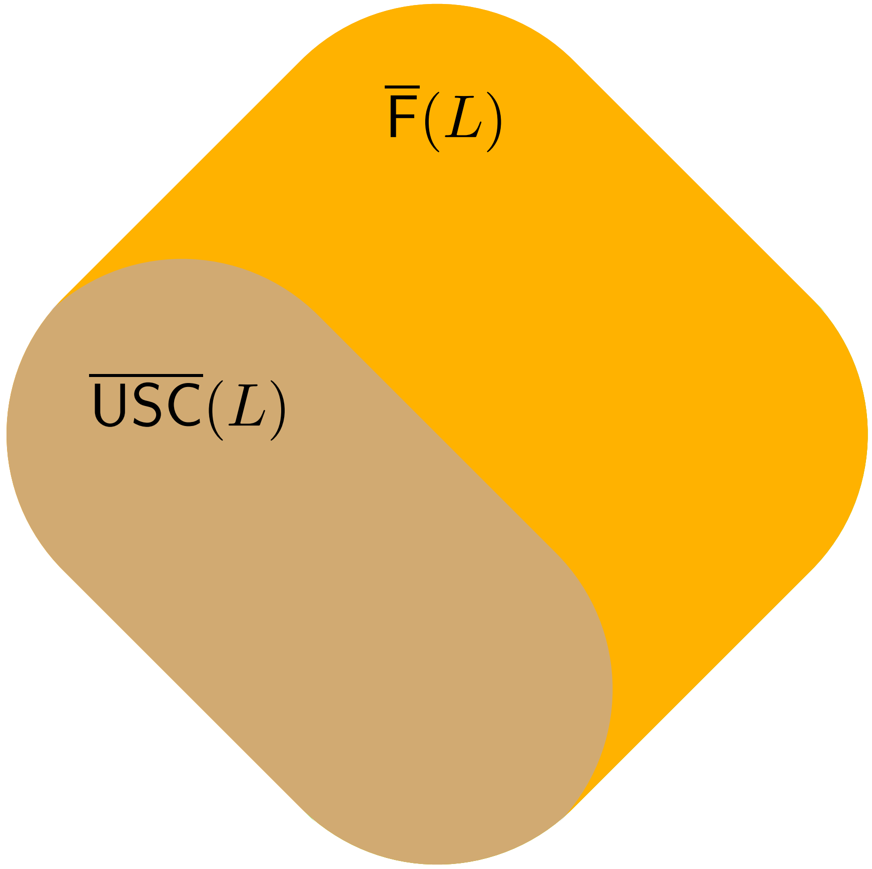

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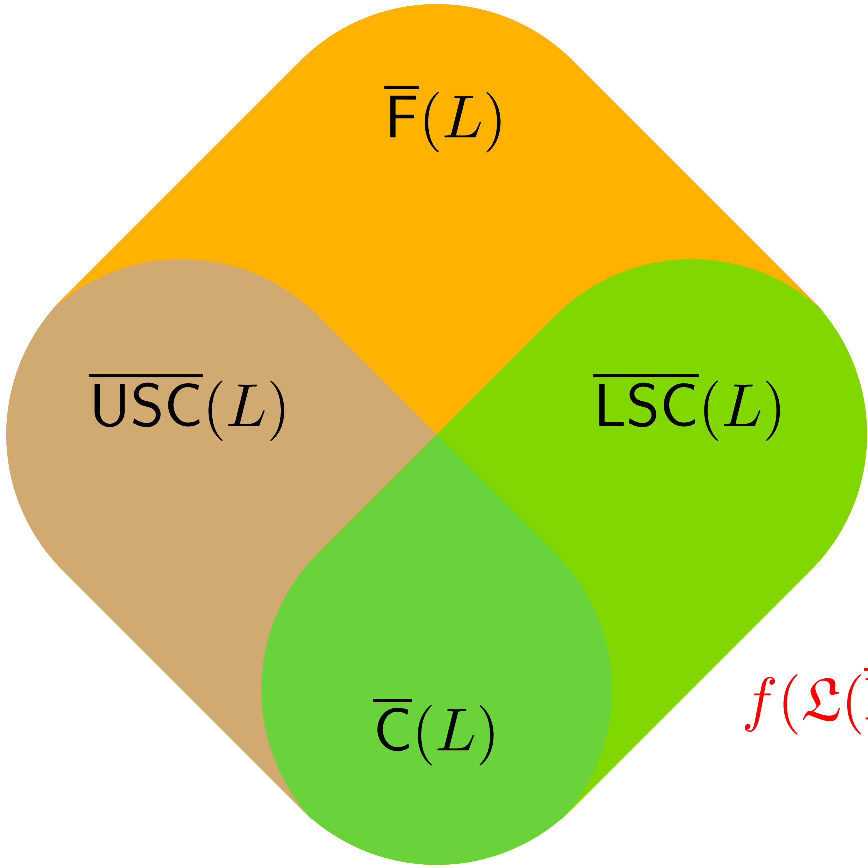

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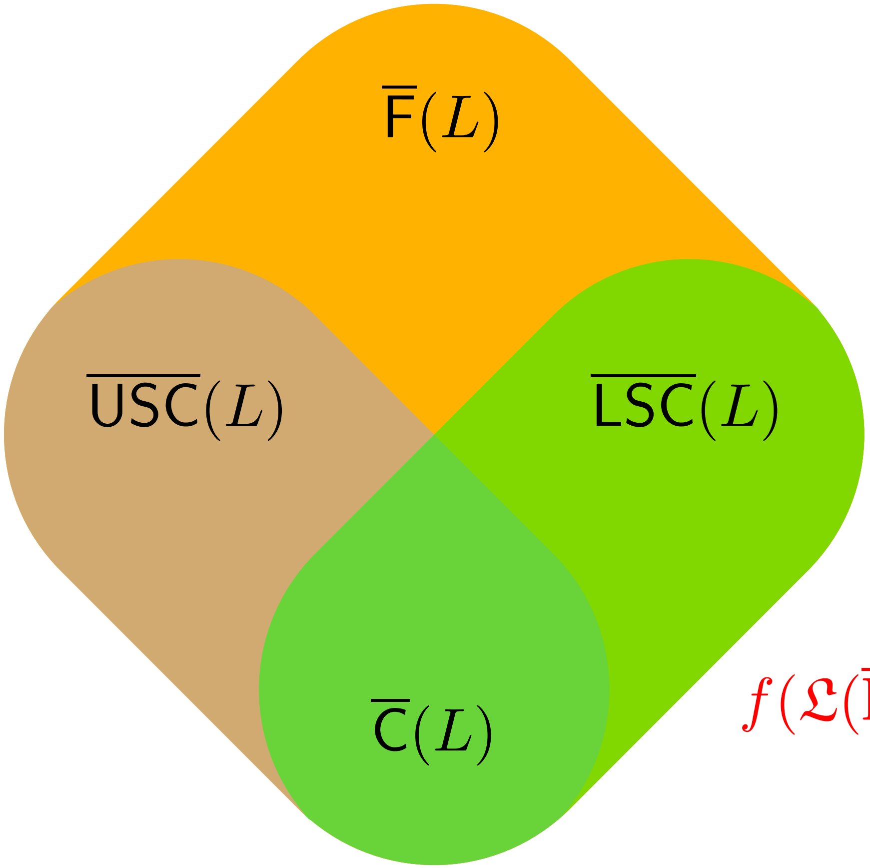

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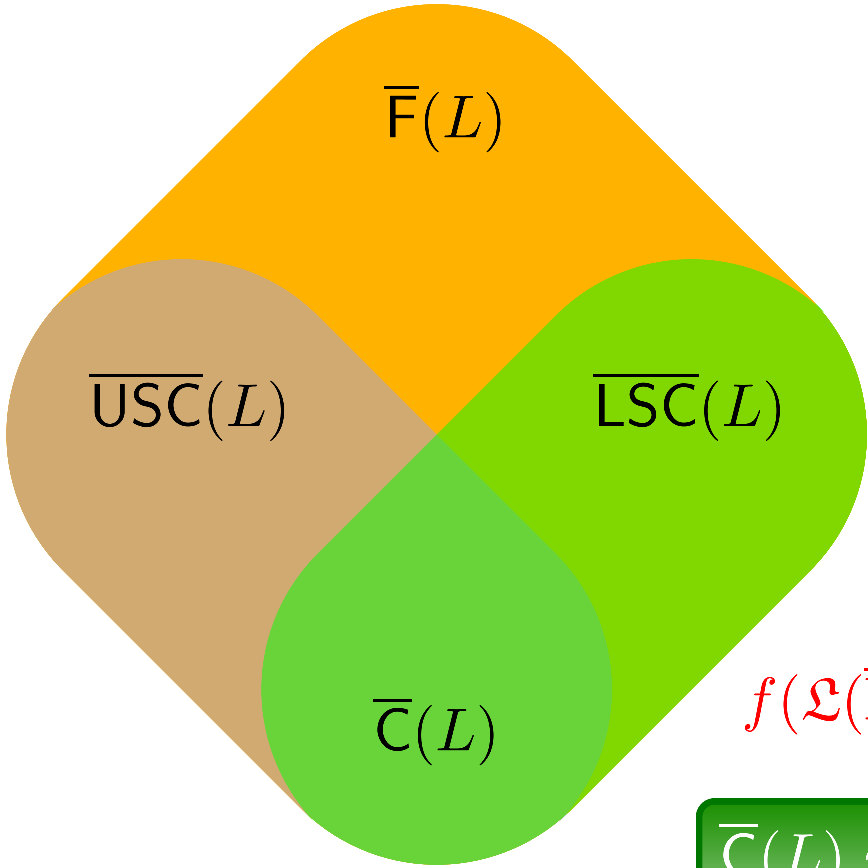

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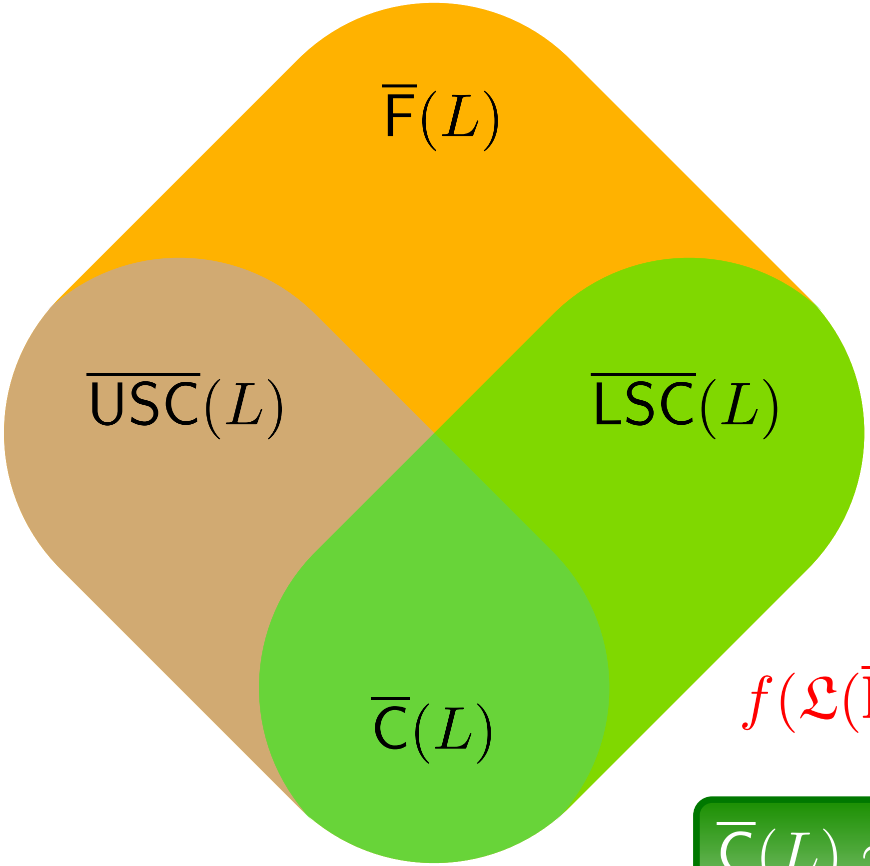

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$$f \leq g \;\equiv\; f(p, -) \leq g(p, -), \; \forall p \in \mathbb{Q}$$

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$\uparrow$

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$\boxed{\mathbf{r}}$ $\quad \mathcal{C}_r = \{c^r_p \mid p \in \mathbb{Q}\} = \{\dots, 1, 1, \dots, 1, \dots, 0, 0, \dots, 0, \dots\}$

$\uparrow$

$p = r$

scale $\rightsquigarrow \mathbf{r} \in \mathsf{C}(L)$

$$\mathbf{r}(p, -) = \bigvee_{s>p} c^r_s = c^r_p, \quad \mathbf{r}(-, q) = \begin{cases} 1 & \text{if } q > r \\ 0 & \text{if } q \leq r \end{cases}$$

$\boxed{+\infty}$ $\quad \mathcal{C}_{+\infty} = \{\dots, 1, 1, \dots, 1, \dots\}$

extended scale $\rightsquigarrow +\infty \in \overline{\mathsf{C}}(L)$

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$f \vee g,\ f \wedge g,\ -f,\ \underset{(\lambda>0)}{\lambda \cdot f}$: EASY!

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PRODUCT $f \cdot g$

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$$c_p^{f\cdot g} = \bigvee_{r>0} f(r,-) \wedge g(\tfrac{p}{r},-) \ (p \geq 0)$$

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$p > 0$: $(f \cdot g)(—,q) = \bigvee_{r>0} f(—,r) \wedge g(—,\frac{q}{r})$, $\qquad p \leq 0$: $(f \cdot g)(—,q) = 0$

ALGEBRA in $\overline{\mathsf{F}}(L)$

$f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)$

$\overline{\mathsf{F}}(L)$	$\vee, \wedge$				
$\overline{\mathsf{LSC}}(L)$					
$\overline{\mathsf{USC}}(L)$					
$\overline{\mathsf{C}}(L)$					

ALGEBRA in $\overline{\mathsf{F}}(L)$

$f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)$

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ALGEBRA in $\overline{\mathsf{F}}(L)$

$f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)$

					comp.
$\overline{\mathsf{F}}(L)$	$\vee, \wedge$	$-f$	$\lambda \cdot f$ $(\lambda > 0)$	$f + g$	
$\overline{\mathsf{LSC}}(L)$					
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$\overline{\mathsf{C}}(L)$	sublat.				

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$\overline{\mathsf{LSC}}(L)$	sublat.	$\in \overline{\mathsf{USC}}(L)$			
$\overline{\mathsf{USC}}(L)$	sublat.	$\in \overline{\mathsf{LSC}}(L)$			
$\overline{\mathsf{C}}(L)$	sublat.	closed			

ALGEBRA in $\overline{\mathsf{F}}(L)$

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$\overline{\mathsf{F}}(L)$	$\vee, \wedge$	$-f$	$\lambda \cdot f$ ($\lambda > 0$)	comp. $f + g$	comp. $f \cdot g$
$\overline{\mathsf{LSC}}(L)$	sublat.	$\in \overline{\mathsf{USC}}(L)$	closed	closed	
$\overline{\mathsf{USC}}(L)$	sublat.	$\in \overline{\mathsf{LSC}}(L)$	closed	closed	
$\overline{\mathsf{C}}(L)$	sublat.	closed	closed	closed	

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$\overline{\mathsf{USC}}(L)$	sublat.	$\in \overline{\mathsf{LSC}}(L)$	closed	closed	closed $\in \overline{\mathsf{LSC}}(L)$
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B. BANASCHEWSKI, J. GUTIÉRREZ GARCÍA & J. P.
Extended real functions in pointfree topology, *submitted*

ALMOST REAL FUNCTIONS

locale L

$$\boxed{\mathsf{D}(L) = \{f \in \overline{\mathsf{C}}(L) \mid a_f \text{ is dense}\}}$$

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(the reality of f)

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(the reality of f)

$L = \mathcal{O}X$: $D(X) = \{f : X \to \overline{\mathbb{R}} \mid f^{-1}(\mathbb{R}) \text{ is dense in } X\}.$

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$\mathsf{D}(L) = \{f \in \overline{\mathsf{C}}(L) \mid a_f$ is **dense**$\}$ $\quad (a_f^* = 0)$

$\mathsf{D}(L)$ is a sublattice with inversion of $\overline{\mathsf{C}}(L)$

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$\therefore$ $\boxed{f \vee g \in \mathsf{D}(L).}$ By inversion $\boxed{f \wedge g = -((-f) \vee (-g)) \in \mathsf{D}(L).}$

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(4) δ_L is an isomorphism iff L is extremal disconnectedness; then the partial operations are total and $\mathsf{D}(L)$ becomes an

order-complete archimedean f-ring with unit