Sublocales and a Boolean extension of a frame

Jorge Picado (Univ. Coimbra, Portugal)

– joint work with Aleš Pultr (Charles Univ.)

cmuc
Centre for Mathematics
University of Coimbra

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

BACKGROUND: The category of LOCALES

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

- MORPHISMS:

$$\begin{array}{c} L \\ \big\downarrow f \\ M \end{array}$$

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

- MORPHISMS: $L \xrightarrow{f} M$

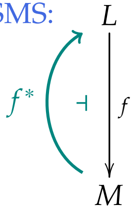
  - $f(\bigwedge S) = \bigwedge f[S]$

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

- MORPHISMS:



- $f(\bigwedge S) = \bigwedge f[S]$

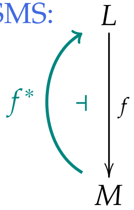
BACKGROUND: The category of LOCALES

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

- MORPHISMS:



- $f(\bigwedge S) = \bigwedge f[S]$
- $f(a) = 1 \Rightarrow a = 1$

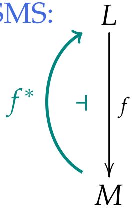
BACKGROUND: The category of LOCALES

- OBJECTS: locales = frames (=cHa)

complete lattices

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

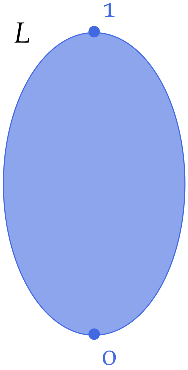
- MORPHISMS:



- $f(\bigwedge S) = \bigwedge f[S]$
- $f(a) = 1 \Rightarrow a = 1$
- $f(f^*(a) \to b) = a \to f(b)$

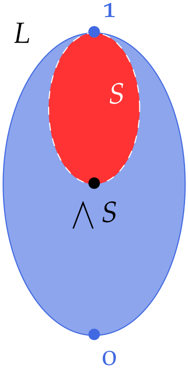
BACKGROUND: Sublocales

$S \subseteq L$ is a SUBLOCALE of L if:



$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

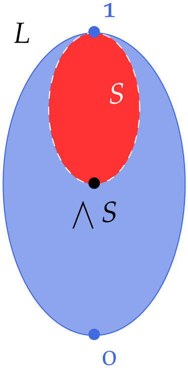


BACKGROUND: Sublocales

$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

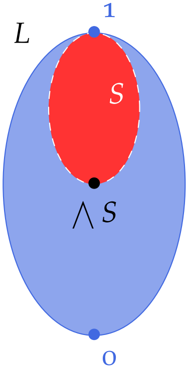
(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.



$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.



Motivation for the definition:

Proposition

$S \subseteq L$ is a sublocale iff the embedding $j_S : S \subseteq L$ is a localic map.

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap,$$

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L,$$

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\},$$

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\}, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\}, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition

This lattice is a **coframe** !

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\}, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition

This lattice is a **coframe** !

Special sublocales:

$a \in L, \quad \mathfrak{c}(a) = \uparrow a$ CLOSED

BACKGROUND: The lattice of sublocales

sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\}, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition

This lattice is a **coframe** !

Special sublocales:

$a \in L$, $\quad \mathfrak{c}(a) = \uparrow a$ CLOSED

$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN sublocales of L, ordered by $\subseteq$:

$$\bigwedge = \bigcap, \quad \mathbf{1} = L, \quad \mathbf{0} = \{1\}, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition

This lattice is a **coframe**!

Special sublocales:

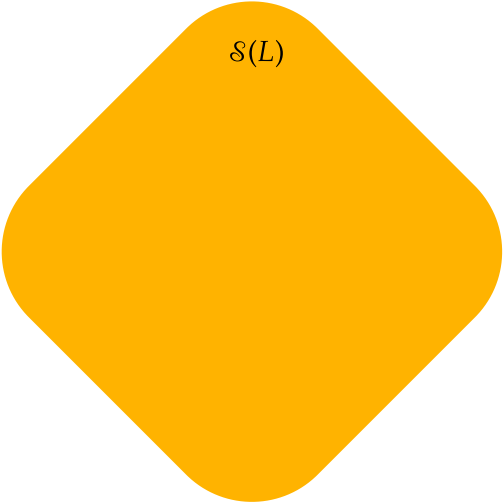
$a \in L$, $\quad \mathfrak{c}(a) = {\uparrow} a$ CLOSED

$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN

} complemented

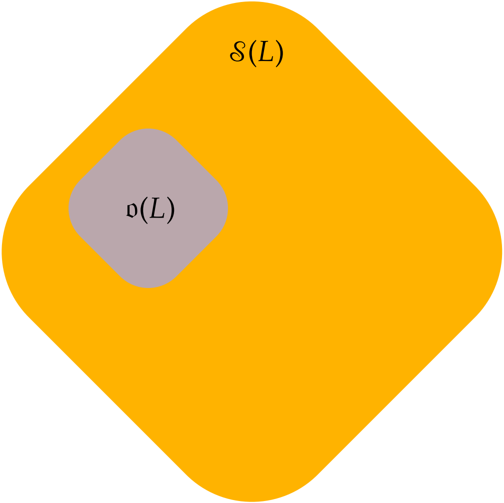
BACKGROUND: The lattice of sublocales

It is a coframe!



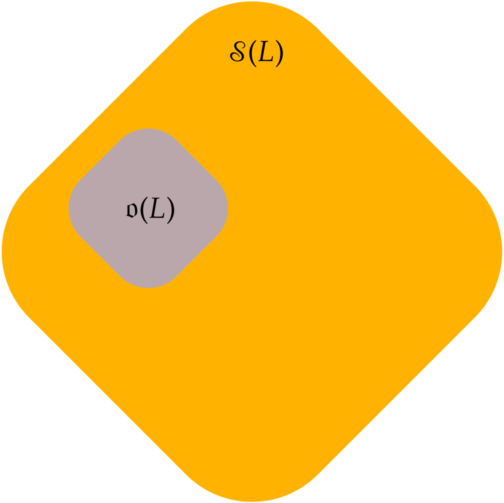
BACKGROUND: The lattice of sublocales

It is a coframe!



BACKGROUND: The lattice of sublocales

It is a coframe!

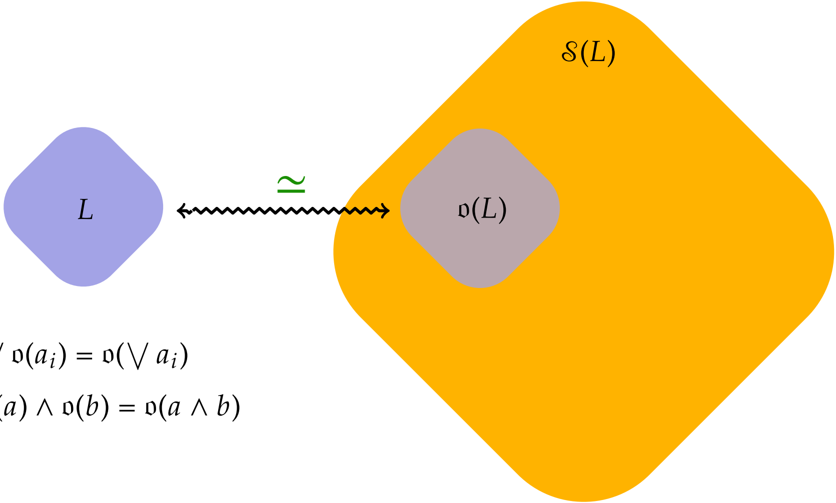


$\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$

$\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$

BACKGROUND: The lattice of sublocales

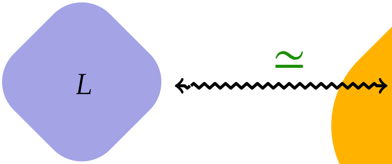
It is a coframe!

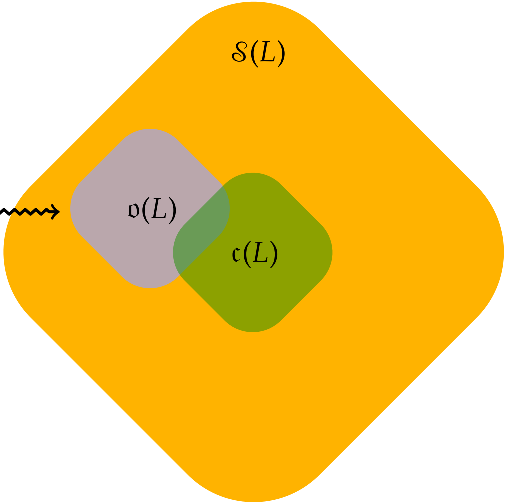


$\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$

$\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$

It is a coframe!

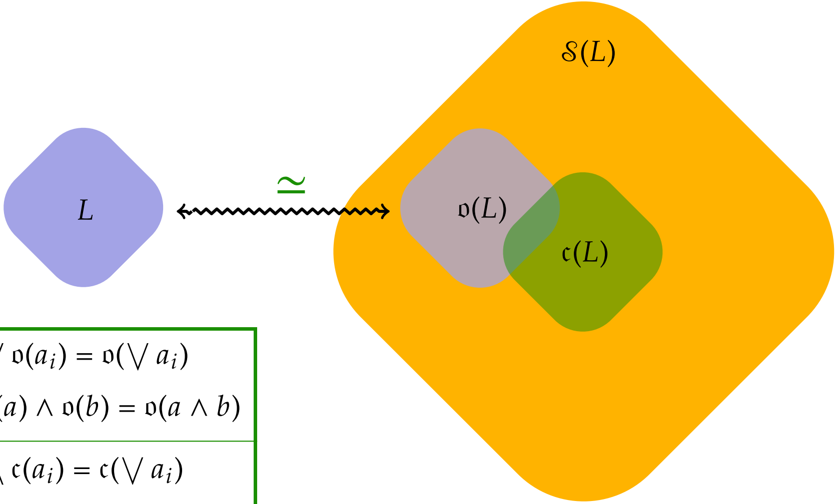




$$\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$$

$$\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

It is a coframe!



$\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i)$

$\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$

$\bigwedge \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee a_i)$

$\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv \forall a, b \ (a \neq b \Rightarrow \exists c\colon a \vee c = 1 \neq b \vee c)$

Subfit frame $\equiv \forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv$ $\forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

Spatial case: $L = \Omega(X)$ (some topological space X)

Subfit space $\equiv$ $\forall U \in \Omega(X), \forall x \in U, \exists y \in \overline{\{x\}} : \overline{\{y\}} \subseteq U.$

Subfit frame $\equiv \forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

Spatial case: $L = \Omega(X)$ (some topological space X)

Subfit space $\equiv \forall U \in \Omega(X), \forall x \in U, \exists y \in \overline{\{x\}} : \overline{\{y\}} \subseteq U.$

$\supsetneq$

T_1-spaces

$T_1 = \text{subfit} + T_D$

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv$ $\forall a, b \ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

Fit frame $\equiv$ $\forall a, b \ (a \neq b \Rightarrow \exists c : a \vee c = 1$ and $c \to b \neq b)$

(hereditary subfitness: every sublocale is subfit)

MOTIVATION: T_1 pointfreely (the ‘fitness club’)

Subfit frame $\equiv \forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\rotatebox{90}{$\subsetneq$}$ $\Leftrightarrow$ **Every open sublocale is a join of closed sublocales**

Fit frame $\equiv \forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1$ and $c \to b \neq b)$

(hereditary subfitness: every sublocale is subfit)

$\Leftrightarrow$ **Every closed sublocale is a meet of open sublocales**

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv$ $\forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

Fit frame $\equiv$ $\forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1$ and $c \to b \neq b)$

(hereditary subfitness: every sublocale is subfit)

$\Leftrightarrow$ Every closed sublocale is a meet of open sublocales

$\Leftrightarrow$ Every sublocale is a meet of open sublocales

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv$ $\forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\supsetneq$ $\qquad \Leftrightarrow$ **Every open sublocale is a join of closed sublocales**

Fit frame $\equiv$ $\forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1$ and $c \to b \neq b)$

(hereditary subfitness: every sublocale is subfit)

$\Leftrightarrow$ **Every closed sublocale is a meet of open sublocales**

$\Leftrightarrow$ Every sublocale is a meet of open sublocales

QUESTION: What about the dual property "every sublocale is a join of closed sublocales"?

MOTIVATION: T_1 pointfreely (the 'fitness club')

Subfit frame $\equiv \forall a, b\ (a \neq b \Rightarrow \exists c : a \vee c = 1 \neq b \vee c)$

$\Leftrightarrow$ Every open sublocale is a join of closed sublocales

ANSWER: characterizes the SCATTERED FRAMES

(the L with Boolean $\mathcal{S}(L)$)

- R.N. Ball, J.P., A. Pultr,
  On an aspect of scatteredness in the pointfree setting,
  Portugaliæ Math. 73 (2016) 139–152.

QUESTION: What about the dual property "every sublocale is a join of closed sublocales"?

To study the system

$$\mathcal{S}_{\mathfrak{c}}(L)$$

of all the sublocales that are joins of closed ones, for a general frame L.

$$\mathcal{S}_{\mathfrak{c}}(L) \hookrightarrow \mathcal{S}(L)$$

(sup-sublattice embedding)

- J. P., Aleš Pultr, A. Tozzi
  Joins of closed sublocales, *submitted*.

Basic fact about $\mathcal{S}_{\mathfrak{c}}(L)$

Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

Basic fact about $\mathcal{S}_{\mathfrak{c}}(L)$

Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

frame

$\emptyset \neq A \subseteq L$: $\quad \mathfrak{U}(L)$

$A = \uparrow A$

Basic fact about $\mathcal{S}_{\mathfrak{c}}(L)$

Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

$\emptyset \neq A \subseteq L$:
$A = \uparrow A$

frame

$$\mathfrak{U}(L) \xrightarrow{J} \mathcal{S}(L)$$

$$A \longmapsto \bigvee\{\mathfrak{c}(a) = \uparrow a \mid \mathfrak{c}(a) \subseteq A\}$$

Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

$\emptyset \neq A \subseteq L$:
$A = \uparrow A$

frame

$$\mathfrak{U}(L) \underset{U}{\overset{J}{\rightleftarrows}} \mathcal{S}(L) \quad (\bot)$$

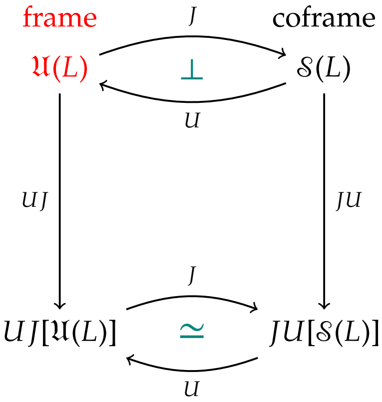
$$A \longmapsto \bigvee \{\mathfrak{c}(a) = \uparrow a \mid \mathfrak{c}(a) \subseteq A\}$$

$$\bigcup \{\uparrow a \mid \uparrow a \subseteq S\} \longleftarrow S$$

Basic fact about $\mathcal{S}_{\mathfrak{c}}(L)$

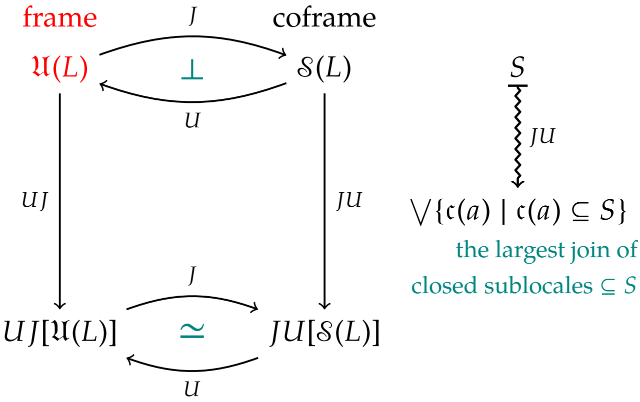
Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.



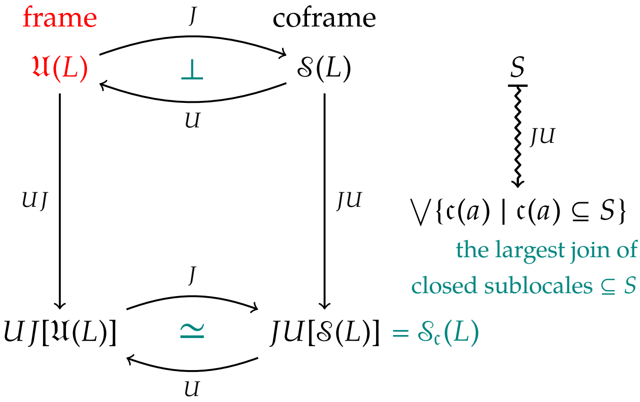
Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.



Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

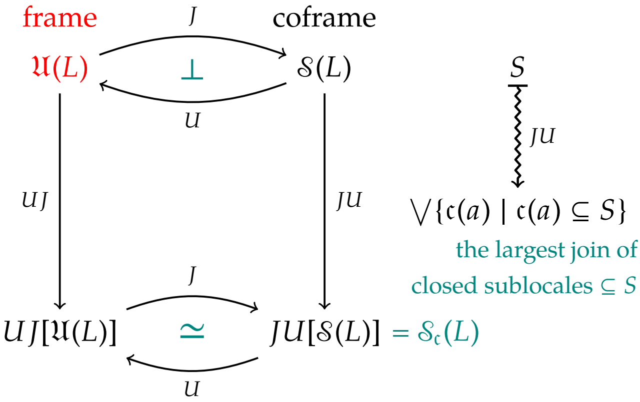


Proposition

For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

Lemma

UJ is a nucleus



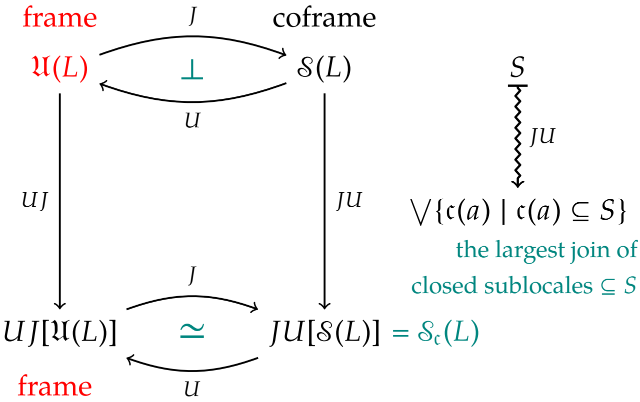
Basic fact about $\mathcal{S}_{\mathfrak{c}}(L)$

Proposition

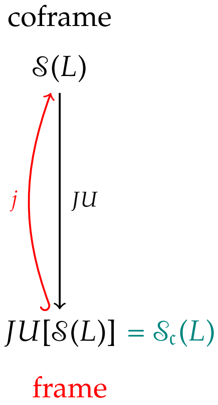
For every frame L, $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame.

Lemma

UJ is a nucleus



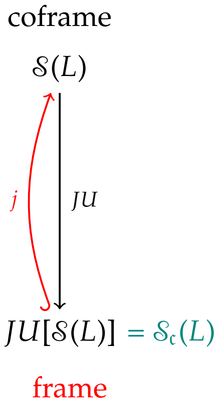
MAIN GOAL now



WHEN do we have more?

WHEN is

- $\mathcal{S}_{\mathfrak{c}}(L)$ a coframe?

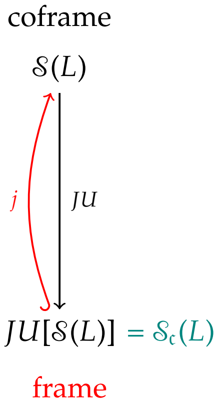


WHEN do we have more?

WHEN is

- $\mathcal{S}_\mathfrak{c}(L)$ a coframe?
- j an embedding of a subcolocale?

(and JU the corresponding conucleus)



The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_\mathfrak{c}(L)$ $\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_\mathfrak{c}(L)$.

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\mathfrak{c}(x) \smallsetminus T = \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))$$ (by o-codim.)

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\mathfrak{c}(x) \smallsetminus T = \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) \qquad \text{(by o-codim.)}$$

$$= \bigvee_i [\mathfrak{c}(x) \smallsetminus (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))]$$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\mathfrak{c}(x) \smallsetminus T = \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) \qquad \text{(by o-codim.)}$$

$$= \bigvee_i [\mathfrak{c}(x) \smallsetminus (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))] \qquad (-) \smallsetminus S \colon L^{\mathrm{op}} \to L \dashv (-) \smallsetminus S \colon L \to L^{\mathrm{op}}$$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \setminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \setminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\mathfrak{c}(x) \setminus T = \mathfrak{c}(x) \setminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) \qquad \text{(by o-codim.)}$$

$$= \bigvee_i [\mathfrak{c}(x) \setminus \underbrace{(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))}_{\text{complemented}}] \qquad (-) \setminus S \colon L^{op} \to L \dashv (-) \setminus S \colon L \to L^{op}$$

$$= \bigvee_i [\mathfrak{c}(x) \cap \mathfrak{c}(a_i) \cap \mathfrak{o}(b_i)]$$

The subfit case

L: subfit. $\quad \mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\begin{aligned}
\mathfrak{c}(x) \smallsetminus T &= \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) && \text{(by o-codim.)} \\
&= \bigvee_i [\mathfrak{c}(x) \smallsetminus \underbrace{(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))}_{\text{complemented}}] && (-) \smallsetminus S : L^{\mathrm{op}} \to L \dashv (-) \smallsetminus S : L \to L^{\mathrm{op}} \\
&= \bigvee_i [\mathfrak{c}(x) \cap \mathfrak{c}(a_i) \cap \mathfrak{o}(b_i)] \\
&= \bigvee_i [\mathfrak{c}(x \vee a_i) \cap \bigvee_j \mathfrak{c}(d^i_j)] && \textsc{Subfit}: \mathfrak{o}(b_i) = \bigvee_j \mathfrak{c}(d^i_j)
\end{aligned}$$

The subfit case

L: subfit. $\quad \mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 1

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $x \in L$, we have $\mathfrak{c}(x) \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$\mathfrak{c}(x) \smallsetminus T = \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) \qquad \text{(by o-codim.)}$$

$$= \bigvee_i [\mathfrak{c}(x) \smallsetminus \underbrace{(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))}_{\text{complemented}}] \qquad (-) \smallsetminus S \colon L^{\mathrm{op}} \to L \dashv (-) \smallsetminus S \colon L \to L^{\mathrm{op}}$$

$$= \bigvee_i [\mathfrak{c}(x) \cap \mathfrak{c}(a_i) \cap \mathfrak{o}(b_i)]$$

$$= \bigvee_i [\mathfrak{c}(x \vee a_i) \cap \bigvee_j \mathfrak{c}(d^i_j)] \qquad \text{SUBFIT: } \mathfrak{o}(b_i) = \bigvee_j \mathfrak{c}(d^i_j)$$

$$= \bigvee_{i,j} \mathfrak{c}(x \vee a_i \vee d^i_j) \in \mathcal{S}_{\mathfrak{c}}(L). \qquad \blacksquare$$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$ $\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 2

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $S \in \mathcal{S}_{\mathfrak{c}}(L)$, we have $S \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$ $\mathcal{S}(L)$ coframe=coHa:

$$(-) \smallsetminus S \dashv S \vee (-)$$

Lemma 2

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $S \in \mathcal{S}_{\mathfrak{c}}(L)$, we have $S \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$S \smallsetminus T = \left(\bigvee_i \mathfrak{c}(x_i)\right) \smallsetminus T$$

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Lemma 2

Let L be subfit. Then, for any $T \in \mathcal{S}(L)$ and $S \in \mathcal{S}_{\mathfrak{c}}(L)$, we have $S \smallsetminus T \in \mathcal{S}_{\mathfrak{c}}(L)$.

Proof:

$$S \smallsetminus T = (\bigvee_i \mathfrak{c}(x_i)) \smallsetminus T = \bigvee_i \underbrace{(\mathfrak{c}(x_i) \smallsetminus T)}_{\text{Lemma 1}} \in \mathcal{S}_{\mathfrak{c}}(L).$$

■

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Proposition

Let L be subfit. Then:

(1) $\mathcal{S}_{\mathfrak{c}}(L)$ is a subcolocale of $\mathcal{S}(L)$ (with JU the associated conucleus).

The subfit case

L: subfit. $\quad \mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$ $\quad \mathcal{S}(L)$ coframe=coHa:

$$(-) \smallsetminus S \dashv S \vee (-)$$

Proposition

Let L be subfit. Then:

(1) $\mathcal{S}_{\mathfrak{c}}(L)$ is a subcolocale of $\mathcal{S}(L)$ (with JU the associated conucleus).

(2) $\mathcal{S}_{\mathfrak{c}}(L)$ is a Boolean algebra.

(3) $JU\colon \mathcal{S}(L) \to \mathcal{S}_{\mathfrak{c}}(L)$, $S \mapsto L \smallsetminus (L \smallsetminus S)$, is the Booleanization of $\mathcal{S}(L)$.

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \smallsetminus S \dashv S \vee (-)$

Proposition

Let L be subfit. Then:

(1) $\mathcal{S}_{\mathfrak{c}}(L)$ is a subcolocale of $\mathcal{S}(L)$ (with JU the associated conucleus).

(2) $\mathcal{S}_{\mathfrak{c}}(L)$ is a Boolean algebra.

(3) $JU\colon \mathcal{S}(L) \to \mathcal{S}_{\mathfrak{c}}(L)$, $S \mapsto L \smallsetminus (L \smallsetminus S)$, is the Booleanization of $\mathcal{S}(L)$.

In fact, we have MORE!

The subfit case

L: subfit. $\mathfrak{o}(L) \subseteq \mathcal{S}_{\mathfrak{c}}(L)$

$\mathcal{S}(L)$ coframe=coHa:

$(-) \setminus S \dashv S \vee (-)$

Theorem

~~Let L be subfit. Then:~~ TFAE for any frame L:

(1) $\mathcal{S}_{\mathfrak{c}}(L)$ is a subcolocale of $\mathcal{S}(L)$ (with JU the associated conucleus).

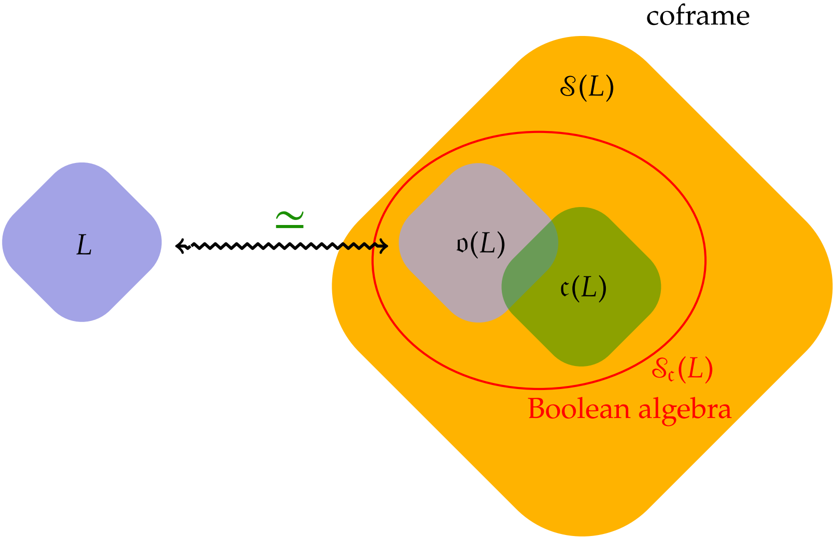
(2) $\mathcal{S}_{\mathfrak{c}}(L)$ is a Boolean algebra.

(3) $JU : \mathcal{S}(L) \to \mathcal{S}_{\mathfrak{c}}(L)$, $S \mapsto L \setminus (L \setminus S)$, is the Booleanization of $\mathcal{S}(L)$.

(4) L is subfit.

In fact, we have MORE! 🙂

CONCLUSION: in the subfit case we have a Boolean extension of L



The spatial case

$L = \Omega(X)$ some space X

NOTE:

- L may have sublocales that are not spatial!
- Even a spatial sublocale of L is not necessarily $\Omega(Y)$ for a subspace $Y \subseteq X$.

The spatial case

$L = \Omega(X)$ some space X

NOTE:

- L may have sublocales that are not spatial!
- Even a spatial sublocale of L is not necessarily $\Omega(Y)$ for a subspace $Y \subseteq X$.

those that are: "spatially induced sublocales"

The spatial case

$L = \Omega(X)$ some space X

NOTE:

- L may have sublocales that are not spatial!
- Even a spatial sublocale of L is not necessarily $\Omega(Y)$ for a subspace $Y \subseteq X$.

those that are: "spatially induced sublocales"

X is T_1 "L is T_1-spatial"

$\mathcal{S}_{\mathfrak{c}}(\Omega(X)) = \{\text{induced sublocales of } \Omega(X)\}$

Booleanization of $\mathcal{S}(\Omega(X))$: precisely $\mathcal{P}(X)$.

(the classical subspaces of X)

$L = \Omega(X)$ some space X

NOTE:
- L may have sublocales that are not spatial!
- Even a spatial sublocale of L is not necessarily $\Omega(Y)$ for a subspace $Y \subseteq X$.

those that are: "spatially induced sublocales"

X is subfit, not T_1

We have still the theorem of course BUT

$\mathcal{S}_\mathfrak{c}(\Omega(X))$ is not any more the system $\mathcal{P}(X)$ of all subspaces.

[lack of T_D: subspaces are not perfectly represented by spatial sublocales]

APPLICATION: real-valued functions on L

frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ ring $\mathsf{F}(L)$

$$\boxed{\mathsf{F}(L) = \mathsf{C}(\mathcal{S}(L)^{op})}$$

- J. Gutiérrez García, T. Kubiak, J. P.,
  Localic real functions: a general setting,
  J. Pure Appl. Algebra 213 (2009) 1064-1074.

APPLICATION: real-valued functions on L

frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ ring $\mathsf{F}(L)$

$$\boxed{\mathsf{F}(L) = \mathsf{C}(\mathcal{S}(L)^{op})}$$

- J. Gutiérrez García, T. Kubiak, J. P.,
  Localic real functions: a general setting,
  J. Pure Appl. Algebra 213 (2009) 1064-1074.

• very expedient mimicking of the classical theory:

generalizations of function insertion theorems
function extension theorems, etc.

Disadvantages:

- $\mathcal{S}(L)^{op}$ can be very big; in particular for spaces, $\mathcal{S}(\Omega(X))^{op}$ is typically much bigger than $\mathcal{P}(X)$.

APPLICATION: real-valued functions on L

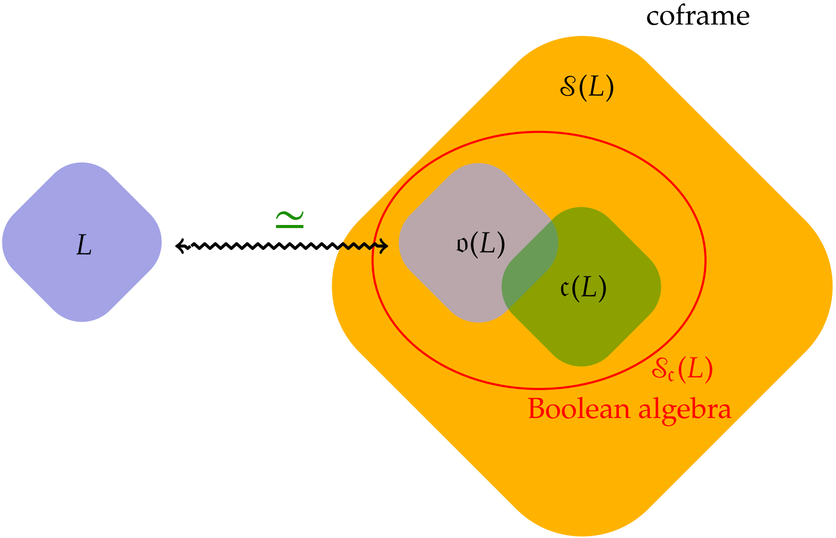
DISADVANTAGES:

- $\mathcal{S}(L)^{op}$ can be very big; in particular for spaces, $\mathcal{S}(\Omega(X))^{op}$ is typically much bigger than $\mathcal{P}(X)$.
- The theory is not quite conservative. When applied to semicontinuity in classical spaces it is satisfactory, but the general not necessarily classical functions are represented only by analogy.

DISADVANTAGES:

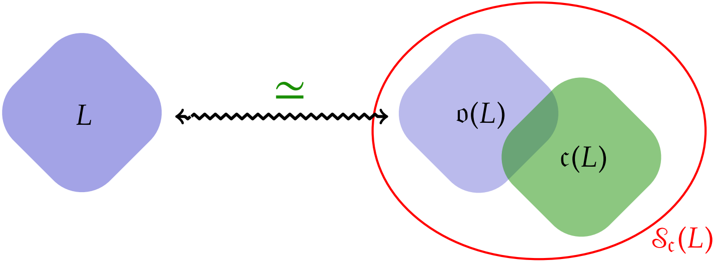
- $\mathcal{S}(L)^{op}$ can be very big; in particular for spaces, $\mathcal{S}(\Omega(X))^{op}$ is typically much bigger than $\mathcal{P}(X)$.
- The theory is not quite conservative. When applied to semicontinuity in classical spaces it is satisfactory, but the general not necessarily classical functions are represented only by analogy.
- The construction is not idempotent, that is, $\mathcal{S}(\mathcal{S}(L)^{op})^{op}$ is typically bigger than $\mathcal{S}(L)^{op}$, as if the discontinuous functions were not discontinuous enough, and needed a further extension to get a representation of "more discontinuous ones" (and again and again).

APPLICATION: real-valued functions on L

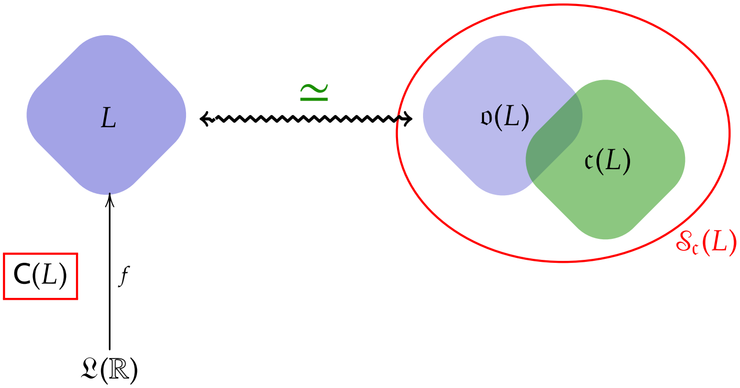


APPLICATION: real-valued functions on L

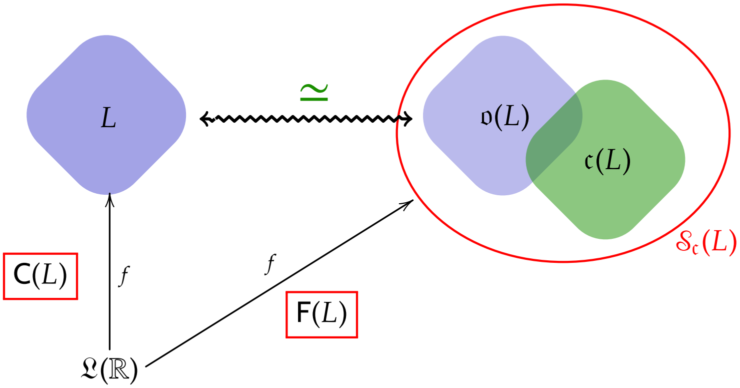
New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.



New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.

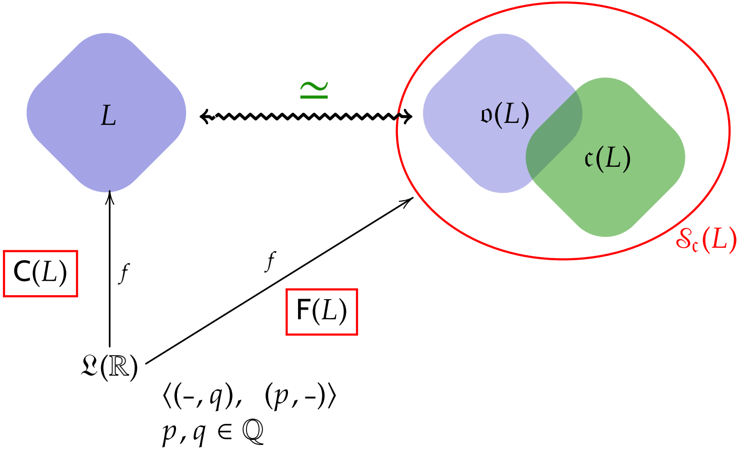


New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.

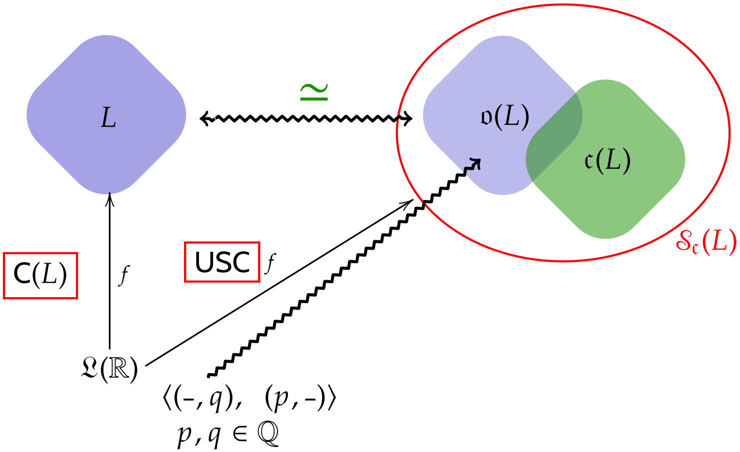


APPLICATION: real-valued functions on L

New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.

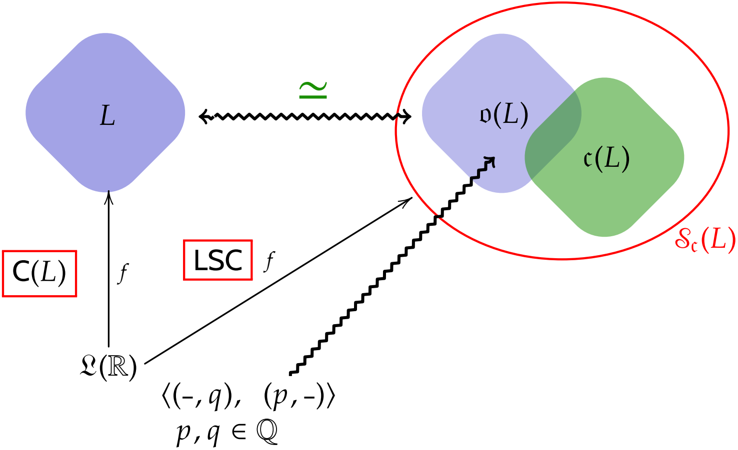


New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.

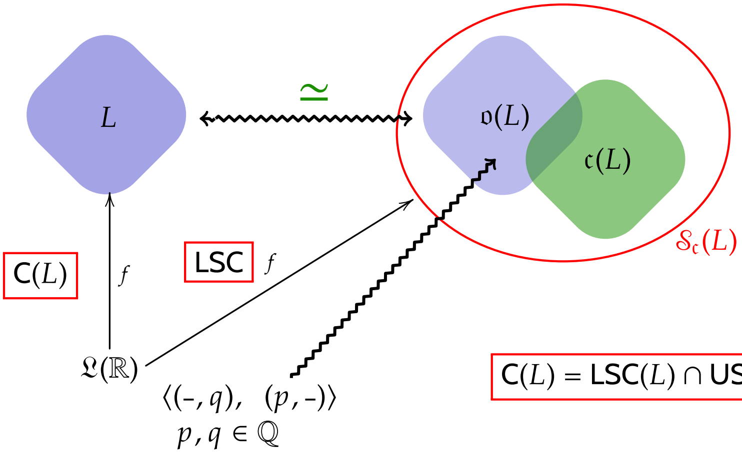


APPLICATION: real-valued functions on L

New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.



New approach: use the frame $\mathcal{S}_{\mathfrak{c}}(L)$ instead of $\mathcal{S}(L)^{op}$.

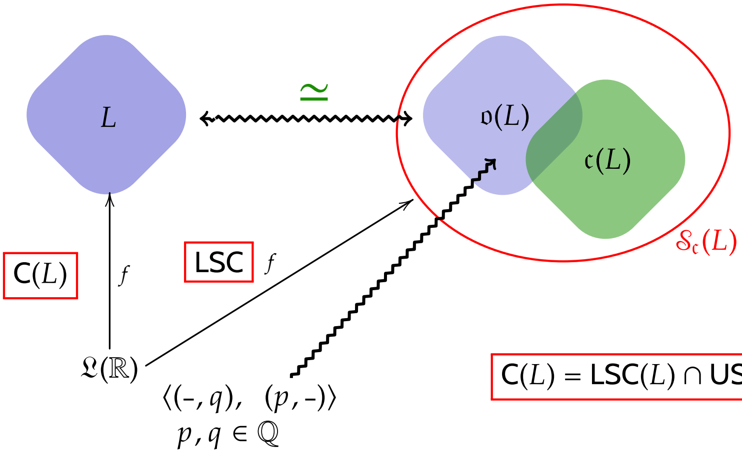


C(L) = LSC(L) ∩ USC(L)

APPLICATION: real-valued functions on L

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.



$\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L)$

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.

- J. P., Aleš Pultr,
  A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Math.*, to appear.

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.

- J. P., Aleš Pultr,
  A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Math.*, to appear.

ADVANTAGES:

- If $L = \Omega(X)$ is T_1-spatial then $\mathcal{S}_{\mathfrak{c}}(L) \cong \mathcal{P}(X)$.

APPLICATION: real-valued functions on L

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.

- J. P., Aleš Pultr,
  A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Math.*, to appear.

ADVANTAGES:

- If $L = \Omega(X)$ is T_1-spatial then $\mathcal{S}_{\mathfrak{c}}(L) \cong \mathcal{P}(X)$.
- If L is subfit (and we are mostly concerned with the much stronger regularity) then $\mathcal{S}_{\mathfrak{c}}(L)$ is Boolean and hence can be viewed as a "discrete cover" of L.

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.

- J. P., Aleš Pultr,
  A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Math.*, to appear.

ADVANTAGES:

- If $L = \Omega(X)$ is T_1-spatial then $\mathcal{S}_\mathfrak{c}(L) \cong \mathcal{P}(X)$.
- If L is subfit (and we are mostly concerned with the much stronger regularity) then $\mathcal{S}_\mathfrak{c}(L)$ is Boolean and hence can be viewed as a "discrete cover" of L.
- The theory is now conservative for T_1 spaces, already starting with the representation of general mappings.

Proposition

If $L = \Omega(X)$ is T_1-spatial, then $\mathsf{LSC}(L) \simeq \mathsf{LSC}(X)$ and $\mathsf{USC}(L) \simeq \mathsf{USC}(X)$.

- J. P., Aleš Pultr,
  A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Math.*, to appear.

ADVANTAGES:

- If $L = \Omega(X)$ is T_1-spatial then $\mathcal{S}_{\mathfrak{c}}(L) \cong \mathcal{P}(X)$.
- If L is subfit (and we are mostly concerned with the much stronger regularity) then $\mathcal{S}_{\mathfrak{c}}(L)$ is Boolean and hence can be viewed as a "discrete cover" of L.
- The theory is now conservative for T_1 spaces, already starting with the representation of general mappings.
- $\mathcal{S}_{\mathfrak{c}}(\mathcal{S}_{\mathfrak{c}}(L)) \cong \mathcal{S}_{\mathfrak{c}}(L)$ and hence the discretization is made once for ever.