Perfectness in Frames

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— joint work with J. Gutiérrez García (UPV-EHU, Bilbao, Spain)
TFAE for a space $X$: (1) $X$ is perfectly normal (= perfect + normal).

(2) $f \leq g$ \quad \Rightarrow \quad \exists h \in C(X) : f \leq h \leq g \quad \text{and}$

$$f(x) < h(x) < g(x) \quad \text{whenever} \quad f(x) < g(x).$$
MOTIVATION: MICHAEL’S STRICT INSERTION

Ann. Math. 63 (1956)

TFAE for a space $X$: (1) $X$ is perfectly normal ( = perfect + normal).

(2) $\underbrace{f}_{\text{USC}} \leq \underbrace{g}_{\text{LSC}} \quad \Rightarrow \quad \exists h \in C(X) : f \leq h \leq g$ and

\[ f(x) < h(x) < g(x) \quad \text{whenever} \quad f(x) < g(x). \]
TFAE for a locale \( L \): (1) \( L \) is perfectly normal.

(2) \( \underbrace{f}_{\text{USC}} \leq \underbrace{g}_{\text{LSC}} \Rightarrow \exists h \in C(L) : f \leq h \leq g \) and

\[ \nu(f, h) = \nu(h, g) = \nu(f, g). \]

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\nu(f, g) := \bigvee_{p \in \mathbb{Q}} (f(-, p) \land g(p, -))
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\Rightarrow \quad \exists h \in C(L) : f \leq h \leq g
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\end{array}
\]
\begin{align*}
\nu(f, h) &= \nu(h, g) = \nu(f, g).
\end{align*}

\[
\nu(f, g) := \bigvee_{p \in \mathbb{Q}} (f(\neg, p) \land g(p, \neg)) \quad f < g \equiv \nu(f, g) = 1
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- PERFECTNESS?
TFAE for a locale \( L \): (1) \( L \) is perfectly normal.

\[
(2) \quad f \leq g \quad \Rightarrow \quad \exists h \in \text{C}(L) : f \leq h \leq g \quad \text{and} \quad \uptau(f, h) = \uptau(h, g) = \uptau(f, g).
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- **PERFECTNESS?**

Charalambous 1974, Gilmour 1984 (\( \sigma \)-frames):
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f < g \equiv \nu(f,g) = 1
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- PERFECTNESS?

Charalambous 1974, Gilmour 1984 ($\sigma$-frames):

\[
\forall a \in L \ \exists (a_n)_\mathbb{N} \subseteq L : a = \bigvee a_n \quad \text{and} \quad a_n < a \ \forall n.
\]
• to study perfectness further.
AIMS (work in progress)

- to study perfectness further.

- to understand better the role of perfectness in insertion of functions.
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- to study perfectness further.
- to understand better the role of perfectness in insertion of functions.
- to unify several insertion results.
Every closed set is a $G_\delta$-set
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\[ (= \bigcap_{n \in \mathbb{N}} U_n ) \]
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Every open set is an $F_\sigma$-set \( (= \bigcup_{n \in \mathbb{N}} F_n) \)
PERFECT SPACES

Heath & Michael 1971

A. Every closed set is a $G_\delta$-set \( (= \bigcap_{n \in \mathbb{N}} U_n) \)

B. Every open set is an $F_\sigma$-set \( (= \bigcup_{n \in \mathbb{N}} F_n) \)

PERFECTLY NORMAL SPACES = PERFECT + NORMAL
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PERFECTLY NORMAL SPACES = PERFECT + NORMAL

\[ \forall U \in \mathcal{O}(X) \exists (U_n)_{n \in \mathbb{N}} \subseteq \mathcal{O}(X): U = \bigcup_{n \in \mathbb{N}} U_n \text{ and } \overline{U_n} \subseteq U \ \forall n. \]
the frame of sublocales

\[ S(L) \]
the frame of sublocales

\[ S(L) \]

\[ cL \]

\[ L \]
BACKGROUND AND NOTATION

The frame of sublocales

\[ S(L) \]

\[ \text{cL} \]

\[ cL := \{ c(a) \mid a \in L \} \]

\[ \bigvee_{i \in I} c(a_i) = c(\bigvee_{i \in I} a_i) \]

\[ c(a) \land c(b) = c(a \land b) \]
BACKGROUND AND NOTATION

The frame of sublocales

- $L$
- $cL$
- $B(S(L))$
- $S(L)$
BACKGROUND AND NOTATION

The frame of sublocales

\[ S(L) \]

\[ \mathcal{c}L \]

\[ \mathcal{o}L \]

\[ B(S(L)) \]

\[ \mathcal{o}L := \{ \mathcal{o}(a) \mid a \in L \} \]

\[ \bigwedge_{i \in I} \mathcal{o}(a_i) = \mathcal{o}(\bigvee_{i \in I} a_i) \]

\[ \mathcal{o}(a) \lor \mathcal{o}(b) = \mathcal{o}(a \land b) \]
Every closed sublocale is a $G_\delta$-sublocale:
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$$\forall a \in L \ \exists (a_n)_N \subseteq L : c(a) = \bigvee_{n \in \mathbb{N}} o(a_n)$$
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Every closed sublocale is a $G_\delta$-sublocale:

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COUNTER-EXAMPLE (a spatial one): the cofinite topology on $\mathbb{N}$.

($T_1$-space, subfit frame, not fit)
Perfectness in Frames

A. Every closed sublocale is a $G_\delta$-sublocale:

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Counter-Example (a spatial one): the cofinite topology on $\mathbb{N}$.

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PERFECTNESS IN LOC

A. Every closed sublocale is a $G_\delta$-sublocale:

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B. Every open sublocale is an $F_\sigma$-sublocale:

$$\forall a \in L \exists (a_n)_\mathbb{N} \subseteq L : \sigma(a) = \bigwedge_{n \in \mathbb{N}} c(a_n)$$

COUNTER-EXAMPLE (a spatial one): the cofinite topology on $\mathbb{N}$.

($T_1$-space, subfit frame, not fit)
SOME RESULTS

coop perfect $\rightarrow$ fit
SOME RESULTS

co-perfect $\rightarrow$ fit

$\downarrow$

perfect $\rightarrow$ subfit
SOME RESULTS

- to which extent (co-)perfect locales model $G_\delta$-spaces (inside $T_0$)?

\[
\begin{array}{ccc}
\text{co-perfect} & \longrightarrow & \text{fit} \\
\downarrow & & \downarrow \\
\text{perfect} & \longrightarrow & \text{subfit}
\end{array}
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• to which extent (co-)perfect locales model $G_\delta$-spaces (inside $T_0$)?

\textbf{TFAE for a $T_0$-space $X$:}

1. $X$ is perfect (=co-perfect).
SOME RESULTS

co-perfect $\rightarrow$ fit

perfect $\rightarrow$ subfit

• to which extent (co-)perfect locales model $G_\delta$-spaces (inside $T_0$)?

TFAE for a $T_0$-space $X$:

1. $X$ is perfect (=co-perfect).

2. $X$ is $T_D$ and the frame $\mathcal{O}X$ is co-perfect.
SOME RESULTS

- to which extent (co-)perfect locales model $G_\delta$-spaces (inside $T_0$)?

**TFAE for a $T_0$-space $X$:**

1. $X$ is perfect (=co-perfect).
2. $X$ is $T_D$ and the frame $O X$ is co-perfect.
3. $X$ is $T_1$ and the frame $O X$ is co-perfect.
PERFECT NORMALITY IN LOC = PERFECT + NORMAL
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PROPOSITION. TFAE for a normal frame $L$: 
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1. $L$ is perfect
PERFECT NORMALITY IN LOC  =  PERFECT + NORMAL

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PROPOSITION. TFAE for a normal frame $L$:

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3. $\forall a \in L \ \exists (a_n)_N \subseteq L : a = \bigvee a_n$ and $a_n < a \ \forall n.$

original GILMOUR’s condition
PROPOSITION. TFAE for a normal frame $L$:

1. $L$ is **perfect** (≡ with regular elements $a_n$.)

2. $L$ is **co-perfect** (≡ with regular elements $a_n$.)

3. $\forall a \in L \; \exists (a_n)_N \subseteq L : a = \sqrt{a_n}$ and $a_n < a \; \forall n.$

original GILMOUR’s condition
PROPOSITION. TFAE for a normal frame $L$:

1. $L$ is perfect \( (\equiv \text{with regular elements } a_n. ) \)

2. $L$ is co-perfect \( (\equiv \text{with regular elements } a_n. ) \)

3. $\forall a \in L \; \exists (a_n)_N \subseteq L : a = \bigvee a_n \text{ and } a_n < a \; \forall n.$

original GILMOUR’s condition \( (\text{each } a \text{ is } G_\delta \text{-regular}) \)
THE ROLE OF NORMALITY IN INSERTION: (weak) insertion

**THEOREM.** TFAE for a frame \( L \):

1. \( L \) is normal.

2. \( \underbrace{f \leq g}_{\text{USC \ LSC}} \implies \exists h \in C(L): f \leq h \leq g. \)

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PERFECTNESS

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THE ROLE OF NORMALITY IN INSERTION: (weak) insertion

PERFECTNESS double

THEOREM. TFAE for a frame $L$:

1. $L$ is normal.

2. $f \leq g$  
   \hspace{1cm} \Rightarrow \hspace{1cm} \exists \ h \in C(L) : \ f \leq h \leq g$.

THEOREM. TFAE for a frame $L$:

1. $L$ is normal. perfect

2. \[
\begin{align*}
&f \leq g \\
&\text{USC} \quad \text{LSC}
\end{align*}
\implies \exists \begin{align*}
&\hat{f} \\
&\text{USC} \quad \text{LSC}
\end{align*}, \begin{align*}
&\hat{g} \\
&\text{USC} \quad \text{LSC}
\end{align*} : f \leq \hat{f} \leq \hat{g} \leq g
\quad \text{and}
\]

\[\iota(f, g) = \iota(f, \hat{f}) = \iota(f, \hat{g}) = \iota(\hat{g}, g).\]
THEOREM. TFAE for a frame $L$:

1. $L$ is normal. perfect

2. $f \leq g$ USCLSC $\Rightarrow \exists \ \hat{f}, \ \hat{g}$ USCLSC : $f \leq \hat{f} \leq \hat{g} \leq g$ and

$$\nu(f, g) = \nu(f, \hat{f}) = \nu(\hat{f}, \hat{g}) = \nu(\hat{g}, g).$$

In particular: $f < g \Rightarrow f < \hat{f} < \hat{g} < g$
**THEOREM.** TFAE for a frame $L$:

1. $L$ is normal + perfect

2. \[
\begin{aligned}
&f \leq g \\
&\text{USC} & \leq & \text{LSC}
\end{aligned}
\] \implies \exists \begin{aligned}
&\hat{f} \\
&\text{USC}
\end{aligned}, \begin{aligned}
&\hat{g} \\
&\text{LSC}
\end{aligned}: f \leq \hat{f} \leq \hat{g} \leq g

and

\[
\nu(f, g) = \nu(f, \hat{f}) = \nu(\hat{f}, \hat{g}) = \nu(\hat{g}, g).
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1. $L$ is normal + perfect

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\underbrace{f}_{\text{USC}} \leq \underbrace{g}_{\text{LSC}} \quad \implies \quad \exists \underbrace{\hat{f}}_{\text{USC}}, \underbrace{\hat{g}}_{\text{LSC}} : f \leq \hat{f} \leq \hat{g} \leq g
\]

In particular: $f < g \implies f < \hat{f} < \hat{g} < g$

and thus $\Rightarrow f < h < g$
A UNIFIED APPROACH: go to $S(L)$

the frame of sublocales
A UNIFIED APPROACH: $\mathcal{A}$-perfectness

the frame of sublocales

$S(L)$

$B(S(L))$

$L$

$\mathcal{A} \subseteq \mathcal{A}^c$
A UNIFIED APPROACH: $\mathcal{A}$-perfectness

the frame of sublocales

$L$ is $\mathcal{A}$-perfect $\equiv \forall A \in \mathcal{A}^c$

$A = \bigwedge_{n \in \mathbb{N}} A_n$ (where each $A_n \in \mathcal{A}$)
A UNIFIED APPROACH: $\mathcal{A}$-normality

the frame of sublocales

$L$ is $\mathcal{A}$-normal $\equiv$ For any $A, B \in \mathcal{A}$,

\[ A \lor B = 1 \Rightarrow \exists U, V \in \mathcal{A}: U \land V = 0, \ A \lor U = 1 = B \lor V. \]
$f : \mathbb{L}(\mathbb{R}) \rightarrow \mathcal{S}(L)$

\[ f \in \text{USC}(L) \iff \forall p < q \; \exists F_{p,q} \in cL : \; f(-,p) \leq F_{p,q} \leq f(-,q). \]
$\mathcal{A}$-SEMICONtinuity and $\mathcal{A}$-CONTINUITY

\[ f : \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L) \]

\[ f \in \text{USC}(L) \iff \forall p < q \exists F_{p,q} \in \mathcal{c}L : f(\mathcal{L}, p) \leq F_{p,q} \leq f(\mathcal{L}, q). \]

\[ \mathcal{A}\text{-USC}(L) \equiv \forall p < q \exists F_{p,q} \in \mathcal{A} : f(\mathcal{L}, p) \leq F_{p,q} \leq f(\mathcal{L}, q). \]
$\mathcal{A}$-semicontinuity and $\mathcal{A}$-continuity

$f : \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)$

$f \in \text{USC}(L) \iff \forall p < q \ \exists F_{p,q} \in cL : f(\neg, p) \leq F_{p,q} \leq f(\neg, q)$.

$\mathcal{A}$-USC($L$) $\equiv \forall p < q \ \exists F_{p,q} \in \mathcal{A} : f(\neg, p) \leq F_{p,q} \leq f(\neg, q)$.

$\mathcal{A}$-LSC($L$) $\equiv \forall p < q \ \exists F_{p,q} \in \mathcal{A} : f(q, \neg) \leq F_{p,q} \leq f(p, \neg)$. 
\[ f \in \text{USC}(L) \iff \forall p < q \ \exists F_{p,q} \in cL : f(-,p) \leq F_{p,q} \leq f(-,q). \]

\[ \mathcal{A}\text{-USC}(L) \equiv \forall p < q \ \exists F_{p,q} \in \mathcal{A} : f(-,p) \leq F_{p,q} \leq f(-,q). \]

\[ \mathcal{A}\text{-LSC}(L) \equiv \forall p < q \ \exists F_{p,q} \in \mathcal{A} : f(q,-) \leq F_{p,q} \leq f(p,-). \]

\[ \mathcal{A}\text{-C}(L) = \mathcal{A}\text{-LSC}(L) \cap \mathcal{A}\text{-USC}(L) \]
$f \in \text{USC}(L) \iff \forall p < q \exists F_{p,q} \in cL : f(\_, p) \leq F_{p,q} \leq f(\_, q)$. 

$\mathcal{A}$-$\text{USC}(L) \equiv \forall p < q \exists F_{p,q} \in \mathcal{A} : f(\_, p) \leq F_{p,q} \leq f(\_, q)$. 

$\mathcal{A}$-$\text{LSC}(L) \equiv \forall p < q \exists F_{p,q} \in \mathcal{A} : f(q, \_) \leq F_{p,q} \leq f(p, \_)$. 

$\mathcal{A}$-$\text{C}(L) = \mathcal{A}$-$\text{LSC}(L) \cap \mathcal{A}$-$\text{USC}(L)$

Clearly: 

$f$ is upper $\mathcal{A}$-semicontinuous iff it is lower $\mathcal{A}^c$-semicon.

$f$ is $\mathcal{A}^c$-continuous iff it is $\mathcal{A}$-continuous.
RESULTS: relative versions

$\mathcal{A}$-perfect normality = $\mathcal{A}$-perfectness + $\mathcal{A}$-normality

\[ \iff \]

(Weak) insertion

for \[ f \leq g \]
\[ \mathcal{A} - \text{USC} \quad \mathcal{A} - \text{LSC} \]
RESULTS: relative versions

\[ \mathcal{A}\text{-perfect normality} = \mathcal{A}\text{-perfectness} + \mathcal{A}\text{-normality} \]

\[ \Leftrightarrow \quad \text{under mild conditions on } \mathcal{A} \]

\[ \text{Strict insertion} = \text{Double insertion} + (\text{Weak) insertion} \]

\[ \text{for } f \preceq g \]

\[ \mathcal{A} - \text{USC} \quad \mathcal{A} - \text{LSC} \]
RESULTS: relative versions

\( \mathcal{A}^c \)-perfect normality = \( \mathcal{A}^c \)-perfectness + \( \mathcal{A}^c \)-normality

\( \mathcal{A}^c \)-perfectness

\( \mathcal{A}^c \)-normality

Strict insertion = Double insertion + (Weak) insertion

under mild conditions on \( \mathcal{A} \)

for

\( f \preceq g \)

\( \mathcal{A} \)-USC \( \mathcal{A} \)-LSC
RESULTS: relative versions

\[ \mathcal{A}^c \text{-perfect normality} = \mathcal{A}^c \text{-perfectness} + \mathcal{A}^c \text{-normality} \]

\[ \text{Strict insertion} = \text{Double insertion} + \text{(Weak) insertion} \]

\[ \text{for } \begin{cases} f \leq g \end{cases} \]

\[ \mathcal{A}^c \text{-USC} \quad \mathcal{A}^c \text{-LSC} \]

\[ \mathcal{A}^c \text{-LSC} \quad \mathcal{A}^c \text{-USC} \]
RESULTS: relative versions

\[ A^C \text{-perfect normality} = A^C \text{-perfectness} + A^C \text{-normality} \]

Strict insertion = Double insertion + (Weak) insertion

\[ \text{for } f \leq g \]
\[ A^C \text{-USC} \quad A^C \text{-LSC} \]

\[ A^C \text{-USC} \quad A^C \text{-LSC} \]
$A_1 = \{ c(a) : a \in L \}$
EXAMPLES

- $\mathcal{A}_1$-normal frames: normal
- $\mathcal{A}_1^c$-normal frames: extremally disconnected

$\mathcal{A}_1 = \{ c(a) : a \in L \}$
• $\mathcal{A}_1$-normal frames: normal
• $\mathcal{A}_1^c$-normal frames: extremally disconnected
• $\mathcal{A}_1$-perfect frames: perfect
• $\mathcal{A}_1^c$-perfect frames: Boolean

$\mathcal{A}_1 = \{c(a) : a \in L\}$
EXAMPLES

- $\mathcal{A}_1$-normal frames: normal
- $\mathcal{A}_1^c$-normal frames: extremally disconnected
- $\mathcal{A}_1$-perfect frames: perfect
- $\mathcal{A}_1^c$-perfect frames: Boolean
- upper $\mathcal{A}_1$-semicontinuous functions: upper semicontinuous
- lower $\mathcal{A}_1$-semicontinuous functions: lower semicontinuous
- $\mathcal{A}_1$-continuous functions: continuous

$\mathcal{A}_1 = \{c(a) : a \in L\}$
Examples

\[ A_2 = \{c(a^*): a \in L\} \]
**EXAMPLES**

- $\mathcal{A}_2$-normal frames: mildly normal
- $\mathcal{A}_2^c$-normal frames: extremally disconnected

- $\mathcal{A}_2$-perfectly normal frames: pm-normal = OZ
- $\mathcal{A}_2^c$-perfectly normal frames: extremally disconnected

- upper $\mathcal{A}_2$-semicontinuous functions: normal upper semicontinuous
- lower $\mathcal{A}_2$-semicontinuous functions: normal lower semicontinuous
- $\mathcal{A}_2$-continuous functions: normal continuous

$$ (f^\circ)^- = f \quad | \quad (f^-)^\circ = f $$

Dilworth 1950
EXAMPLES

\[ A_3 = \{c(coz f) : f \in C(L)\} \]
EXAMPLES

- $\mathcal{A}_3$-normal frames: all frames
- $\mathcal{A}_3^c$-normal frames: $F$-frames
- $\mathcal{A}_3$-perfectly normal frames: all frames
- $\mathcal{A}_3^c$-perfectly normal frames: $P$-frames
- upper $\mathcal{A}_3$-semicontinuous functions: zero upper semicontinuous
- lower $\mathcal{A}_3$-semicontinuous functions: zero lower semicontinuous
- $\mathcal{A}_3$-continuous functions: zero continuous

$\mathcal{A}_3 = \{c(\text{coz} f) : f \in C(L)\}$

Stone 1949
EXAMPLES

\[ A_4 = \{ c(a) : a \text{ regular } G_\delta \} \]
EXAMPLES

\[ \mathcal{A}_4 = \{ c(a) : a \text{ regular } G_\delta \} \]

- \( \mathcal{A}_4 \)-normal frames: \( \delta \)-normal
- \( \mathcal{A}_4^c \)-normal frames: \( \delta \)-extremally disconnected

- \( \mathcal{A}_4 \)-perfectly normal frames: ???
- \( \mathcal{A}_4^c \)-perfectly normal frames: ???

- upper \( \mathcal{A}_4 \)-semicontinuous functions: regular upper semicontinuous
- lower \( \mathcal{A}_4 \)-semicontinuous functions: regular lower semicontinuous
- \( \mathcal{A}_4 \)-continuous functions: regular continuous

Lane 1983