PERFECTNESS IN FRAMES

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— joint work with J. Gutiérrez García (UPV-EHU, Bilbao, Spain)

Ann. Math. 63 (1956)

TFAE for a space X: (1) X is perfectly normal (= perfect + normal).

(2) $\underbrace{f}_{\text{USC}} \leqslant \underbrace{g}_{\text{LSC}} \Rightarrow \exists\, h \in \mathsf{C}(X) : f \leqslant h \leqslant g$ and

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$$\forall a \in L \;\; \exists (a_n)_{\mathbb{N}} \subseteq L : \; a = \bigvee a_n \; \text{ and } \; a_n < a \;\; \forall n.$$

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Heath & Michael 1971

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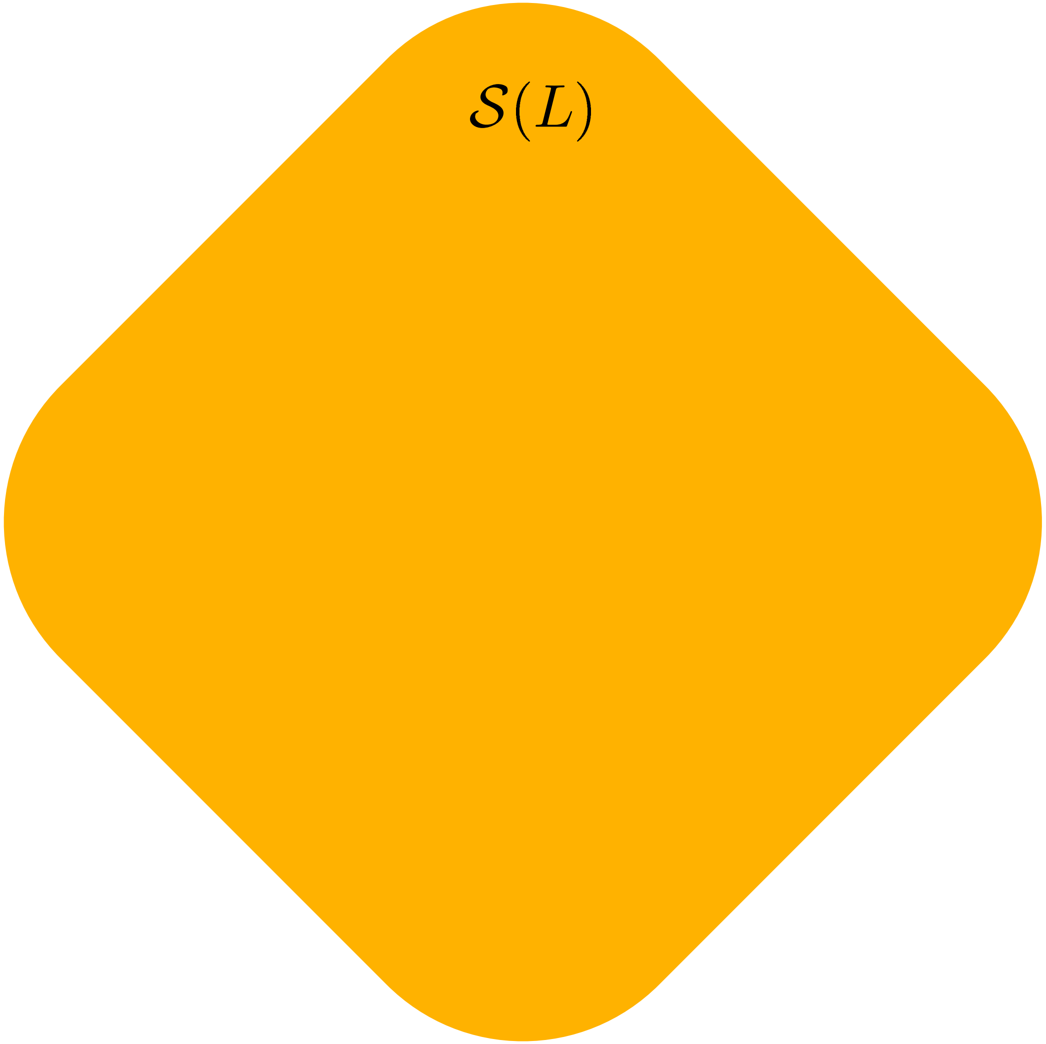
PERFECTLY NORMAL SPACES = PERFECT + NORMAL

$\Leftrightarrow \; \forall\, U \in \mathcal{O}(X)\ \exists\ (U_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}(X) : U = \bigcup_{n\in\mathbb{N}} U_n$ and $\overline{U_n} \subseteq U\ \forall n$.

Michael 1956

the frame of sublocales



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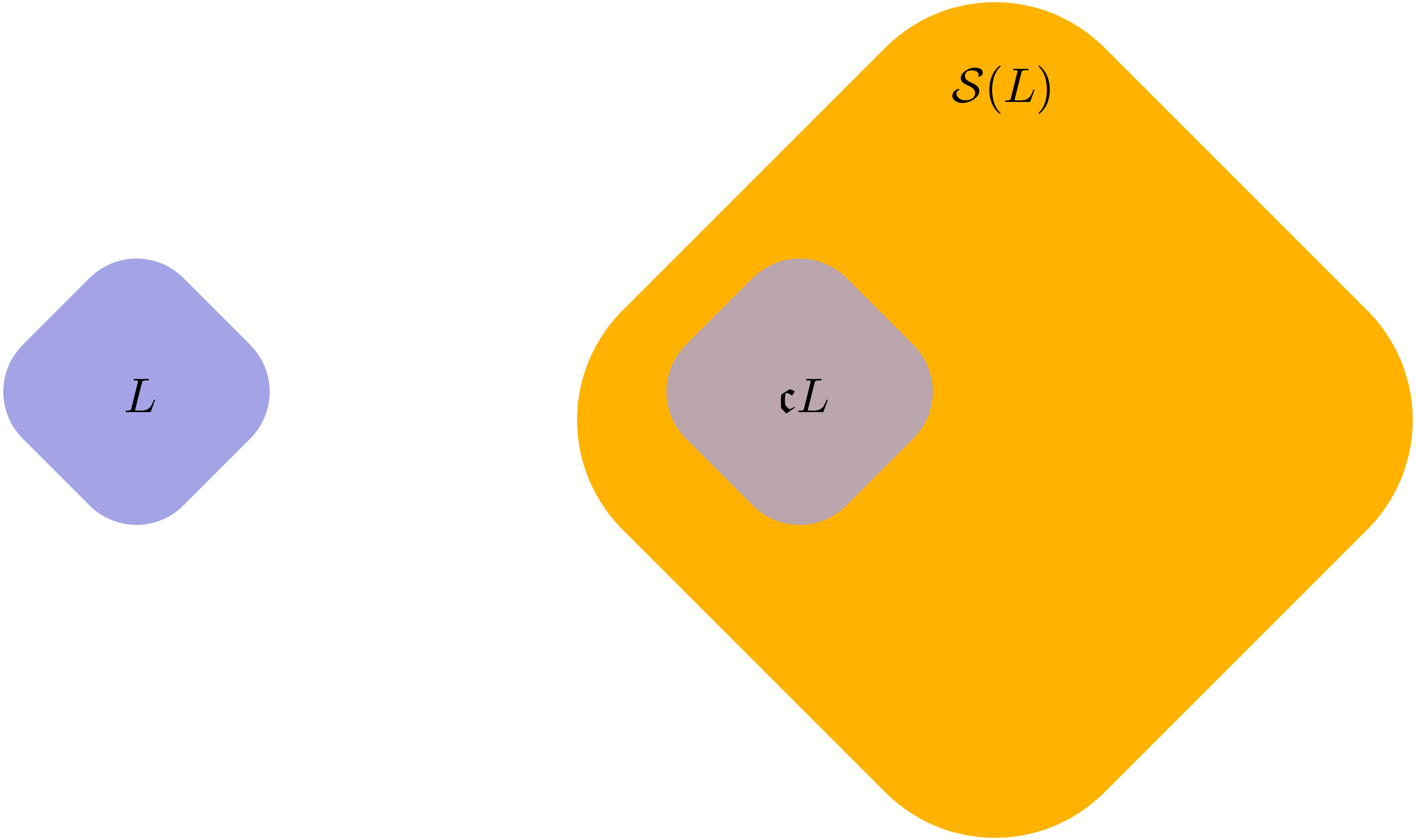
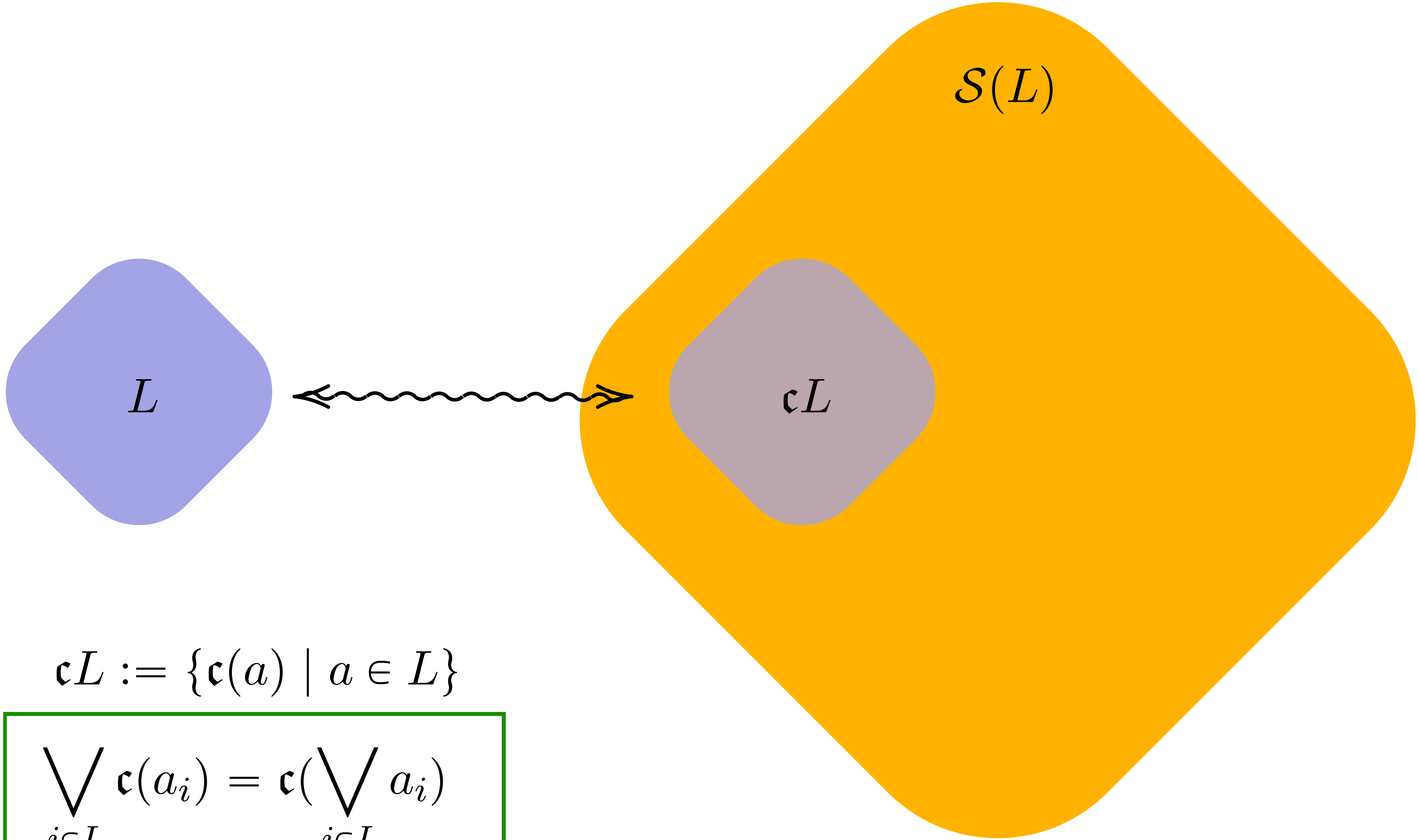


BACKGROUND AND NOTATION

the frame of sublocales



$$\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\bigvee_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i)$$

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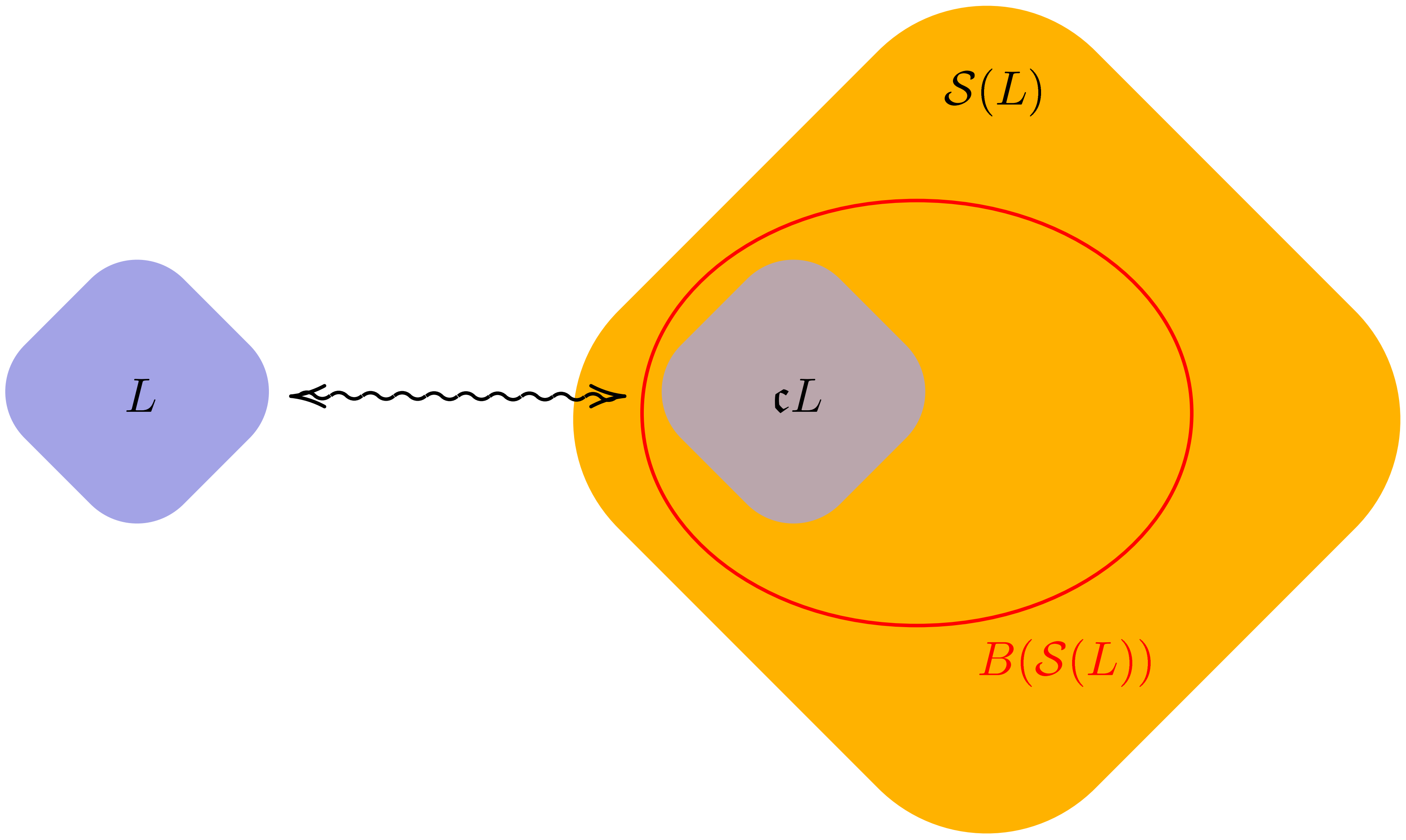

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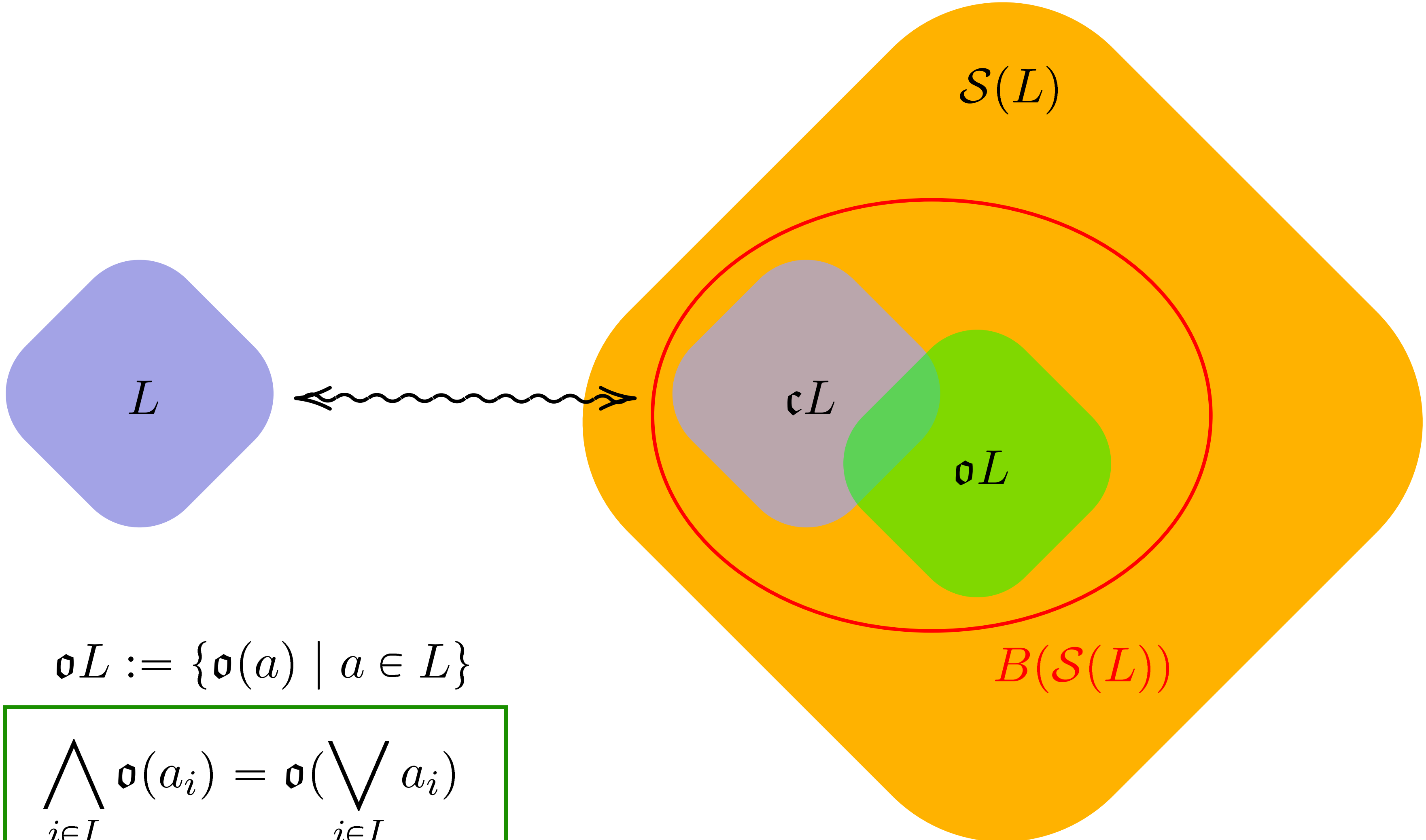

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COUNTER-EXAMPLE (a spatial one): the cofinite topology on $\mathbb{N}$.

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original GILMOUR's condition

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THE ROLE OF NORMALITY IN INSERTION: (weak) insertion

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J. Gutiérrez García & J. P., *JPAA* (2007)

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$$\Uparrow$$

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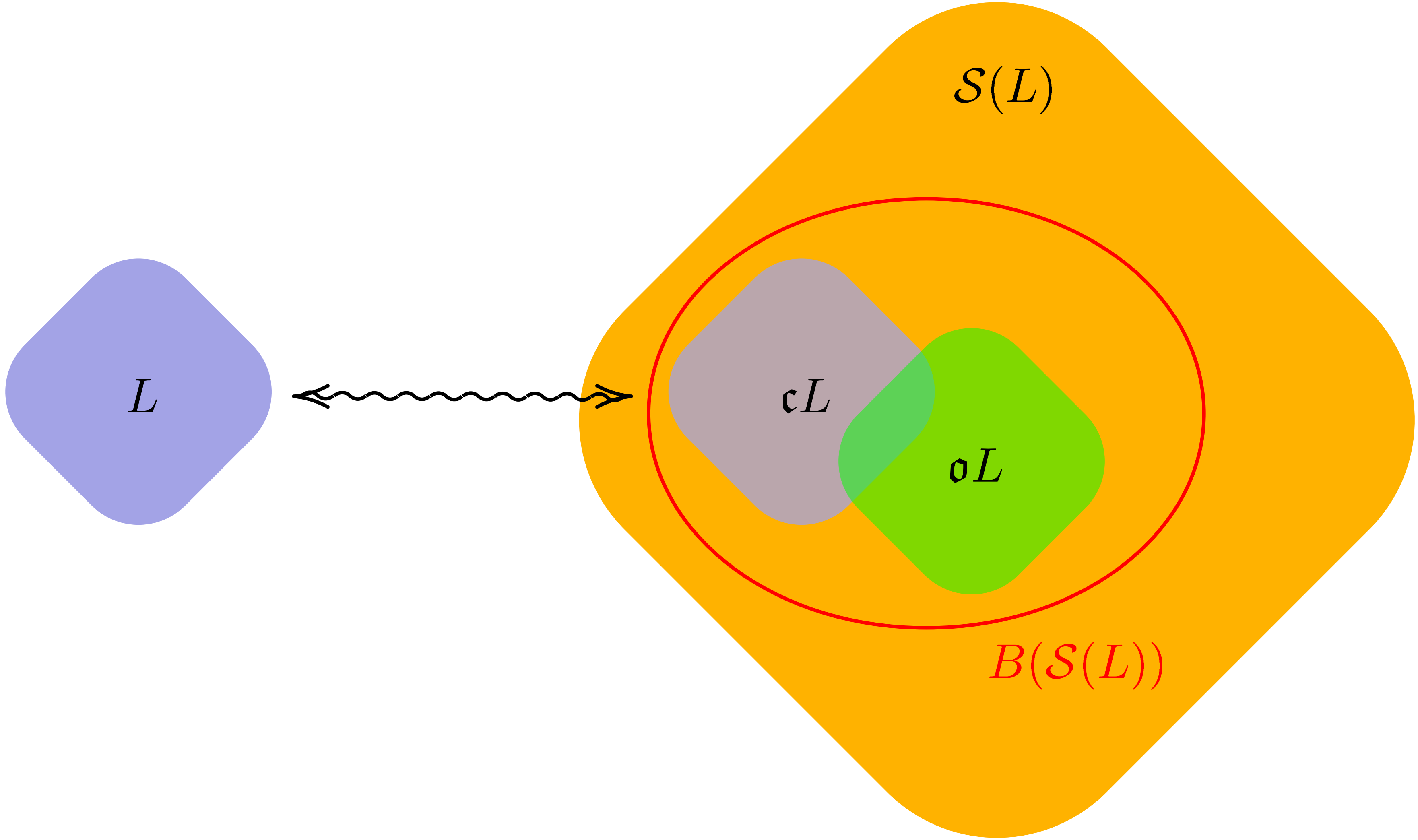
and thus $\Rightarrow \boxed{f < h < g}$

A UNIFIED APPROACH: go to $\mathcal{S}(L)$

the frame of sublocales



A UNIFIED APPROACH: $\mathscr{A}$-perfectness

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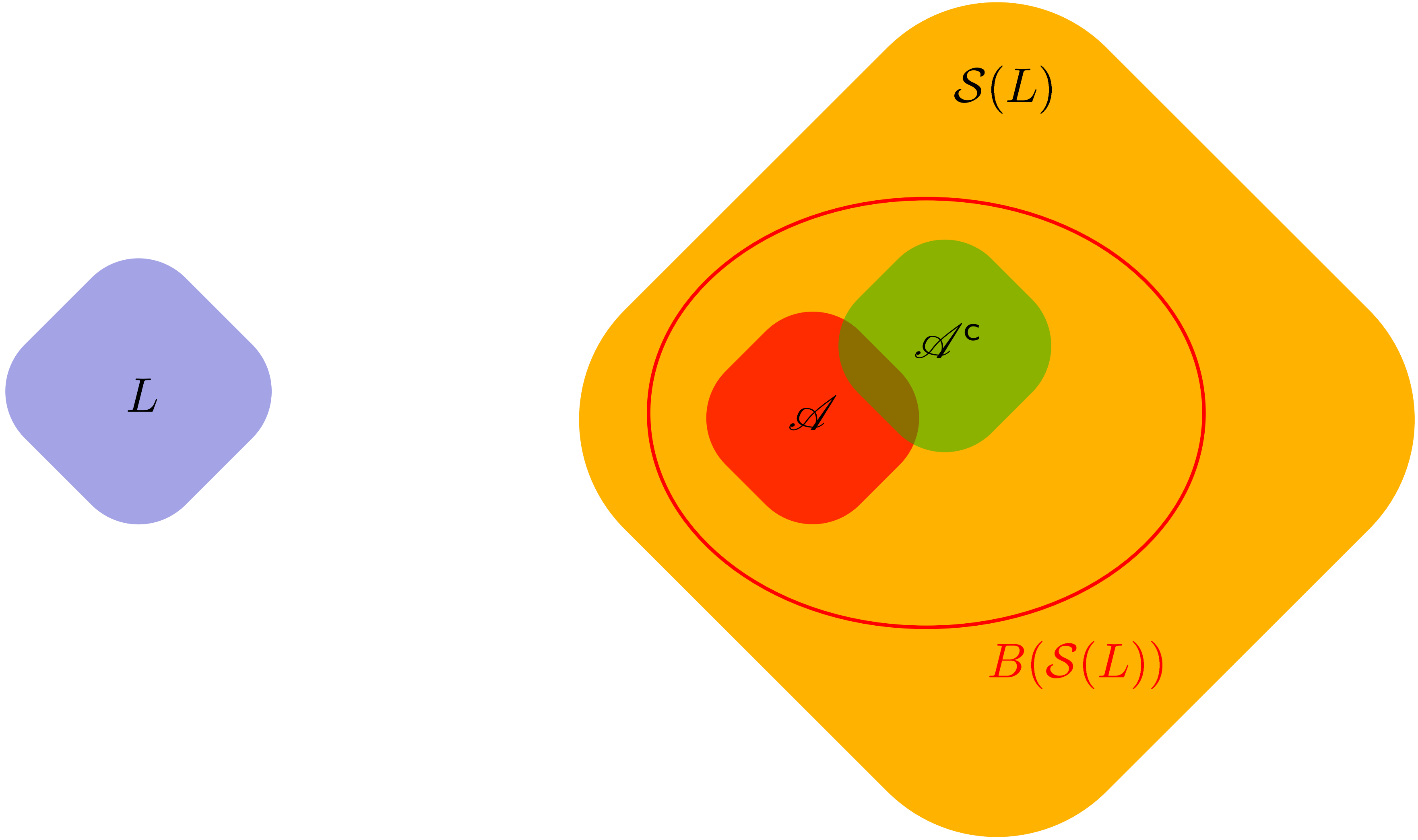

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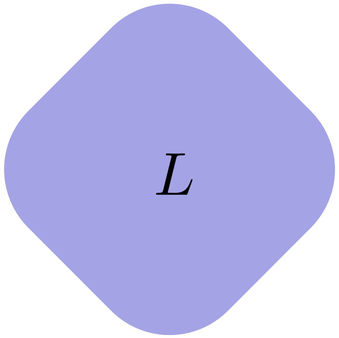
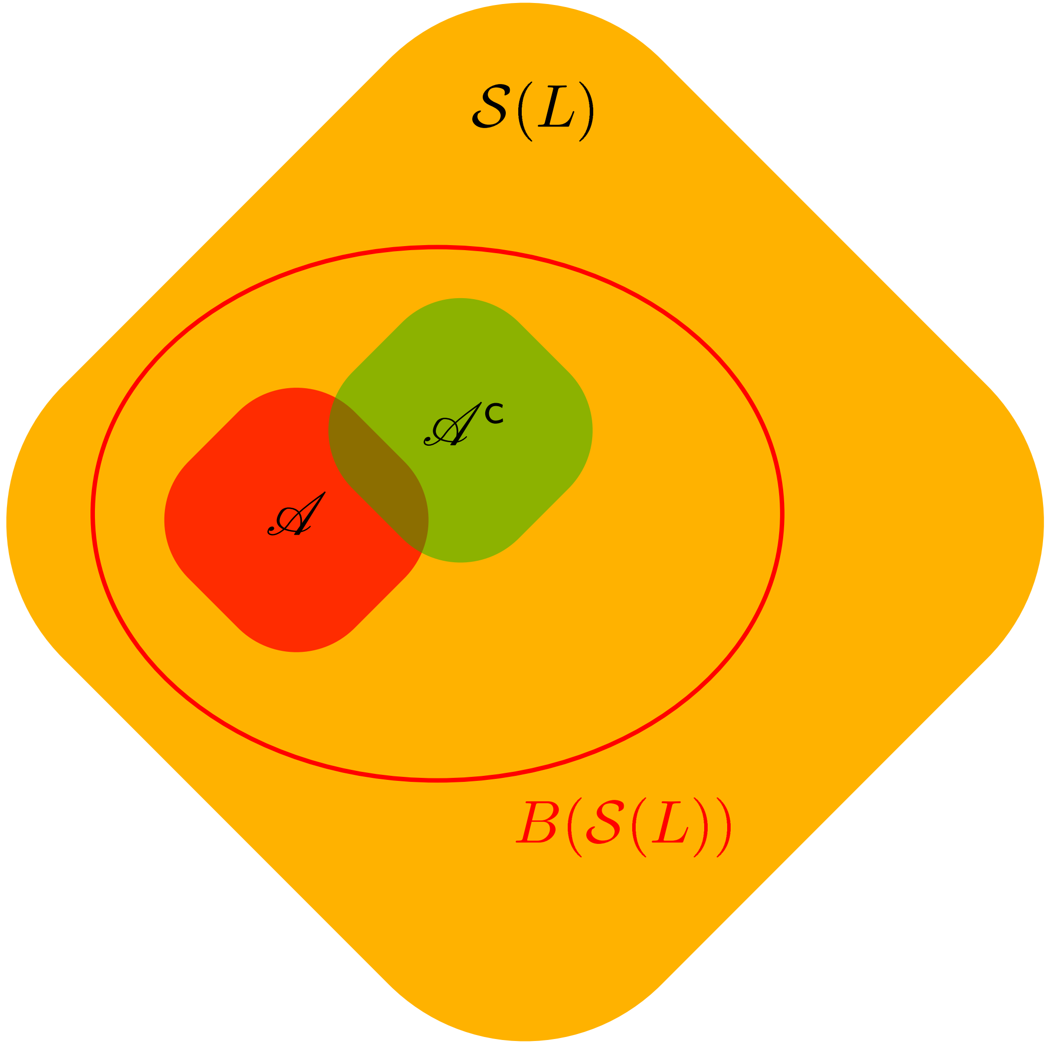


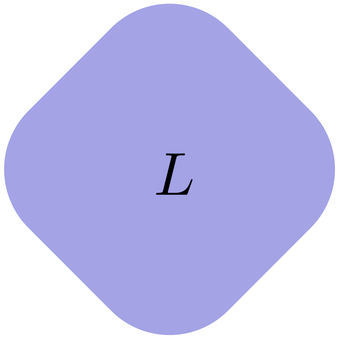


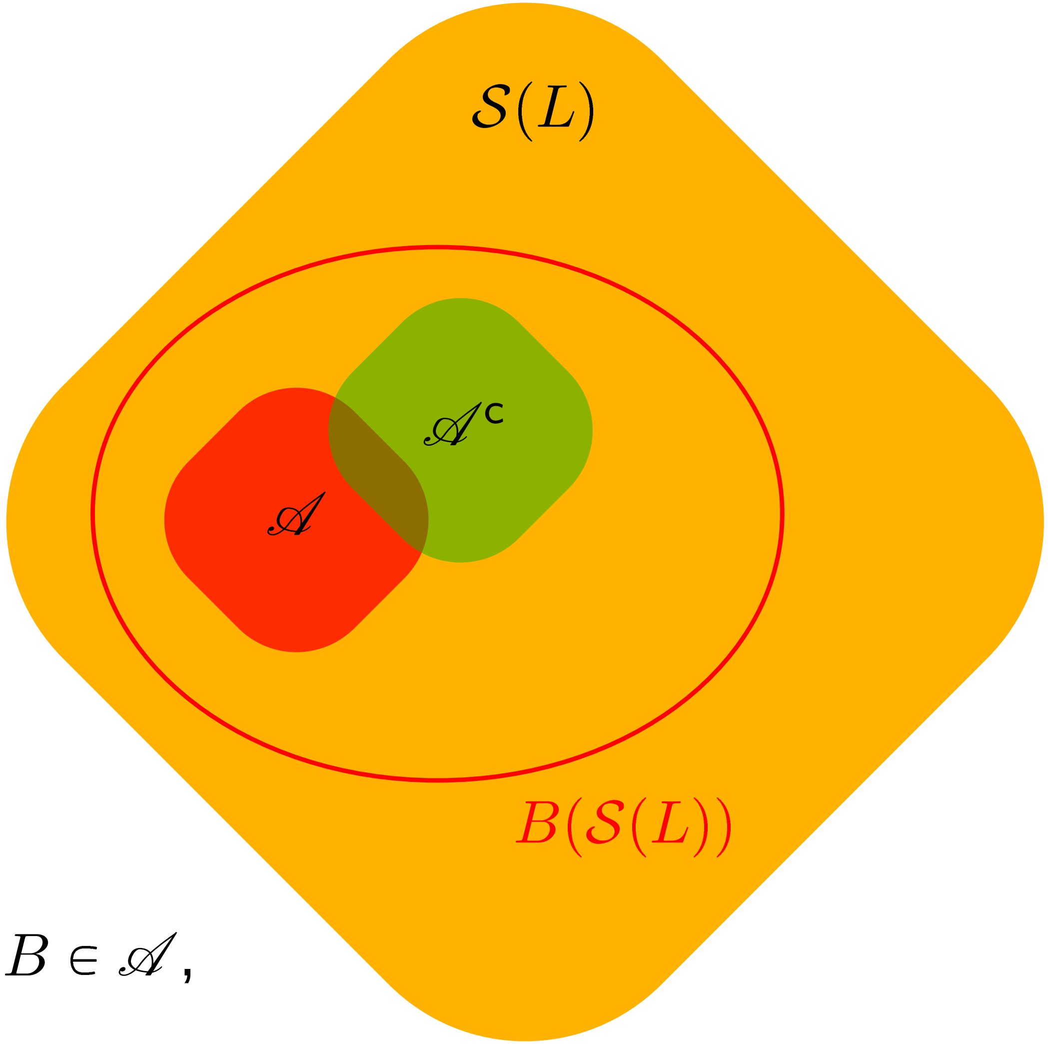
L is $\mathscr{A}$-perfect $\equiv \forall\ A \in \mathscr{A}^{\mathsf{c}}$

$$A = \bigwedge_{n\in\mathbb{N}} A_n \quad \text{(where each } A_n \in \mathscr{A}\text{)}$$

A UNIFIED APPROACH: $\mathscr{A}$-normality

the frame of sublocales





L is $\mathscr{A}$-normal $\equiv$ For any $A, B \in \mathscr{A}$,

$$A \vee B = 1 \;\Rightarrow\; \exists\, U, V \in \mathscr{A} : U \wedge V = 0,\; A \vee U = 1 = B \vee V.$$

$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$

$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L : \;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

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$\mathscr{A}$-SEMICONTINUITY AND $\mathscr{A}$-CONTINUITY

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$$f \in \mathsf{USC}(L) \iff \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L \;:\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\; f(-,p) \leqslant F_{p,q} \leqslant f(-,q).$$

$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \;:\; f(q,-) \leqslant F_{p,q} \leqslant f(p,-).$$

$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

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$$f \in \mathsf{USC}(L) \;\Leftrightarrow\; \forall\, p < q \;\; \exists\, F_{p,q} \in \mathfrak{c}L : \;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$\mathscr{A}\text{-}\mathsf{USC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \; : \;\; f(\text{—}, p) \leqslant F_{p,q} \leqslant f(\text{—}, q).$$

$$\mathscr{A}\text{-}\mathsf{LSC}(L) \equiv \forall\, p < q \;\; \exists\, F_{p,q} \in \mathscr{A} \; : \;\; f(q, \text{—}) \leqslant F_{p,q} \leqslant f(p, \text{—}).$$

$$\mathscr{A}\text{-}\mathsf{C}(L) = \mathscr{A}\text{-}\mathsf{LSC}(L) \cap \mathscr{A}\text{-}\mathsf{USC}(L)$$

Clearly:

> f is upper $\mathscr{A}$-semicontinuous iff it is lower $\mathscr{A}^{c}$-semicont.
>
> f is $\mathscr{A}^{c}$-continuous iff it is $\mathscr{A}$-continuous.

RESULTS: relative versions

$\mathscr{A}$-perfect normality $=$ $\mathscr{A}$-perfectness $+$ $\mathscr{A}$-normality

$\Updownarrow$

(Weak) insertion

for $\underbrace{f}_{\mathscr{A}-\mathsf{USC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{LSC}}$

RESULTS: relative versions

$\mathscr{A}$-perfect normality $=$ $\mathscr{A}$-perfectness $+$ $\mathscr{A}$-normality

$\Updownarrow$ (perfectness) $\Updownarrow$ under mild conditions on $\mathscr{A}$

Strict insertion $=$ Double insertion $+$ (Weak) insertion

for $\underbrace{f}_{\mathscr{A}-\mathsf{USC}} \leqslant \underbrace{g}_{\mathscr{A}-\mathsf{LSC}}$

RESULTS: relative versions

$\mathscr{A}^{c}$-perfect normality $=$ $\mathscr{A}^{c}$-perfectness $+$ $\mathscr{A}^{c}$-normality

$\Updownarrow$ (perfectness) $\Updownarrow$ (normality) under mild conditions on $\mathscr{A}$

Strict insertion $=$ Double insertion $+$ (Weak) insertion

for $\underbrace{f}_{\mathscr{A}^{c}-\text{USC}} \leqslant \underbrace{g}_{\mathscr{A}^{c}-\text{LSC}}$

RESULTS: relative versions

$\mathscr{A}^c$-perfect normality = $\mathscr{A}^c$-perfectness + $\mathscr{A}^c$-normality

$\Updownarrow$ (for $\mathscr{A}^c$-perfectness)

$\Updownarrow$ under mild conditions on $\mathscr{A}$ (for $\mathscr{A}^c$-normality)

Strict insertion = Double insertion + (Weak) insertion

for $\underbrace{f}_{\mathscr{A}^c-\mathrm{USC}} \leqslant \underbrace{g}_{\mathscr{A}^c-\mathrm{LSC}}$

$\mathscr{A}$-LSC $\mathscr{A}$-USC

RESULTS: relative versions

$\mathscr{A}$-Booleanness $\qquad$ $\mathscr{A}$-extremally disc.

$\mathscr{A}^{c}$-perfect normality $=$ $\mathscr{A}^{c}$-perfectness $+$ $\mathscr{A}^{c}$-normality

$\Updownarrow$ $\qquad$ $\Updownarrow$ under mild conditions on $\mathscr{A}$

Strict insertion $=$ Double insertion $+$ (Weak) insertion

for $\underbrace{f}_{\mathscr{A}^{c}-\mathrm{USC}} \leqslant \underbrace{g}_{\mathscr{A}^{c}-\mathrm{LSC}}$

$\mathscr{A}$-LSC $\qquad$ $\mathscr{A}$-USC

EXAMPLES

$$\mathscr{A}_1 = \{\mathfrak{c}(a) : a \in L\}$$

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- $\mathscr{A}_1$-normal frames: normal
- $\mathscr{A}_1^{\mathsf{c}}$-normal frames: extremally disconnected

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- $\mathscr{A}_1^{\mathrm{c}}$-perfect frames: Boolean

EXAMPLES

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- $\mathscr{A}_1$-normal frames: normal
- $\mathscr{A}_1^{\mathsf{c}}$-normal frames: extremally disconnected

- $\mathscr{A}_1$-perfect frames: perfect
- $\mathscr{A}_1^{\mathsf{c}}$-perfect frames: Boolean

- upper $\mathscr{A}_1$-semicontinuous functions: upper semicontinuous
- lower $\mathscr{A}_1$-semicontinuous functions: lower semicontinuous
- $\mathscr{A}_1$-continuous functions: continuous

EXAMPLES

$$\mathscr{A}_2 = \{\mathfrak{c}(a^*) : a \in L\}$$

EXAMPLES

$\mathscr{A}_2 = \{\mathfrak{c}(a^*) : a \in L\}$

- $\mathscr{A}_2$-normal frames: mildly normal
- $\mathscr{A}_2^{\mathsf{c}}$-normal frames: extremally disconnected

- $\mathscr{A}_2$-perfectly normal frames: pm-normal = OZ
- $\mathscr{A}_2^{\mathsf{c}}$-perfectly normal frames: extremally disconnected

- upper $\mathscr{A}_2$-semicontinuous functions: normal upper semicontinuous
- lower $\mathscr{A}_2$-semicontinuous functions: normal lower semicontinuous
- $\mathscr{A}_2$-continuous functions: normal continuous

$(f^\circ)^- = f$ | $(f^-)^\circ = f$

Dilworth 1950

EXAMPLES

$$\mathscr{A}_3 = \{\mathfrak{c}(\mathrm{coz}\, f) \colon\; f \in \mathsf{C}(L)\}$$

EXAMPLES

$\mathscr{A}_3 = \{\mathfrak{c}(\mathrm{coz} f) : f \in \mathsf{C}(L)\}$

- $\mathscr{A}_3$-normal frames: all frames
- $\mathscr{A}_3^{\mathsf{c}}$-normal frames: F-frames

- $\mathscr{A}_3$-perfectly normal frames: all frames
- $\mathscr{A}_3^{\mathsf{c}}$-perfectly normal frames: P-frames

- upper $\mathscr{A}_3$-semicontinuous functions: zero upper semicontinuous
- lower $\mathscr{A}_3$-semicontinuous functions: zero lower semicontinuous
- $\mathscr{A}_3$-continuous functions: zero continuous

Stone 1949

EXAMPLES

$\mathscr{A}_4 = \{\mathfrak{c}(a) \colon a \text{ regular } G_\delta\}$

EXAMPLES

$\mathscr{A}_4 = \{\mathfrak{c}(a) \colon a \text{ regular } G_\delta\}$

- $\mathscr{A}_4$-normal frames: δ-normal
- $\mathscr{A}_4^{\mathsf{c}}$-normal frames: δ-extremally disconnected

- $\mathscr{A}_4$-perfectly normal frames: ???
- $\mathscr{A}_4^{\mathsf{c}}$-perfectly normal frames: ???

- upper $\mathscr{A}_4$-semicontinuous functions: regular upper semicontinuous
- lower $\mathscr{A}_4$-semicontinuous functions: regular lower semicontinuous
- $\mathscr{A}_4$-continuous functions: regular continuous

Lane 1983