A new diagonal separation property in the category of locales

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— joint work with Igor Arrieta (Bilbao) and Aleš Pultr (Prague)

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Universidade de Coimbra

General categorical phenomena [Clementino, Giuli and Tholen, 2004]

$\mathcal{C}$: a category with finite products
$\mathcal{P}$: a property of subobjects

$X \in \mathrm{Obj}(\mathcal{C})$ is *$\mathcal{P}$-separated* if the diagonal $\Delta_X \colon X \rightarrowtail X \times X$ has property $\mathcal{P}$.

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Proposition [CGT, 2004]

Let $\mathcal{P}$ be stable under pullbacks. Then:

(1)

$$\left.\begin{array}{l} X \rightarrowtail Y \text{ is a monomorphism} \\ Y \text{ is } \mathcal{P}\text{-separated} \end{array}\right\} \implies X \text{ is } \mathcal{P}\text{-separated.}$$

(2)

$$\left.\begin{array}{l} f, g \colon X \to Y \in \mathrm{Mor}(\mathcal{C}) \\ Y \text{ is } \mathcal{P}\text{-separated} \end{array}\right\} \implies \mathrm{Eq}(f,g) \rightarrowtail X \text{ has property } \mathcal{P}.$$

Diagonal separation in *Loc*: examples

Pullback stable properties in ***Loc***:
closed sublocales, open sublocales, complemented sublocales,
locally closed sublocales (= closed $\cap$ open)
fitted sublocales (= intersections of open sublocales), . . .

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CLASS $\mathcal{P}$	PROPERTY in ***Top***	PROPERTY in ***Loc***	REFERENCES in ***Loc***
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Fitted	T_1-spaces	$\mathcal{F}$-separated	Arrieta, P. & Pultr 2022

$\mathcal{F}$-separation

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By the general categorical results, we have:

(1) If $L \rightarrowtail M$ is a mono in ***Loc*** and M is $\left|\begin{array}{l}\text{strongly Hausdorff}\\ \mathcal{F}\text{-separated}\end{array}\right|$, then so is L.

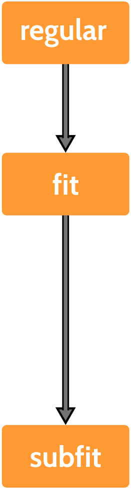
• Monomorphisms in ***Loc*** are fairly wild,
so this tells us more than simply heredity!

(2) A locale M is $\left|\begin{array}{l}\text{strongly Hausdorff}\\ \mathcal{F}\text{-separated}\end{array}\right|$ iff all $Eq(f,g) \rightarrowtail M$ are $\left|\begin{array}{l}\text{closed}\\ \text{fitted}\end{array}\right|$.

Relation with fitness and subfitness

A locale L is subfit $\equiv$ every open sublocale is a join of closed sublocales.

fit $\equiv$ every (closed) sublocale is a meet of opens.



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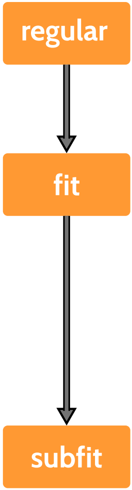

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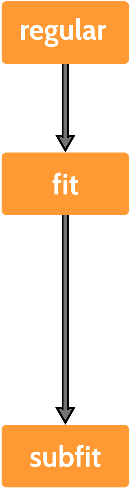
Lemma. *Every fit locale is $\mathcal{F}$-separated*



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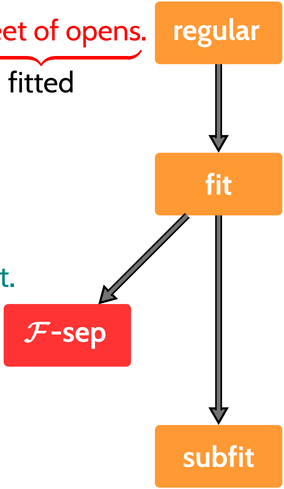

regular

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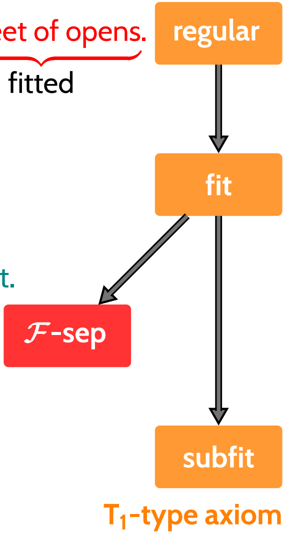

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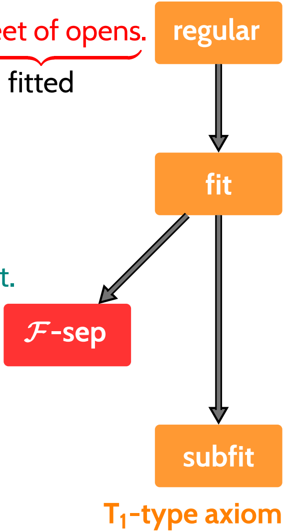

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Subfit is not hereditary; no relation with $\mathcal{F}$-sep.

Dowker-Strauss characterization of strong Hausdorffness

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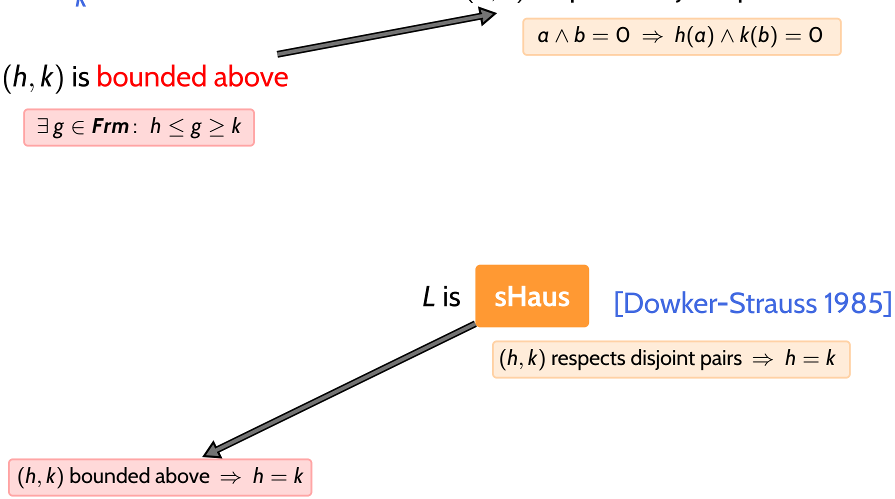

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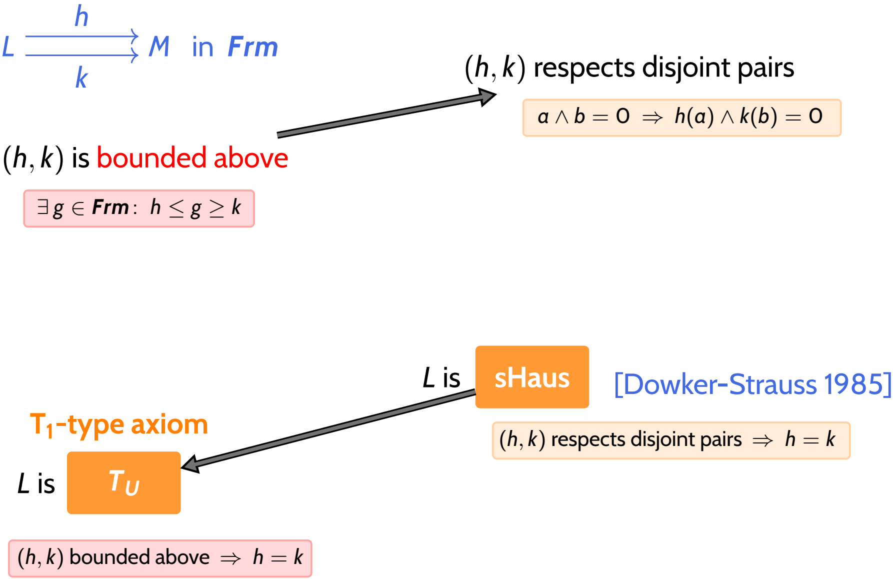

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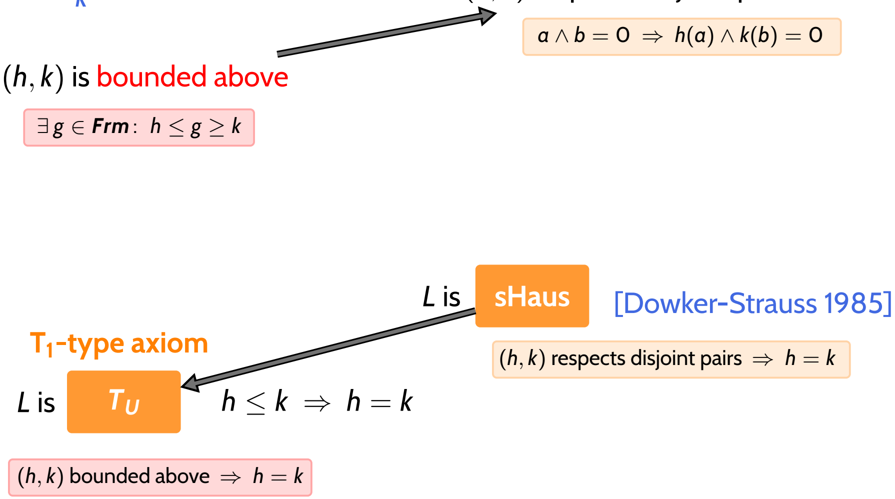

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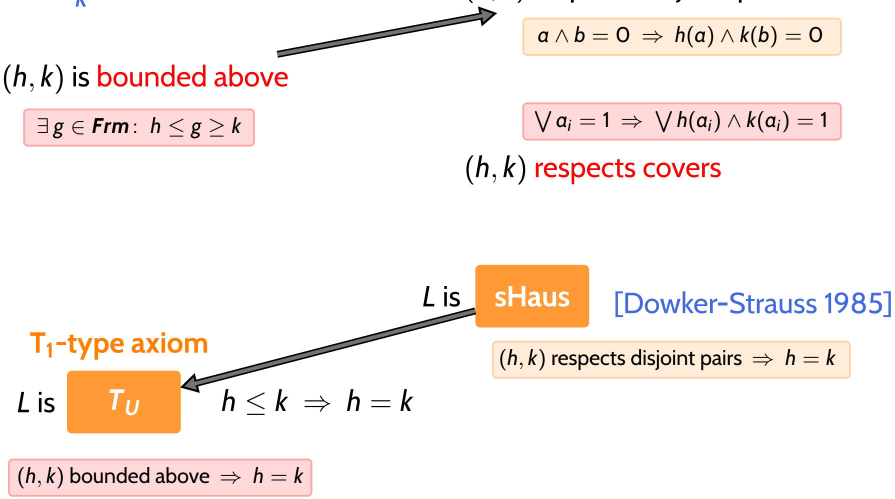

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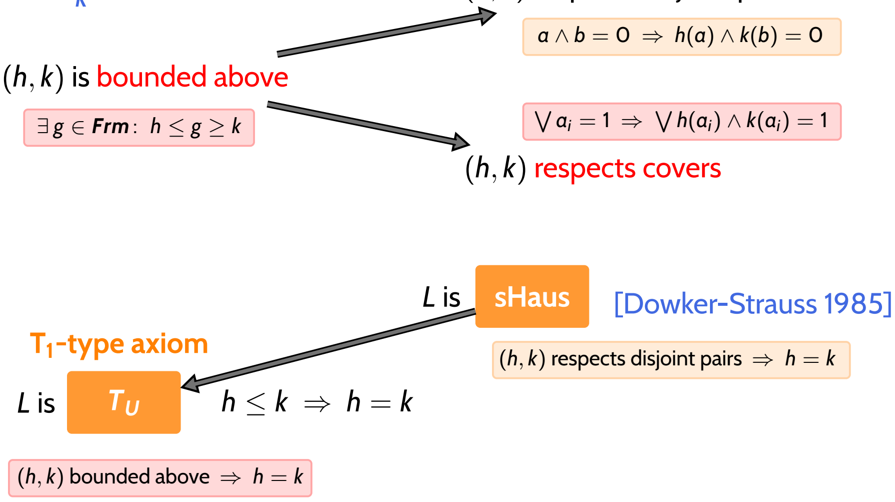

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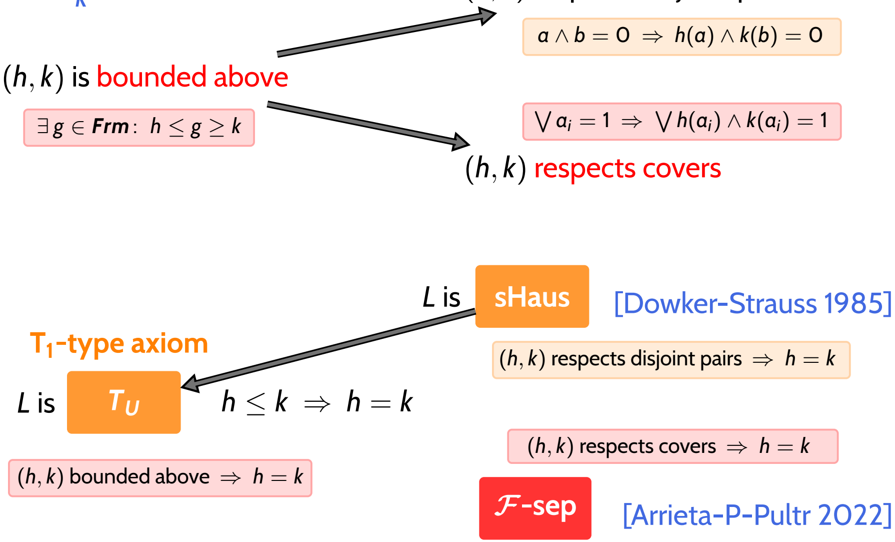

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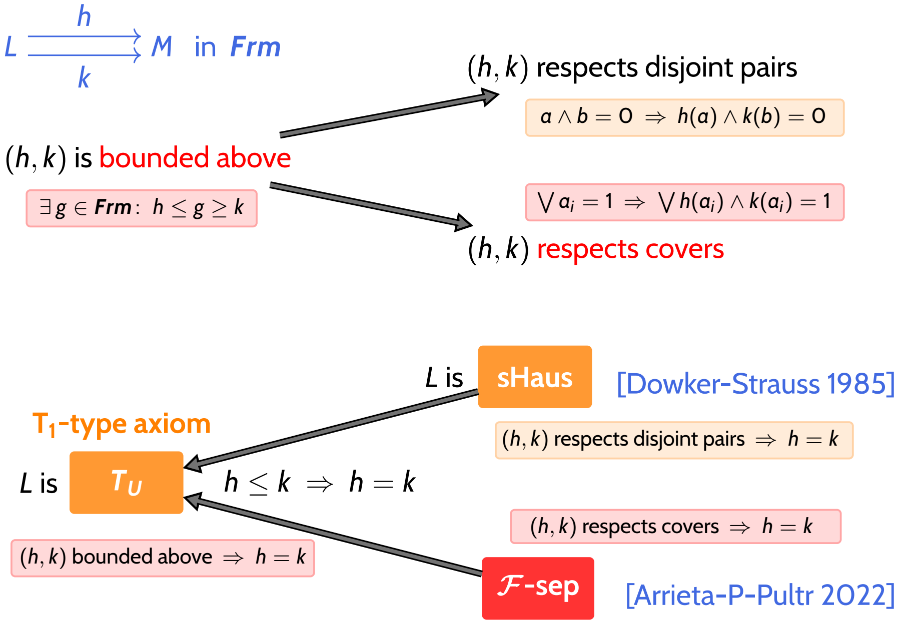

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A mapping $h\colon L \to M$ between frames is a weak homomorphism if

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Furthermore, they say a frame L has property (W) when

(W) Every weak homomorphism $h\colon L \to M$ is a frame homomorphism.

More on strong Hausdorffness

Theorem [Banaschewski-Pultr, 2004)]

A locale L is strongly Hausdorff iff it satisfies T_U + (W).

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The main part of the proof uses the following fundamental result:

Theorem [Joyal-Tierney, 1984]

The category ***Sup*** is symmetric monoidal closed (tensor product: $\otimes$).

If L and M are frames, its coproduct $L \oplus M$ is isomorphic to $L \otimes M$.

More on $\mathcal{F}$-separatedness: a dual counterpart

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A preframe homomorphism is a function which preserves directed joins and finite meets.

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Can we use this in order to obtain a “dual” characterization for $\mathcal{F}$-separatedness?

The idea: almost homomorphisms

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We then say that a frame L satisfies property (A) whenever

(A) Every almost homomorphism $h\colon L \to M$ is a frame homomorphism.

$\mathcal{F}$-separatedness: a dual counterpart

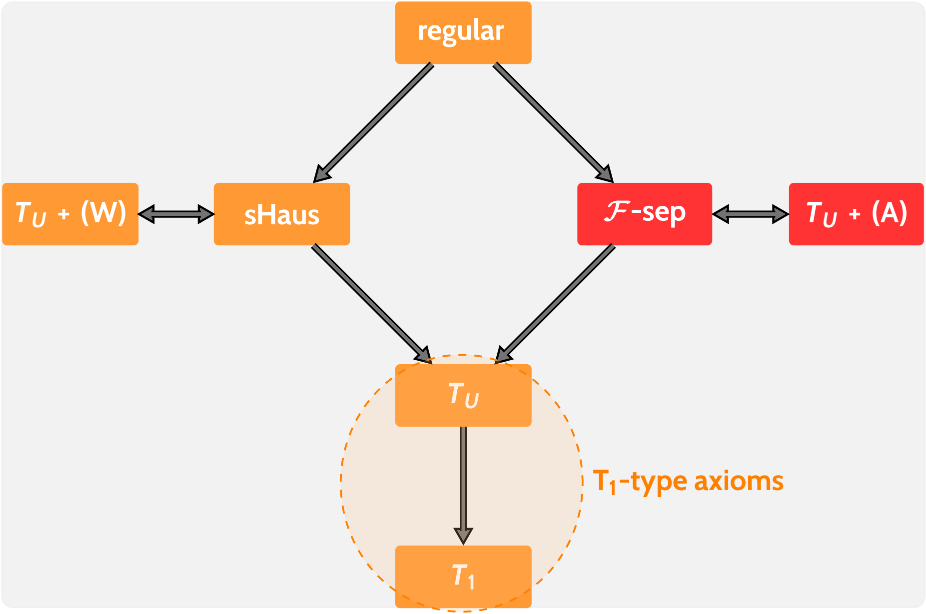
Theorem.

A locale L is $\mathcal{F}$-separated iff it satisfies T_U + (A).

Sketch of proof:

- $\mathcal{F}$-separated $\Rightarrow T_U$: ✓
- $\mathcal{F}$-separated $\Rightarrow$ (A) : hard part; uses the preframe tensor.
- T_U + (A) $\Rightarrow$ $\mathcal{F}$-separated : easy.

CONCLUSION



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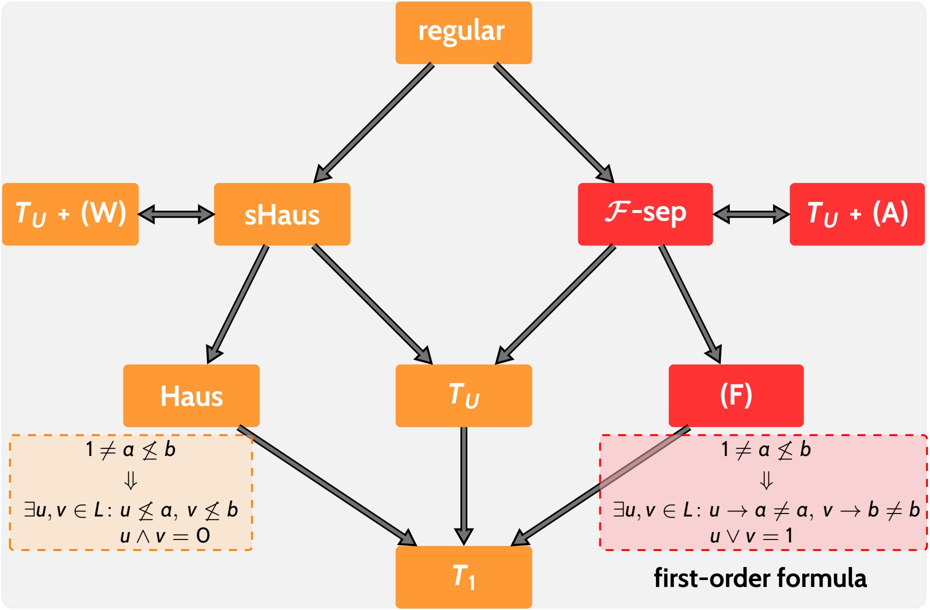

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T_1

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first-order formula

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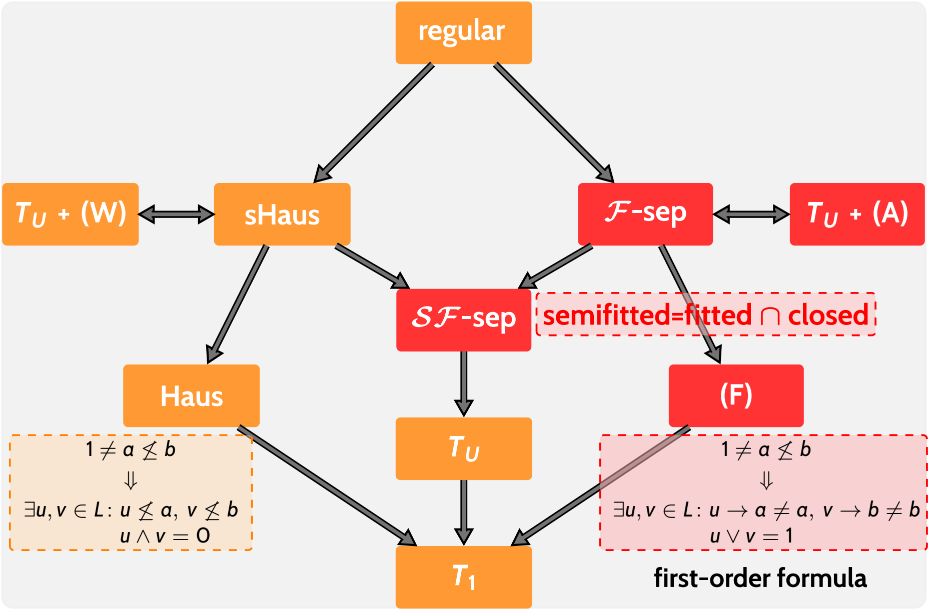

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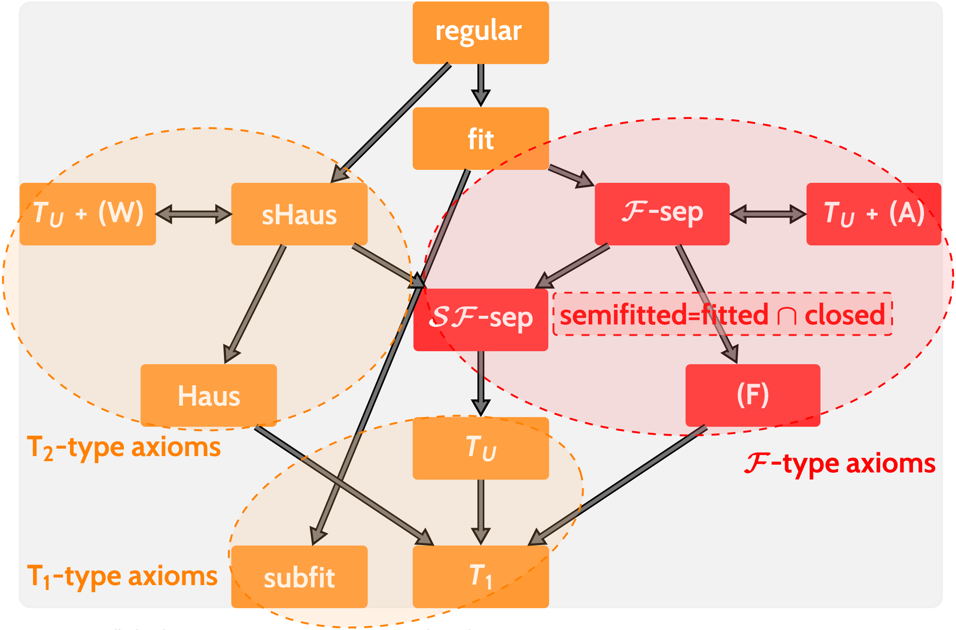

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Aleš Pultr

Separation in Point-Free Topology

Birkhäuser

2021