The geometry of localic maps

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— honouring Themba Dube

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Joint work with M. Erné and A. Pultr

- Marcel Erné, JP, Aleš Pultr
  Adjoint maps between implicative semilattices and continuity of localic maps
  Algebra Universalis 83 (2022) article n. 13, 1-23.

(in the more general setting of IMPLICATIVE (HEYTING) SEMILATTICES)

MOTIVATION: continuity conditions

A map $f : X \to Y$ is continuous

iff $f^{-1}[U]$ is open for every open U (O)

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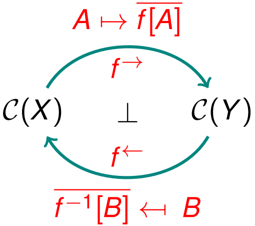

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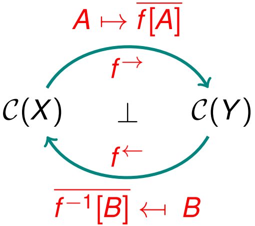

- AIM: to examine this in the pointfree setting.

LOCALIC MAPS

Frame homomorphisms $h\colon M \to L$ preserve arbitrary joins HENCE have uniquely defined right adjoints $f = h_* \colon L \to M$.

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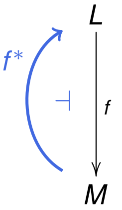

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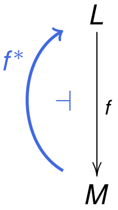

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$$f^* \dashv f, \quad f \colon L \to M, \quad f^* \colon M \to L$$

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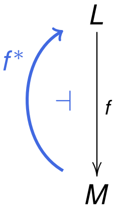

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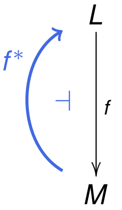
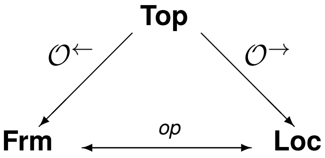


- $f(\bigwedge S) = \bigwedge f[S]$
- $f(f^*(a) \to b) = a \to f(b)$ (Frobenius)
- $f(a) = 1 \Rightarrow a = 1$ (codense)

The "STARTING POINT of pointfree topology"



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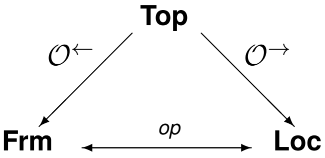

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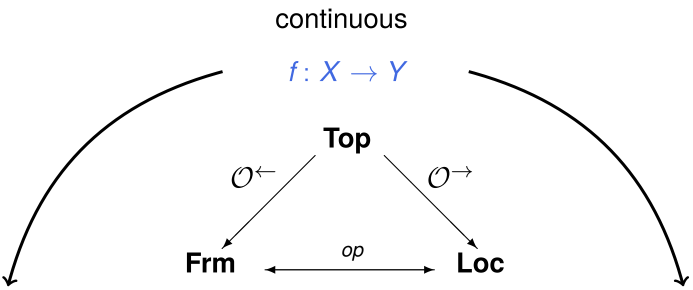

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$\mathcal{O}^{\rightarrow} f : \mathcal{O}X \to \mathcal{O}Y$

$U \;\mapsto\; Y \smallsetminus \overline{f[X \smallsetminus U]}$

frame homomorphism $\dashv$ localic map

BACKGROUND: Sublocales

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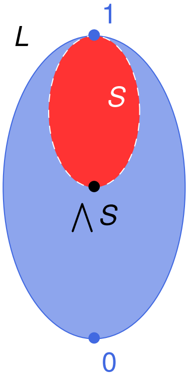

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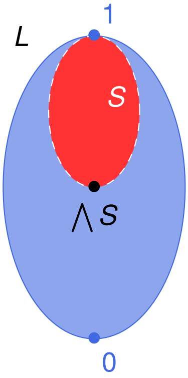

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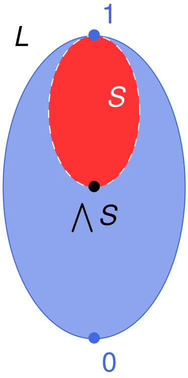

Localic structure of S:

$\bigwedge_S = \bigwedge_L,\ \to_S = \to_L,$ however $\bigsqcup_i s_i = \bigwedge\{s \in S \mid \bigvee_i s_i \leq s\}$.

BACKGROUND: the sublocale lattice

$\mathcal{S}(L)$: sublocales of L, ordered by $\subseteq$

$\mathbf{0} = \{1\}$, $\mathbf{1} = L$, $\bigwedge = \bigcap$, $\boxed{\bigvee_i S_i = \{\bigwedge A \mid A \subseteq \bigcup_i S_i\}}$

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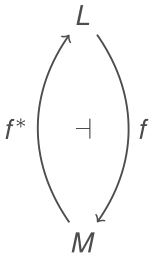

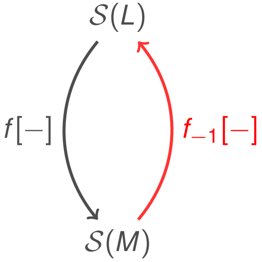

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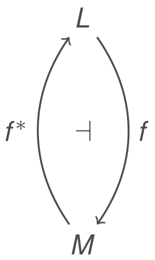

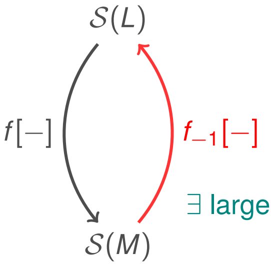

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$\exists$ largest sublocale $f_{-1}[T] \subseteq f^{-1}[T]$

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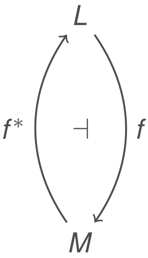

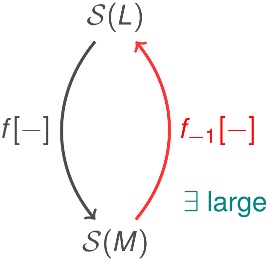

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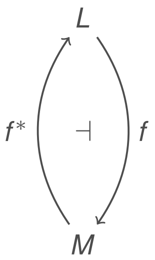

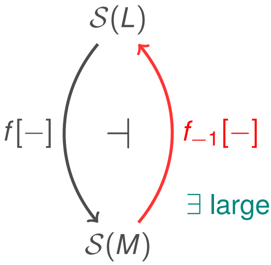

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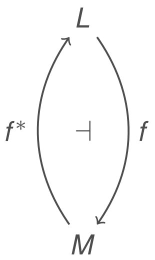

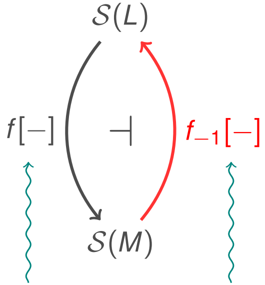

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colocalic map

coframe homomorphism
preserves $(-)^{c}$

BACKGROUND: the fundamental picture

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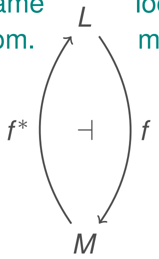

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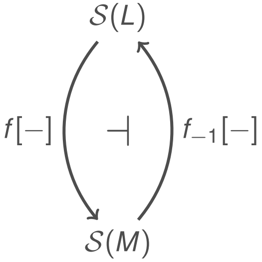

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CONTINUITY conditions

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Continuity Conditions For any localic map f, we have:

(C) $f_{-1}[\mathfrak{c}(a)] = f^{-1}[{\uparrow}a] = \mathfrak{c}(f^*(a))$.

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More generally: for a map $f\colon A \to B$ between posets, $\exists\, f^* \dashv f$

iff preimages of principal filters are again principal filters.

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Proof of (O). $\mathfrak{o}(f^*(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$: $\quad f(f^*(a) \to x) = a \to f(x) \in \mathfrak{o}(a)$.

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Proof of (O). $S \subseteq f^{-1}[\mathfrak{o}(a)] \Rightarrow S \subseteq \mathfrak{o}(f^*(a))$:

INTERMEZZO: the Heyting operator

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H2 $$a \le b \to c \text{ iff } b \le a \to c.$$

INTERMEZZO: the Heyting operator

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H1 $a \to (\bigwedge b_i) = \bigwedge(a \to b_i)$.

H2 $a \leq b \to c$ iff $b \leq a \to c$.

H3 $(\bigvee a_i) \to b = \bigwedge(a_i \to b)$.

INTERMEZZO: the Heyting operator

Properties

H1 $a \to (\bigwedge b_i) = \bigwedge(a \to b_i)$.

H2 $a \leq b \to c$ iff $b \leq a \to c$.

H3 $(\bigvee a_i) \to b = \bigwedge(a_i \to b)$.

H4 $a \to b = a \to (a \wedge b)$.

H5 $a \wedge (a \to b) = a \wedge b$.

H6 $a \wedge b = a \wedge c$ iff $a \to b = a \to c$.

H7 $(a \wedge b) \to c = a \to (b \to c) = b \to (a \to c)$.

H8 $a = (a \vee b) \wedge (b \to a)$.

H9 $a \leq (a \to b) \to b$.

H10 $((a \to b) \to b) \to b = a \to b$.

CONTINUITY conditions

$a \in L, \quad \mathfrak{c}(a) = {\uparrow}a$ CLOSED

$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$ OPEN

} complemented to each other

Continuity Conditions For any localic map f, we have:

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$\Rightarrow$ $(f^*(a) \to s) \to s = 1$, that is, $f^*(a) \to s = s$. ∎

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▶ **QUESTION:** Does conditions (C) & (O) characterize localic maps among plain maps of locales?

$f_{-1}[S]$ **versus** $f^{-1}[S]$

The smallest sublocale that contains a: $\mathfrak{b}(a)$ boolean sublocales

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$$a \in f_{-1}[S] \;\Leftrightarrow\; \mathfrak{b}(a) \subseteq f_{-1}[S] \;\Leftrightarrow\; \mathfrak{b}(a) \subseteq f^{-1}[S] \;\Leftrightarrow\; f(x \to a) \in S,\ \forall\, x \in L$$

$f_{-1}[S]$ **versus** $f^{-1}[S]$

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For other sublocales they differ:

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PROPOSITION

Let S be a sublocale of M that is not closed. Then there is a localic map $f\colon L \to M$ such that $f_{-1}[S] \neq f^{-1}[S]$.

More specifically: one has $f_{-1}[S] \neq f^{-1}[S]$ for any f adjoint to a frame embedding $h\colon M \to L$ such that $h[M]$ is contained in the Boolean part of L.

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COROLLARY

Let S be a sublocale of M. Then

$$f_{-1}[S] = f^{-1}[S] \text{ for all localic maps } f\colon L \to M$$

if and only if S is closed.

(1) ORDER-PRESERVING MAPS

posets X, Y

order-preserving maps $\equiv$ $f\colon X \to Y$ such that

$f^{-1}[\uparrow b]$ is an upper set for every $b \in Y$.

(2) MEET-PRESERVING MAPS

complete lattices L, M

meet-preserving maps $\equiv$ $f : L \to M$ such that

$$\forall\, b \in M \;\; \exists\, a \in L \; : \; f^{-1}[\uparrow b] = \uparrow a.$$

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$$b \overset{\phi}{\longmapsto} a$$

MAIN QUESTION: first observations

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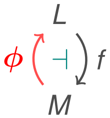

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locales L, M, order-preserving $f\colon L \to M$

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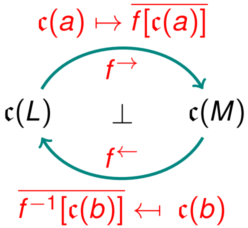

For any subset $S \subseteq L$,

$$\overline{S} = \bigcap\{\mathfrak{c}(a) \in \mathfrak{c}(L) \mid S \subseteq \mathfrak{c}(a)\}$$

MAIN QUESTION: first observations

(3) FROBENIUS CONDITION

locales L, M, meet-preserving $f\colon L \to M$

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LEMMA

TFAE on each $b \in M$:

$f(f^*(b) \to a) = b \to f(a)$ for all $a \in L$

$\Leftrightarrow$ $f[\mathfrak{o}(f^*(b)) \cap \mathfrak{o}(a)] = \mathfrak{o}(b) \cap f[\mathfrak{o}(a)]$ for all $a \in L$

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MAIN QUESTION: the characterisations

THEOREM 1

A map $f\colon L \to M$ between locales is a localic map iff $f^{-1}[\mathbf{0}] = \mathbf{0}$, $f^{-1}[A]$ is closed for every closed A, and $f^{-1}[U] \supseteq f^{-1}[U^{\mathrm{c}}]^{\mathrm{c}}$ for every open U.

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Proof (sketch):

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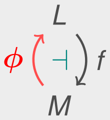
Proof (sketch): $A = \mathfrak{c}(b)$ $\quad b \overset{\phi}{\longmapsto} a$



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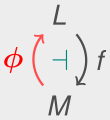

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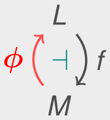

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- Frobenius condition: $f(\phi(a) \to x) = a \to f(x)$.

  This is the technical hard part!... ■

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Hence we can formulate the theorem as follows:

A map $f: L \to M$ between locales is a localic map iff

$f^{-1}[A]$ is closed for every closed A, and $f_{-1}[U] = f^{-1}[U^c]^c$ for every open U.

(and hence it is open)

A MORE SYMMETRIC VERSION (for order-preserving maps)

Let $f\colon L \to M$ be a map between locales, with property

$$\forall \text{ closed } \underbrace{A}_{\mathfrak{c}(a)} \subseteq M \;\; \exists \text{ largest closed sublocale } \underbrace{A'}_{\mathfrak{c}(b) \text{ (unique } b)} \subseteq f^{-1}[A]. \qquad (*)$$

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$$a \overset{\phi}{\longmapsto} b$$

$\phi(a)$ is the smallest $y \in L$ with $\uparrow y \subseteq f^{-1}[\uparrow a]$, i.e. $\phi(a) \leq y \ \Rightarrow \ a \leq f(y)$.

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$$a \xrightarrow{\phi} b$$

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Lemma. ϕ is order-preserving, $1 \leq f\phi$, and $\phi f \leq 1$ iff f is order-preserving

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Then, one gets a version in which also the condition on preimages of closed sets is relaxed:

THEOREM 2

An order-preserving map $f\colon L \to M$ between locales is a localic map iff

– for every closed A, there exists a largest closed sublocale $A' \subseteq f^{-1}[A]$, and

– for every open U, $((U^{\mathrm{c}})')^{\mathrm{c}} \subseteq f^{-1}[U]$. (now: cannot speak of $f_{-1}[-]$)

MORE RESULTS: OPEN localic maps

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Johnstone (2006):

$(*)$ defines hereditary skeletal maps

$(*) \Rightarrow$ OPEN, if $f[L]$ is a complemented sublocale.

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- new proof of Joyal-Tierney Theorem that OPEN$\equiv$CHeyt.

> A localic map $f\colon L \to M$ is open if and only if f^* is a complete Heyting homomorphism.

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$\exists f_! \dashv f^* \dashv f$

(biadjoint morphism)

MORE RESULTS: OPEN localic maps

- new proof of Joyal-Tierney Theorem that OPEN≡CHeyt.

> A localic map $f: L \to M$ is open if and only if f^* is a complete Heyting homomorphism.

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$$f^*(b \to c) = f^*(b) \to f^*(c)$$

$$\Leftrightarrow \ f_!(f^*(b) \wedge a) = b \wedge f_!(a)$$

$$\Leftrightarrow \ f(a \to f^*(c)) = f_!(a) \to c$$

(Frobenius identities)

MORE results: BIADJOINT morphisms

- Completeness without Heyting (i.e. just biadjoint)

THEOREM 4

The left adjoint of a localic map $f : L \to M$ is complete ($\equiv$ biadjoint) iff for every open $U \subseteq L$ there is a unique minimal open $V \subseteq M$ such that $f[U] \subseteq V$.

(quasi-open localic maps)

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Hence $\mathfrak{o}(f_!(u))$ is the minimal such open.

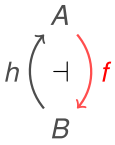
The general setting

Implicative semilattices (Heyting)

- $\wedge$-semilattices with top 1
- $\lambda_a = a \wedge (-)$ have right adjoints $\alpha_a = a \to (-)$

morphisms: residuated maps (=left adjoints) + preserve finite $\wedge$

- left adjoints
  ($\Rightarrow$ preserve existing $\bigvee$)
- preserve $(-) \wedge (-)$
- preserve 1



r-morphisms

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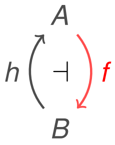

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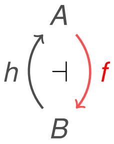

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r-morphisms $\qquad$ l-morphisms (localizations)

► One may not only regard the l-morphisms as abstract continuous maps in a (not necessarily complete) pointfree setting, but may also characterize them by concrete closure-theoretical continuity properties.

► These concepts provide generalizations of continuous and open maps between spaces to an algebraic (not necessarily complete) pointfree setting.

The general setting

CATEGORIES		CATEGORIES
implicative semilattices & implicative biadjoint maps	⇜ dual ⇝	implicative semilattices & open l-morphisms
implicative semilattices & biadjoint maps	⇜ dual ⇝	implicative semilattices & quasi-open l-morphisms

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Congrats Themba!