On the completion of pointfree function rings

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— joint work with J. Gutiérrez García and I. Mozo Carollo (Bilbao, Spain)

POINTFREE TOPOLOGY

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topology in the category $\mathbf{Loc}$
- locales
- localic maps

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- frames: $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$
- frame homomorphisms: preserve
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- frame homomorphisms: preserve $\bigvee, 0$ and $\wedge, 1$

Algebraic category: *presentations by generators and relations.*

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Natural extension:

$$\mathsf{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

ALGEBRAIC OPERATIONS IN $\mathsf{C}(L)$

$f\colon \mathfrak{L}(\mathbb{R}) \to L$

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$$f \leqslant g \quad \text{iff} \quad f(p,—) \leqslant g(p,—)\ \forall p \in \mathbb{Q}$$

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For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p,—) = \bigvee_{r \diamond s > p} f(r,—) \wedge g(s,—) = \cdots$$

$$(f \diamond g)(—,q) = \bigvee_{r \diamond s < q} f(—,r) \wedge g(—,s) = \cdots$$

[JGG & JP, *Rings of real functions in pointfree topology*, TAA 2011]

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BACKGROUND:

Definition. A poset P is Dedekind complete (=conditionally complete)

if every bounded $\emptyset \neq A \subseteq P$ has a supremum and an infimum in P.

(more useful since we are dealing with l.o. groups with no $\perp$ and $\top$)

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BACKGROUND:

A Dedekind completion of P is a join- and meet-dense embedding

$$\Phi\colon P \to \mathcal{D}(P)$$

in a Dedekind complete poset.

$$\forall\, \hat{p} \in \mathcal{D}(P) \quad \boxed{\hat{p} = \bigvee^{\mathcal{D}(P)} \{\Phi(p) \mid \Phi(p) \leqslant \hat{p}\} = \bigwedge^{\mathcal{D}(P)} \{\Phi(p) \mid \Phi(p) \geqslant \hat{p}\}}$$

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Natural candidate:

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

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$h \in \mathsf{C}(L) \Leftrightarrow$

- **(R1)** if $p \geqslant q$, then $h(p,\text{—}) \wedge h(\text{—},q) = 0$
- **(R2)** if $p < q$, then $h(p,\text{—}) \vee h(\text{—},q) = 1$
- **(R3)** $h(p,\text{—}) = \bigvee_{r>p} h(r,\text{—})$ and $h(\text{—},q) = \bigvee_{s<q} h(\text{—},s)$
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$$h \in \mathsf{C}(L) \Leftrightarrow \begin{cases} \textbf{(R1)} \text{ if } p \geqslant q, \text{ then } h(p,-) \wedge h(-,q) \overset{\text{(R1) for } f_i}{=\cdots} = 0 \quad ✓ \\ \textbf{(R2)} \text{ if } p < q, \text{ then } h(p,-) \vee h(-,q) = 1 \\ \textbf{(R3)} \ h(p,-) = \bigvee_{r>p} h(r,-) \text{ and } h(-,q) = \bigvee_{s<q} h(-,s) \\ \textbf{(R4)} \ \bigvee_{p\in\mathbb{Q}} h(p,-) = 1 = \bigvee_{q\in\mathbb{Q}} h(-,q) \end{cases}$$

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(R2) if $p < q$, then $h(p,-) \vee h(-,q) \neq 1$ in general.

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$\mathsf{C}(L)$ fails to be Dedekind complete because of (R2) !

Approach I:

Partial real functions

J. Gutiérrez García, I. Mozo Carollo & JP
On the Dedekind completion of function rings, **Forum Math., in press.**

IDEA: DELETE RELATION (R2)

Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
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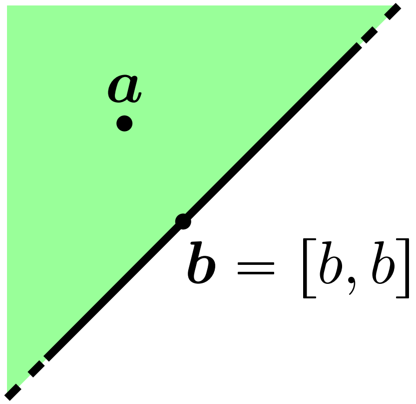

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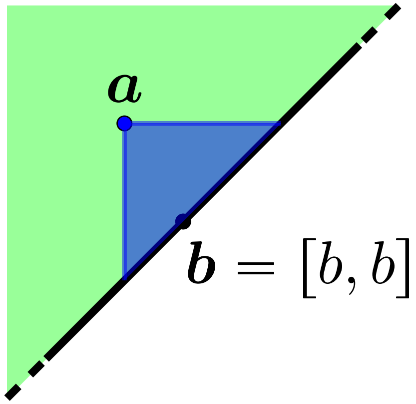

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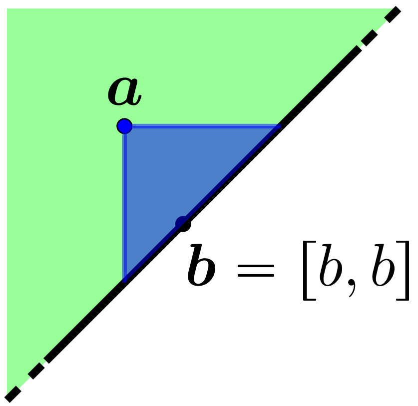

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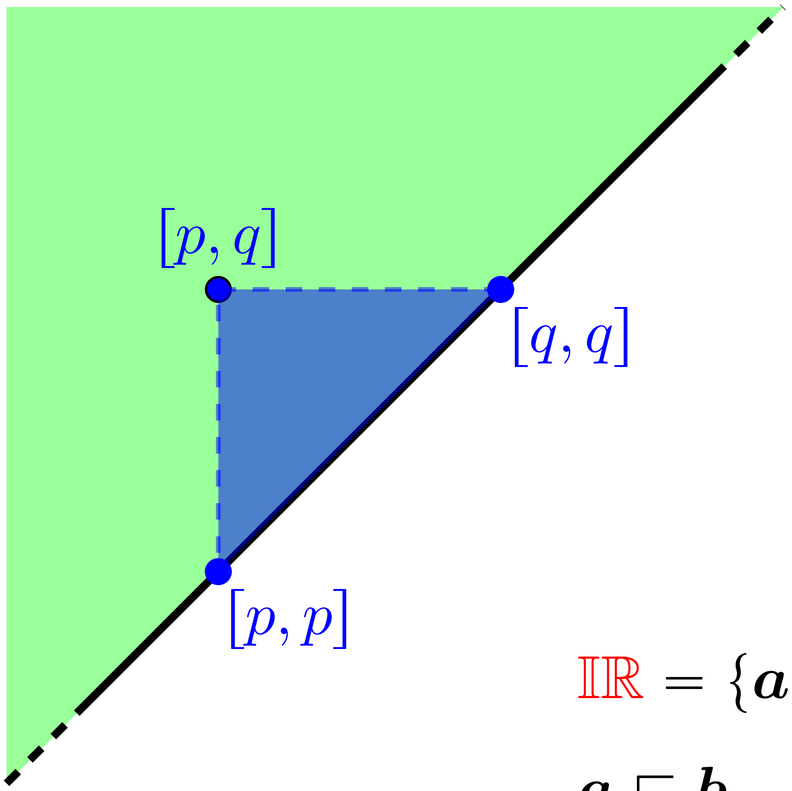

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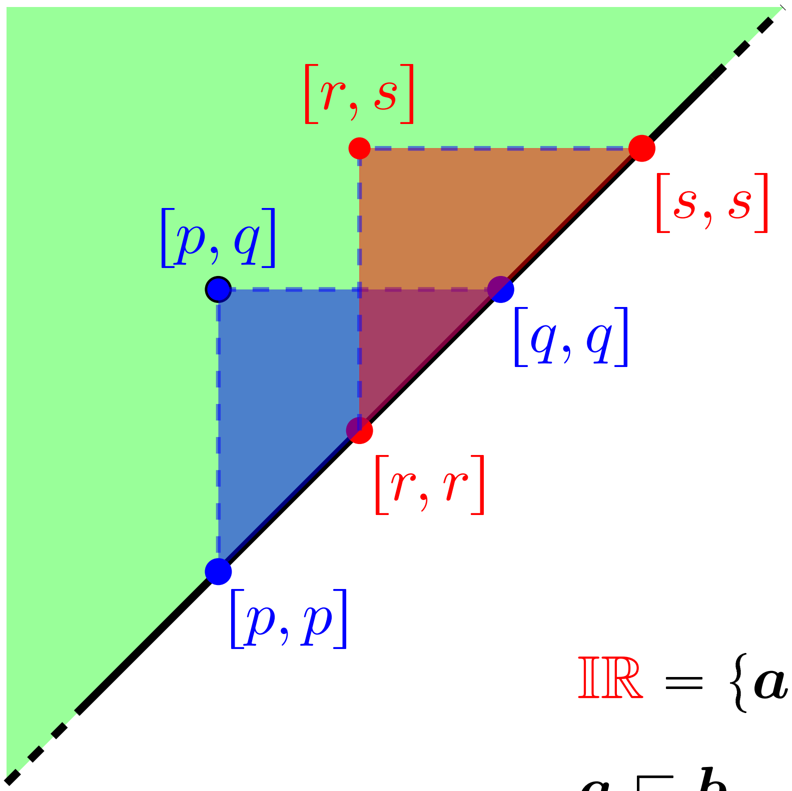

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PARTIAL CONTINUOUS REAL FUNCTIONS POINTFREELY

- The spectrum $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ (interval-domain space).

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

- For $L = \mathfrak{L}(\mathbb{IR})$: $\mathsf{IC}(X) = \mathbf{Top}(X, \mathbb{IR}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathfrak{O}X)$

Natural extension: $\mathsf{IC}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$

$$f\colon \mathfrak{L}(\mathbb{IR}) \to L$$

The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

Generators	$(p,—), (—,q), \quad p,q \in \mathbb{Q}$
Relations	(R1) $(p,—) \wedge (—,q) = 0$ whenever $p \geqslant q$
	(R2) $(p,—) \vee (—,q) = 1$ whenever $p < q$
	(R3) $(p,—) = \bigvee_{r>p}(r,—)$ and $(—,q) = \bigvee_{s<q}(—,s)$
	~~(R4) $\bigvee_{p\in\mathbb{Q}}(p,—) = 1 = \bigvee_{q\in\mathbb{Q}}(—,q)$~~

A SIDE NOTE

The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

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Similarly, we have the extended continuous real functions:

$$\overline{\mathrm{C}}(L)=\mathbf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}),L)$$

B. BANASCHEWSKI, J. GUTIÉRREZ GARCÍA & J. P.
Extended real functions in pointfree topology, *J. Pure Appl. Algebra* **216 (2012)**

DEDEKIND COMPLETENESS OF $\mathsf{C}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{C}(L)$ and $f \in \mathsf{C}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{C}(L)$?

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Natural candidate (again):

$$h(p,-) = \bigvee_{i\in I} f_i(p,-) \quad \text{and} \quad h(-,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(-,s)\Big)$$

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$h \in \mathsf{C}(L) \Leftrightarrow$

- (R1) if $p \leqslant q$, then $h(\text{—},p) \wedge h(q,\text{—}) = 0$
- (R3) $h(p,\text{—}) = \bigvee_{r>p} h(r,\text{—})$ and $h(\text{—},q) = \bigvee_{s<q} h(\text{—},s)$
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- (R3) $h(p,—) = \bigvee_{r>p} h(r,—)$ and $h(—,q) = \bigvee_{s<q} h(—,s)$ ✓
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q)$ ✓

Hence $h \in \mathsf{IC}(L)$. Moreover, $h = \bigvee_{i\in I}^{\mathsf{IC}(L)} f_i$.

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THEOREM. $\mathsf{IC}(L)$ is Dedekind complete.

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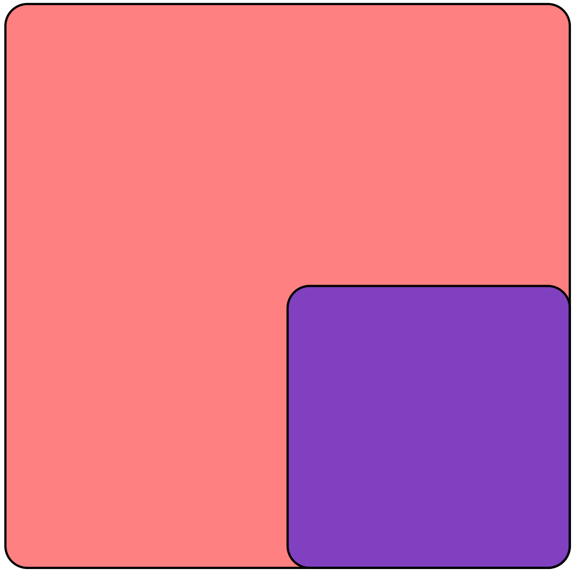
Of course, we may consider $\mathsf{C}(L)$ as a subset of $\mathsf{IC}(L)$:



$$\mathsf{C}(L) = \{h \in \mathsf{IC}(L) \mid h(p,\text{—}) \vee h(\text{—},q) = 1 \text{ for every } p < q\}$$

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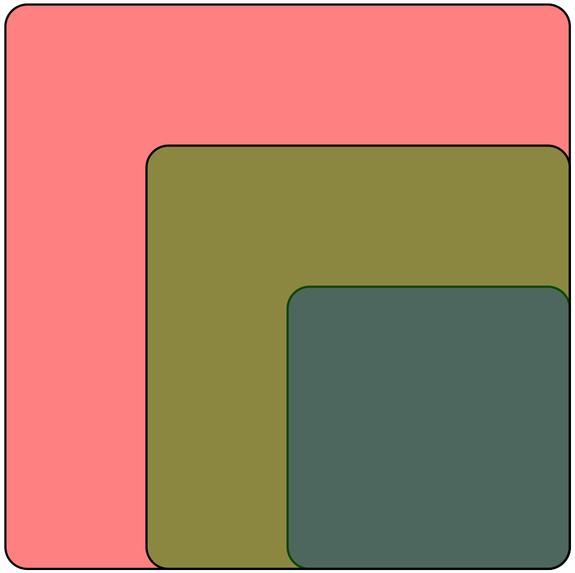

$\mathcal{D}(\mathsf{C}(L)) = $?

$$\mathsf{C}(L) = \{h \in \mathsf{IC}(L) \mid h(p,—) \vee h(—,q) = 1 \text{ for every } p < q\}$$

Since $\mathsf{IC}(L)$ is Dedekind complete, it must contain the Dedekind completion of $\mathsf{C}(L)$.

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From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

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(R2) $\underbrace{h(p,\text{—}) \vee h(\text{—},q) = 1}_{\forall p<q} \quad \Rightarrow \quad$ (R2′) $\begin{cases} h(p,\text{—})^* \leqslant h(\text{—},q) \\ h(\text{—},q)^* \leqslant h(p,\text{—}) \end{cases} \quad \forall p < q$

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COROLLARY. $\mathsf{C}(L)$ is Dedekind complete iff L is extr. disconnected.

[Banaschewski & Hong, 2003]

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- $\overline{\mathsf{C}}(L)$: extended functions

• • •

Approach II:

Semicontinuous real functions

J. Gutiérrez García, I. Mozo Carollo & JP
Normal semicontinuity and
the Dedekind completion of pointfree function rings, **2014, submitted.**

- Any $f: X \longrightarrow \mathbb{R}$

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

Motivation

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

  i.e. $\mathsf{F}(X) \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathfrak{T}))$

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Natural extension: $\boxed{\mathsf{F}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))}$

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- $\mathcal{S}(L)$ is NOT e.d. in general and so

  $\mathsf{F}(L)$ is NOT necessarily Dedekind complete...

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It is a co-frame!

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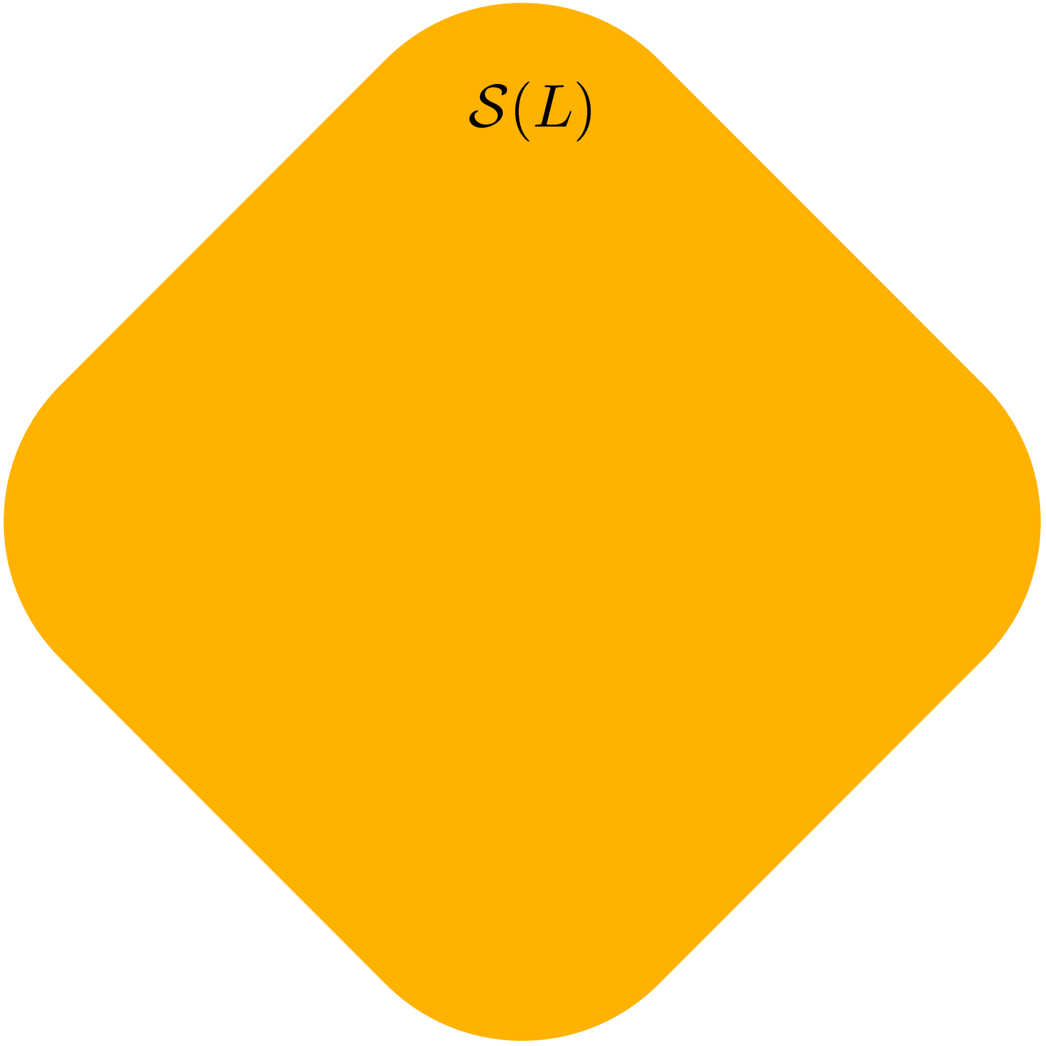
$\mathcal{S}(L)$: the DUAL FRAME



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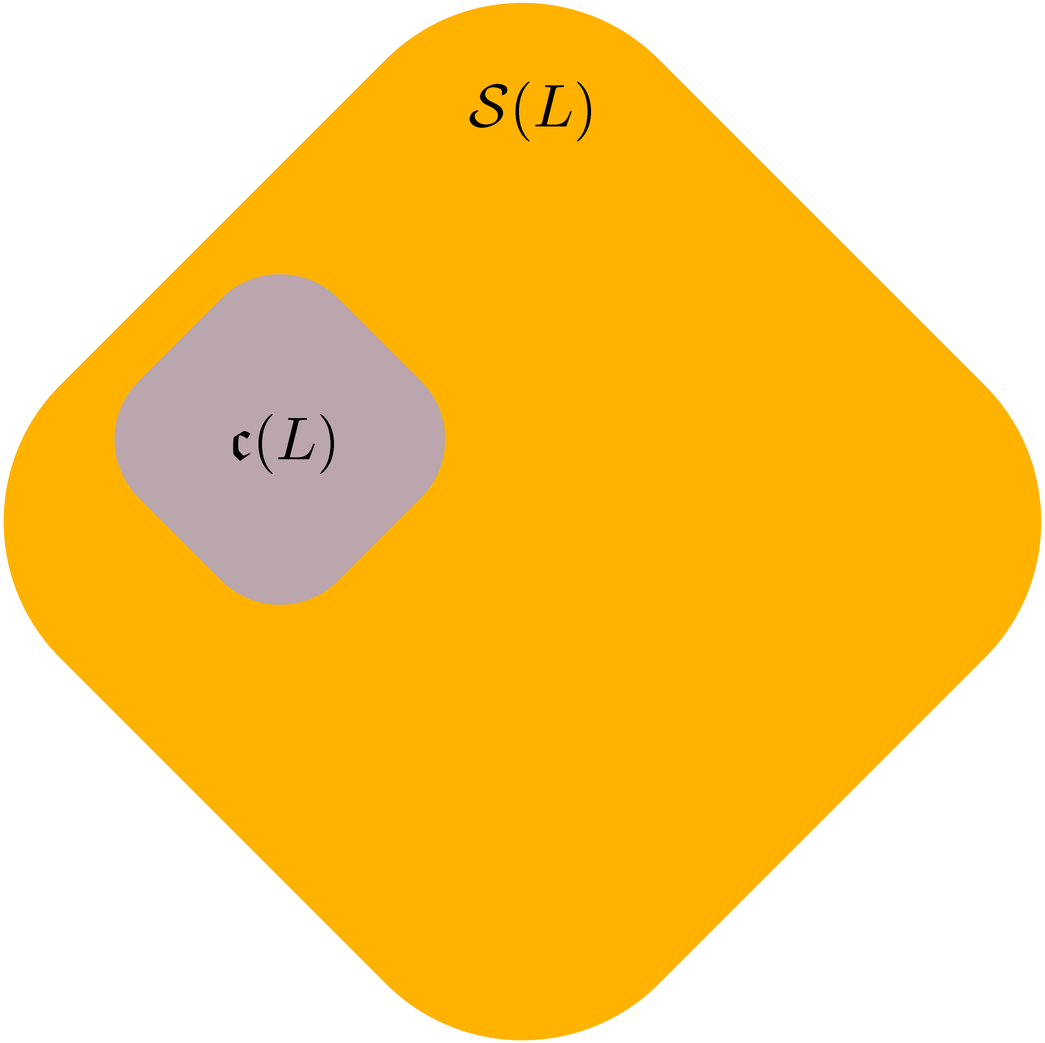

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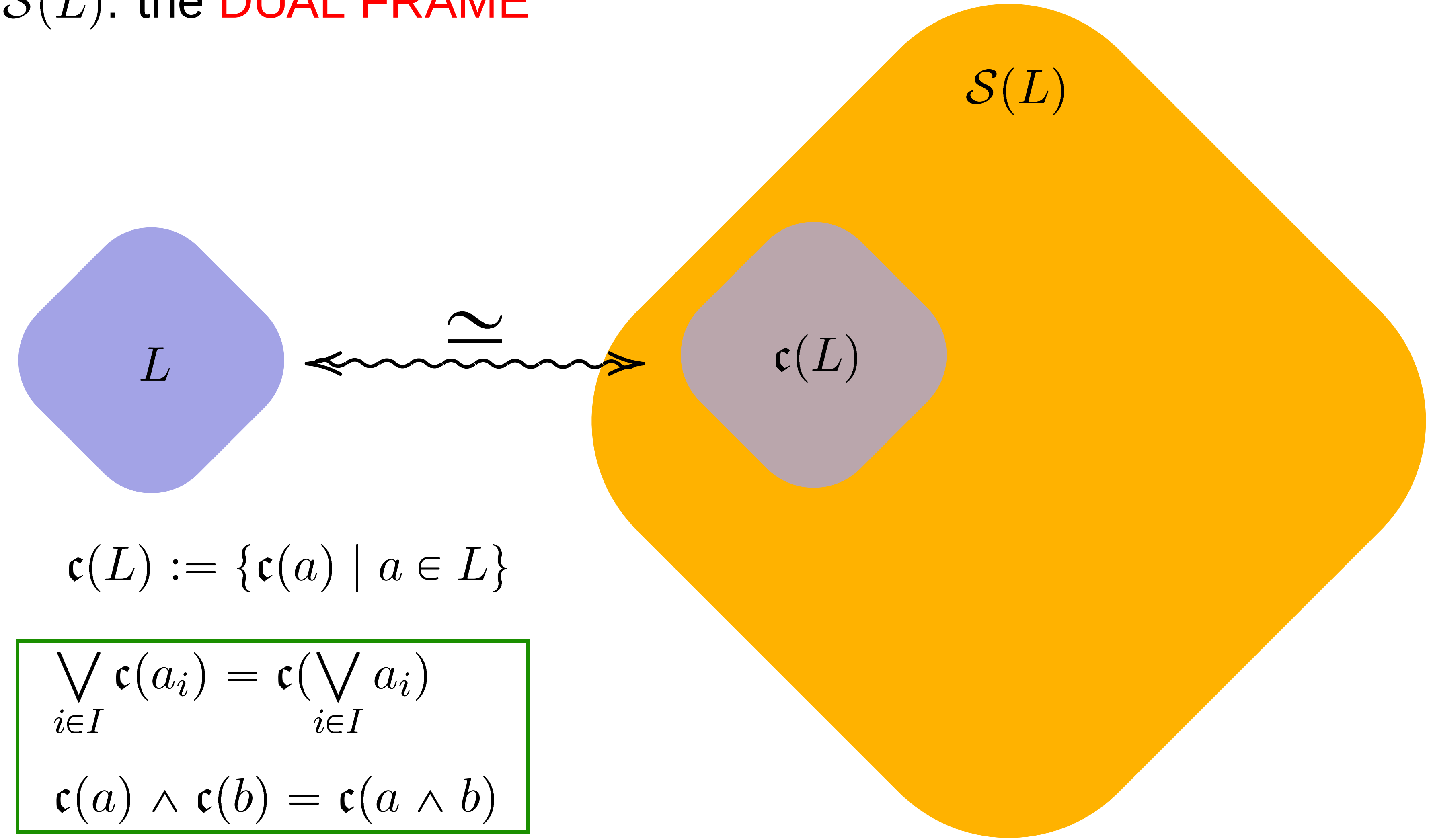

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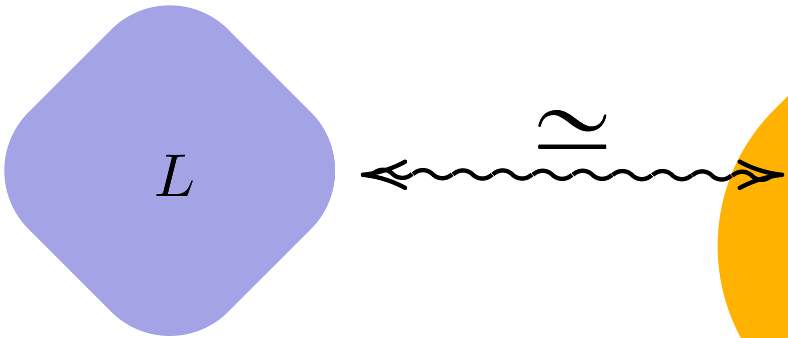

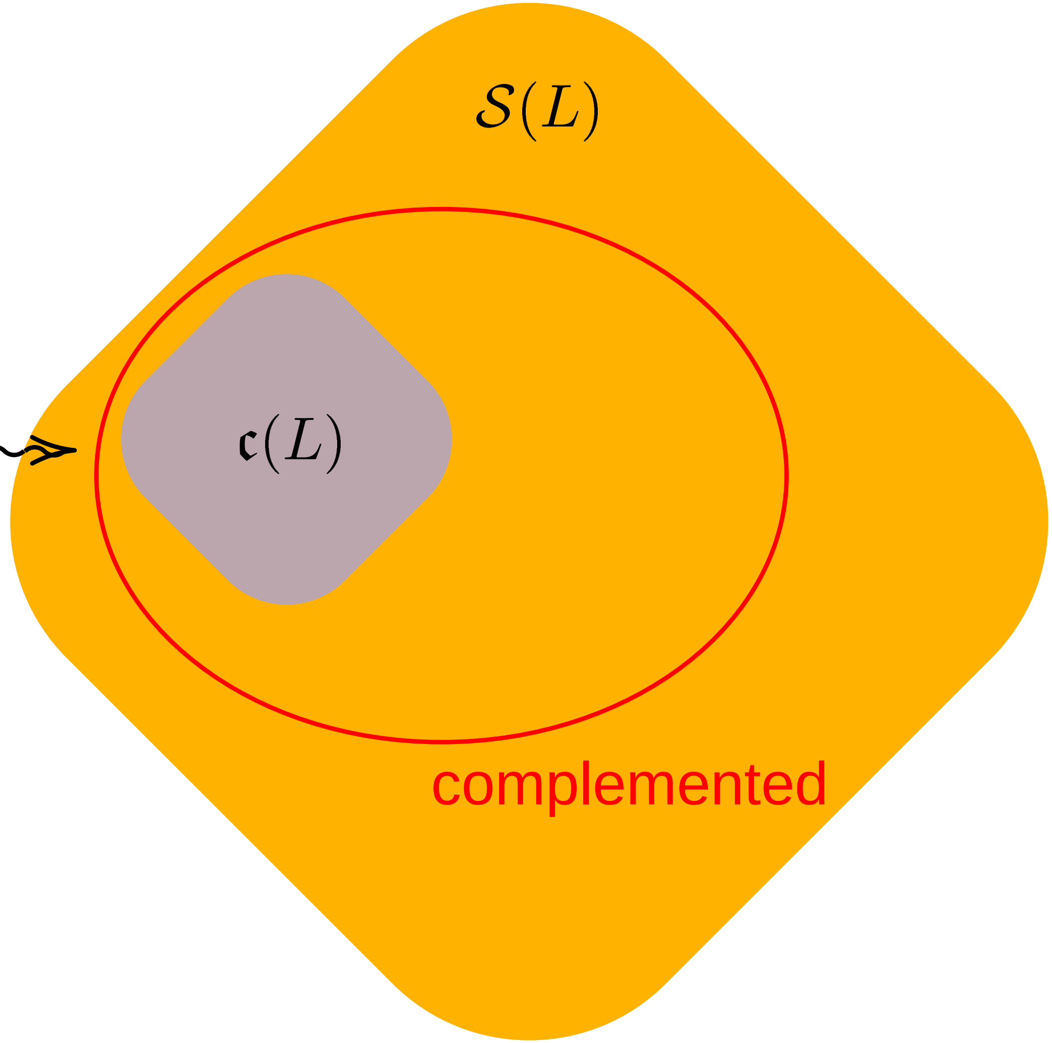

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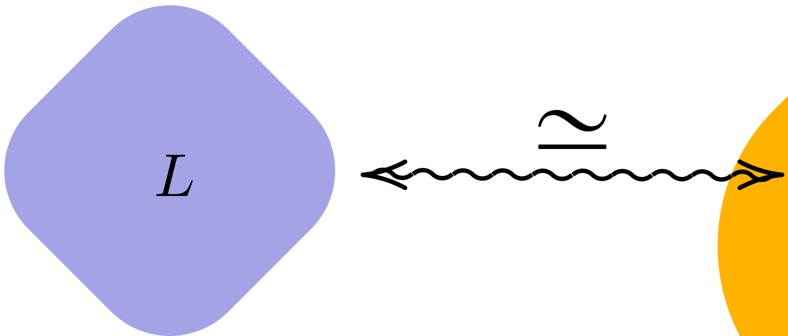

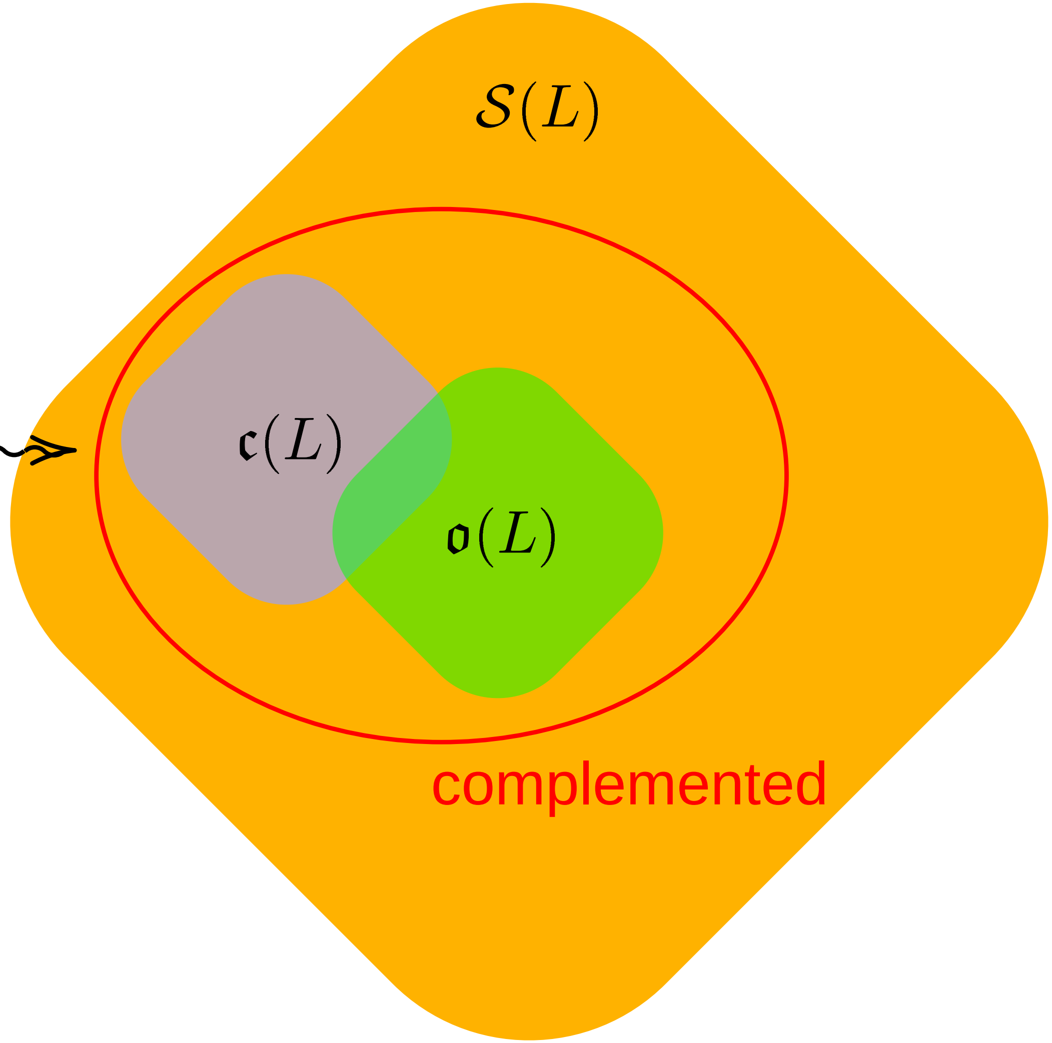

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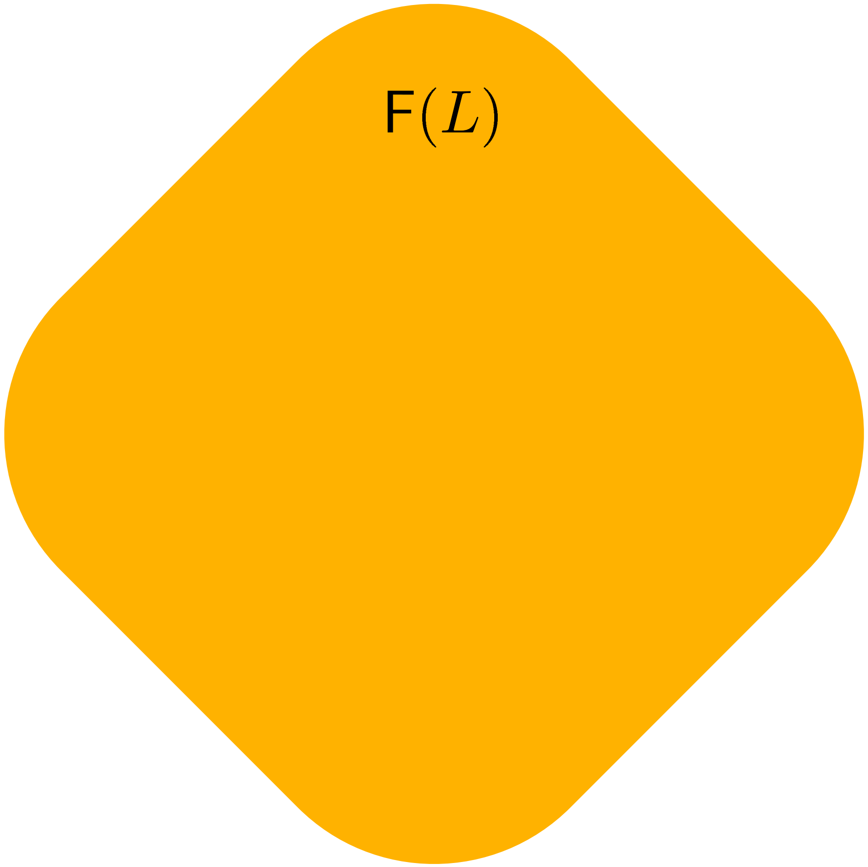

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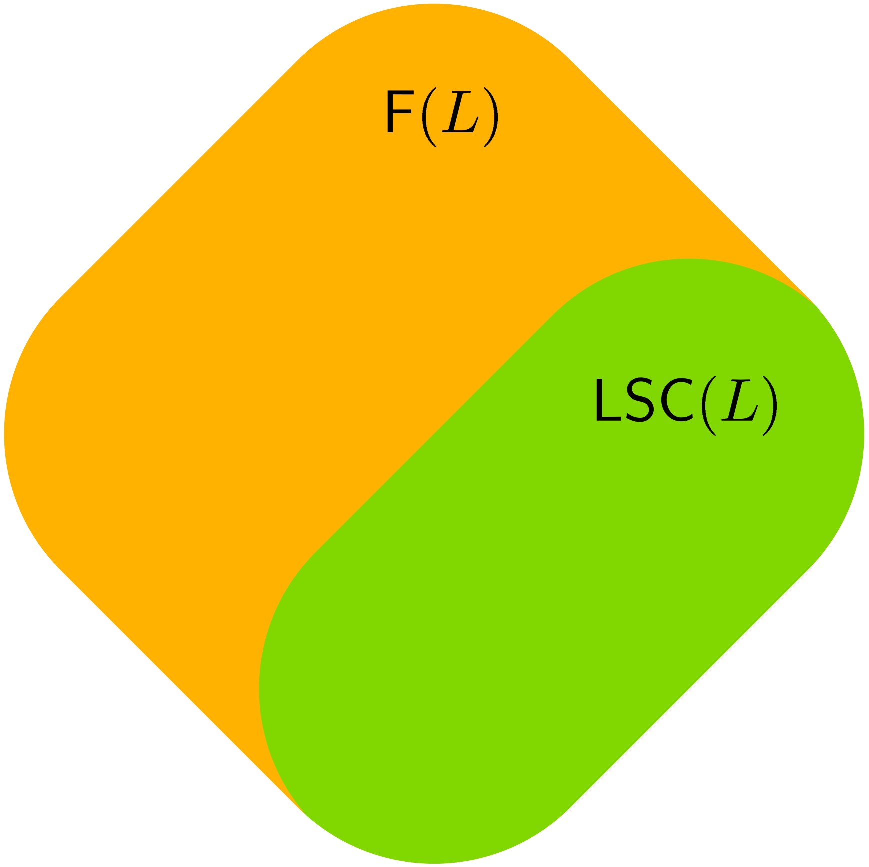

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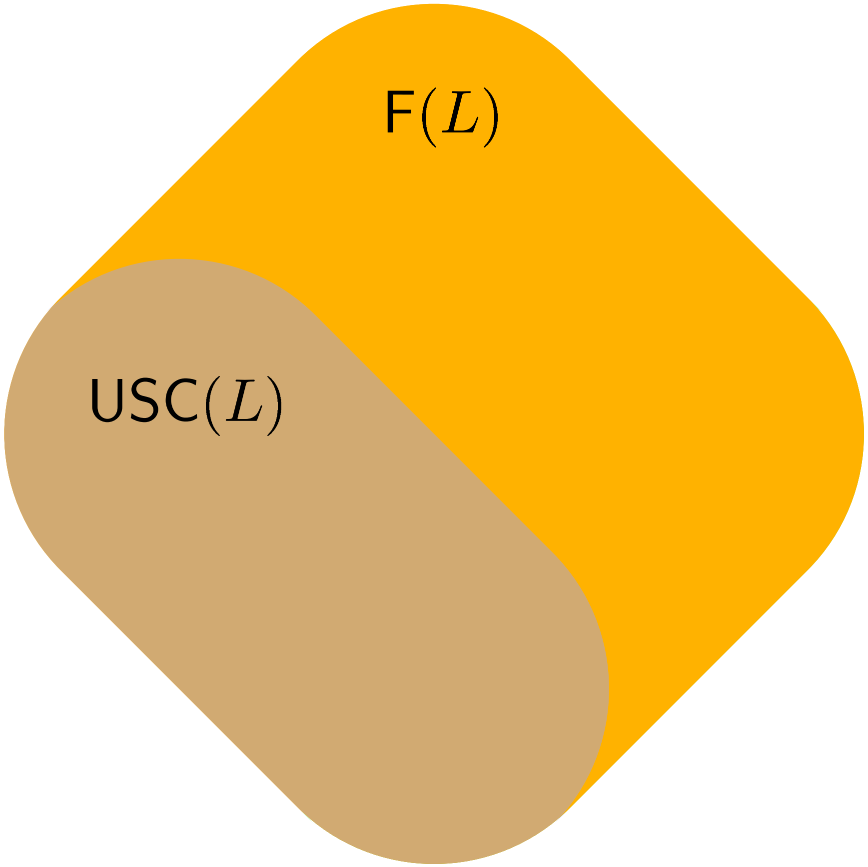

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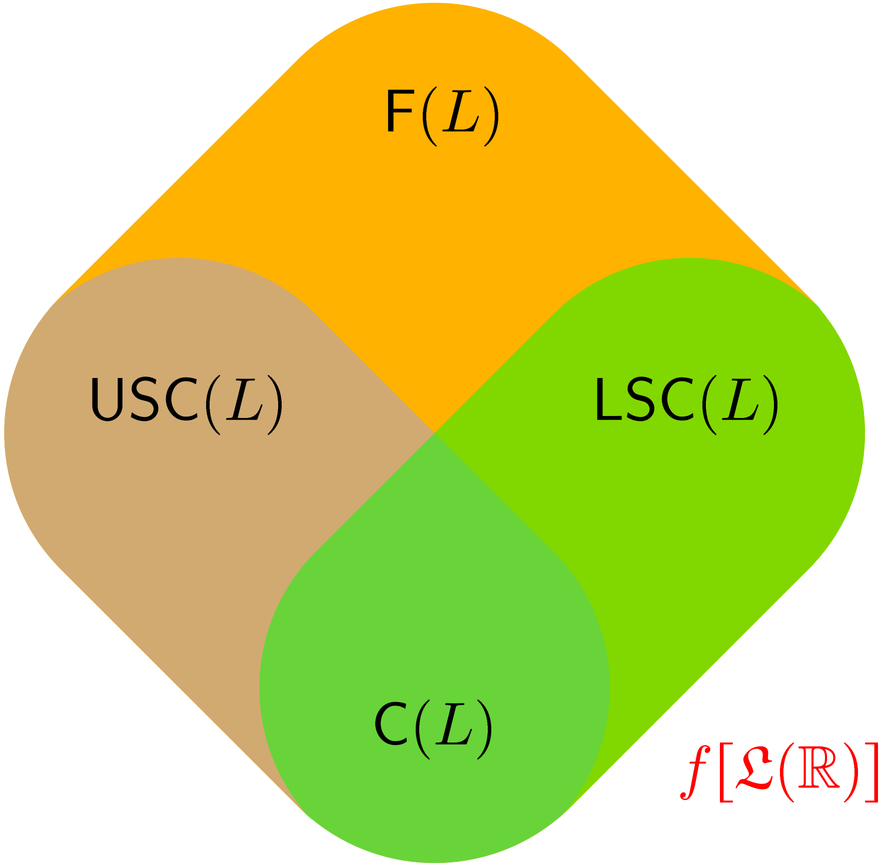

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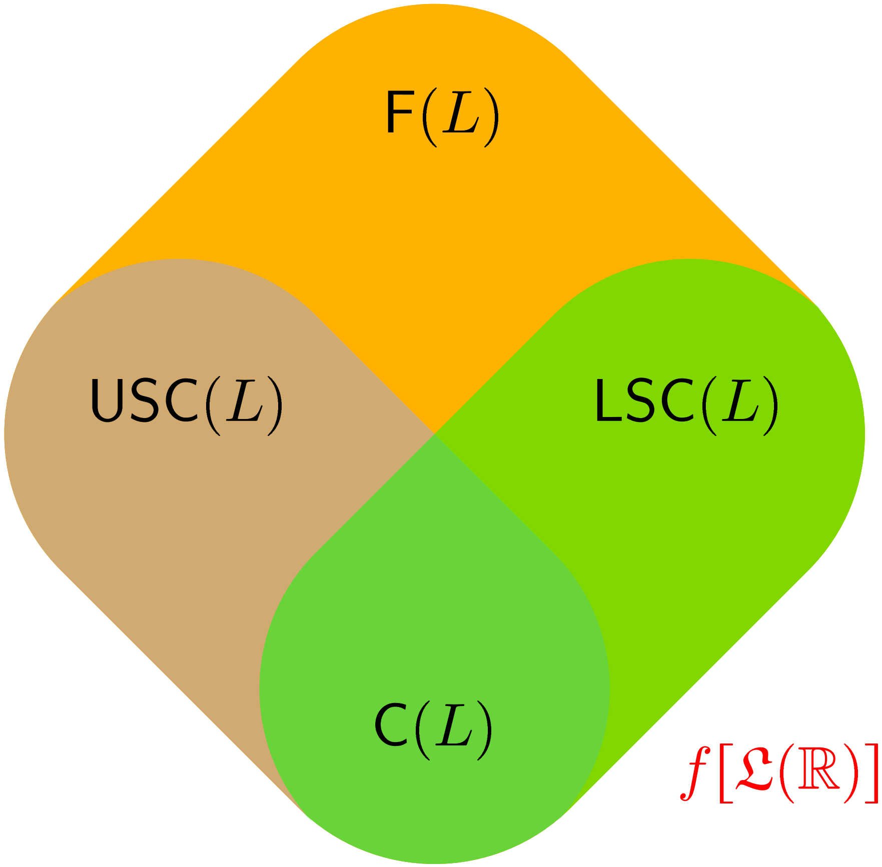

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J. Gutiérrez García, T. Kubiak & J. P.
Localic real functions: a general setting, *J. Pure Appl. Algebra* **213 (2009) 1064-1074**

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

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Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p, -) = \bigvee_{r>p} \overline{f(r, -)}$$

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- Dually: the upper regularization $f^- = -(-f)^\circ$

J. GUTIÉRREZ GARCÍA, T. KUBIAK & J. P.
Lower and upper regularizations of frame semicontinuous real functions, *Alg. Univ.* (2009)

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

- BOUNDED: $\exists\, p < q \colon f(p, \text{—}) = 1 = f(\text{—}, q)$ $\qquad \mathsf{F}^*(L)$

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[classically: Dilworth, 1950]

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PROPOSITION. TFAE for a frame L:

(1) L is weak cb.

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THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset P with no $\bot$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

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THEOREM. Let L be a completely regular frame. Then:

(1) $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}^{cb}(L)$. (2) $\mathcal{D}(\mathsf{C}^*(L)) \simeq \mathsf{NLSC}^*(L)$.

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COROLLARIES.

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THE COMPLETION AS A FUNCTION RING

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

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bounded case

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For every completely regular frame L, there exists a (unique) completely regular and extremally disconnected frame $\mathfrak{G}(L)$ and a proper essential embedding $\gamma_L \colon L \hookrightarrow \mathfrak{G}(L)$.

[B. Banaschewski, 1988]

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$$L \xrightarrow{\gamma_L} \mathfrak{G}(L) \xrightarrow{\forall f} M$$

embedding $\Rightarrow$ f is an embedding

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