

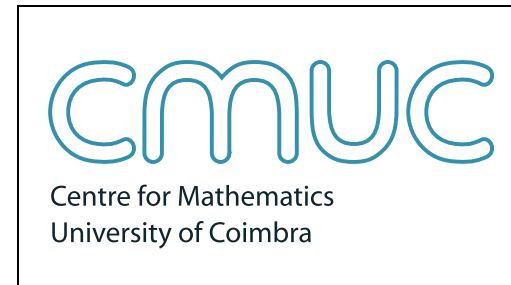
On (binary) localic products and localic groups

Jorge Picado

Department of Mathematics

University of Coimbra

PORTUGAL



— *joint work with Aleš Pultr (Charles University, Prague, CZ)*

- Complete lattices L satisfying

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

(= complete Heyting algebras)

THE SETTING

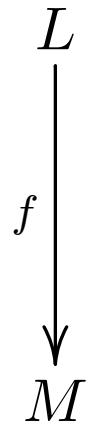
locales (or frames)

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Loc



Frm

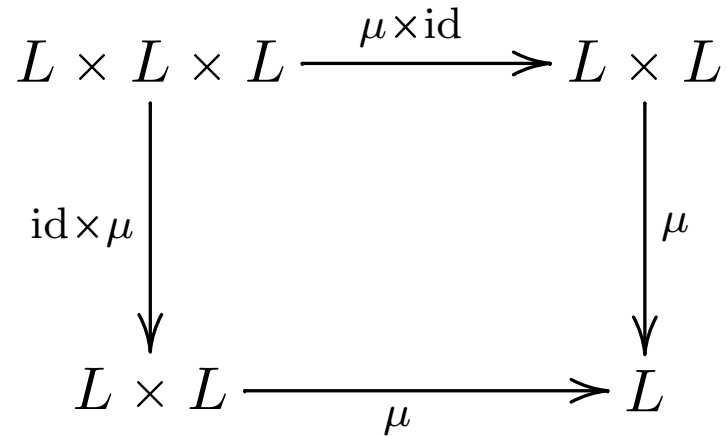
preserves \bigvee (incl. the bottom 0)

\wedge (incl. the top 1)

- $\mu : L \times L \rightarrow L$

“multiplication”

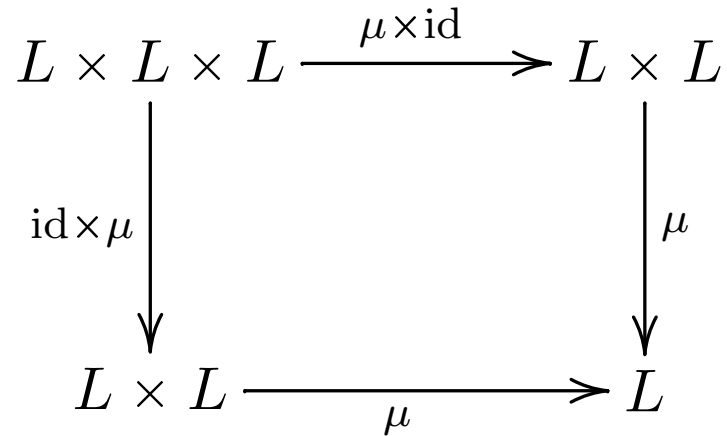
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$$a(bc) = (ab)c$$

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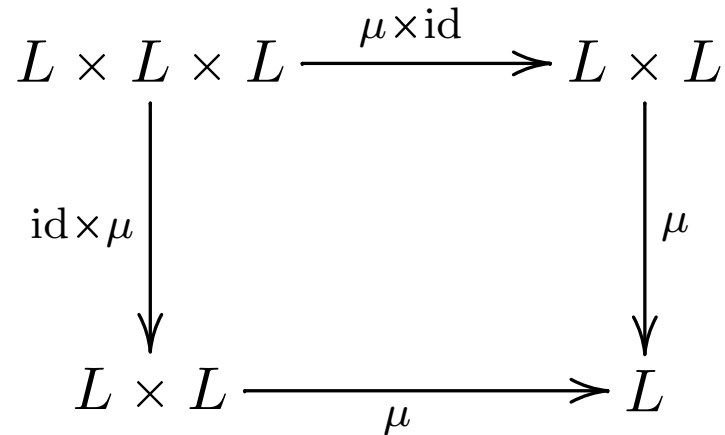


$$a(bc) = (ab)c$$

- $\varepsilon : \mathbf{2} = \{0, 1\} \rightarrow L$

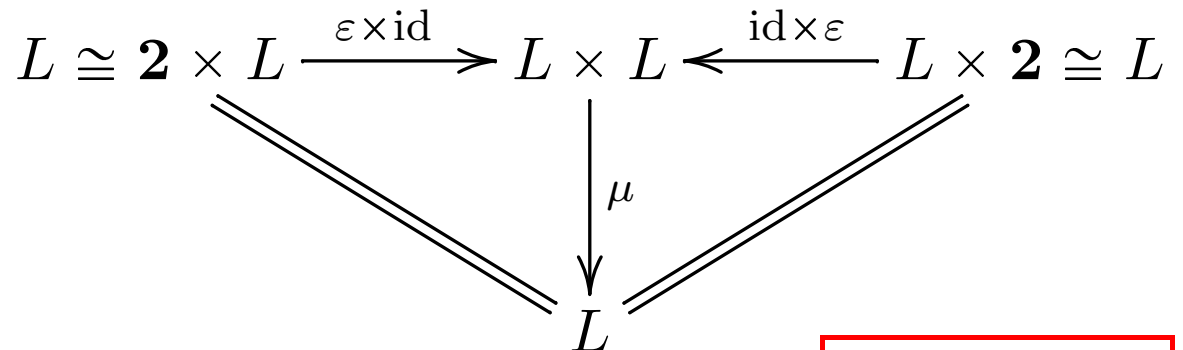
“unit”

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$$\varepsilon a = a = a \varepsilon$$

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“multiplication”

$$\begin{array}{ccc}
 L \times L \times L & \xrightarrow{\mu \times \text{id}} & L \times L \\
 \downarrow \text{id} \times \mu & & \downarrow \mu \\
 L \times L & \xrightarrow{\mu} & L
 \end{array}$$

$$a(bc) = (ab)c$$

- $\iota : L \rightarrow L$

“inverse”

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 \downarrow \text{id} \times \iota & & \downarrow ! & & \downarrow \iota \times \text{id} \\
 L \times L & & \mathbf{2} & & L \times L \\
 \downarrow \mu & & \downarrow \varepsilon & & \downarrow \mu \\
 L & & L & & L
 \end{array}$$

$$aa^{-1} = \varepsilon = a^{-1}a$$

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It is an improvement of classical TopGrp: **Closed Subgroup Theorem...**

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J. Isbell, I. Kříž, A. Pultr, J. Rosický, *LNM* 1348 (1987) 154-172

BACKGROUND: BINARY PRODUCTS IN Loc

Frm The *coproduct* $L \otimes M$ of L and M :

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$$1 \otimes b \longleftarrow b$$

UNIFORMITIES (Tukey type)

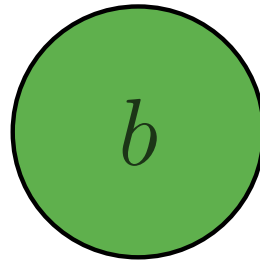
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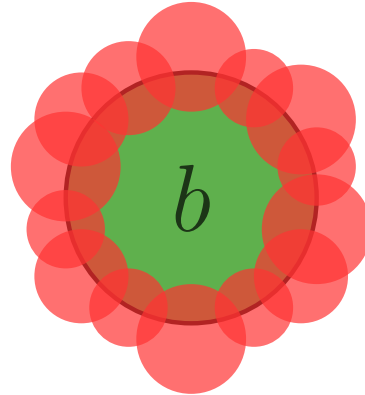
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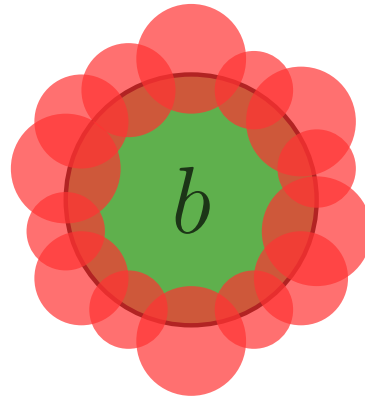


“the **star** of $b \in L$ in U ”

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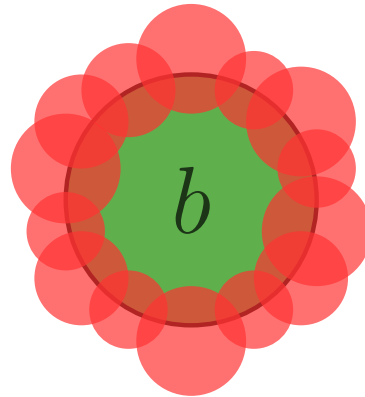
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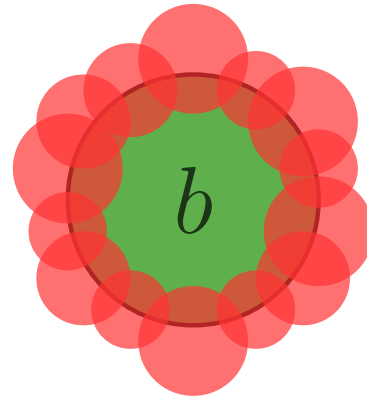


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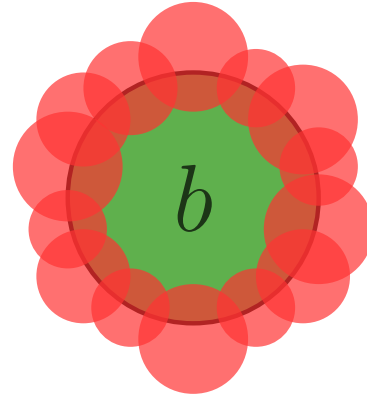
\mathcal{U} = system of covers

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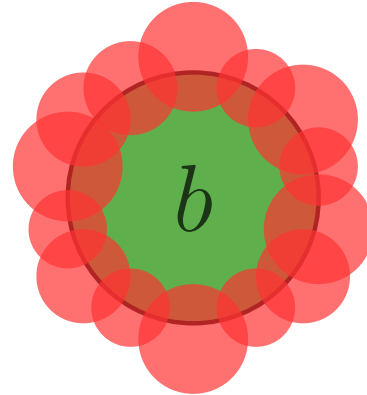
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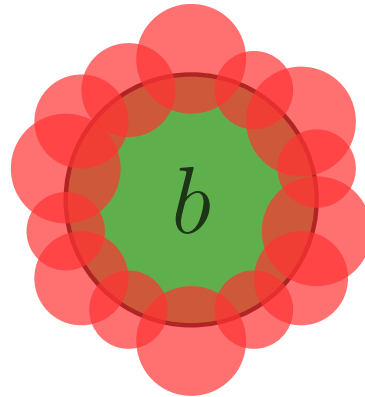
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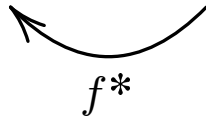
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Uniform locale (L, \mathcal{U}) : non-void filter \mathcal{U} of covers such that

- (U1) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$,
- (U2) for each $a \in L$, $a = \bigvee \{b \in L \mid b \triangleleft_{\mathcal{U}} a\}$.

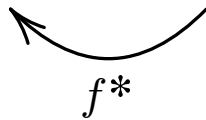
Uniform maps: $f : (L, \mathcal{U}) \rightarrow (M, \mathcal{V})$

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frame homomorphism

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frame homomorphism

$f^*[V] \in \mathcal{U}$ for all $V \in \mathcal{V}$

Neighbourhoods of the unit:

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“left” uniformity $\mathcal{U}_l(L)$

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Any localic group is complete in its two-sided uniformity.

B. BANASCHEWSKI & J. VERMEULEN

On the completeness of localic groups, CMUC 40 (1999) 293-307

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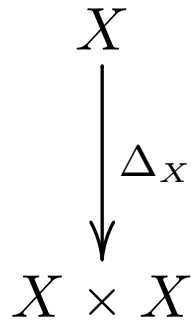
QUESTION: are $L \mapsto (L, \mathcal{U}_l(L))$ and $L \mapsto (L, \mathcal{U}_r(L))$ functorial?

Spaces

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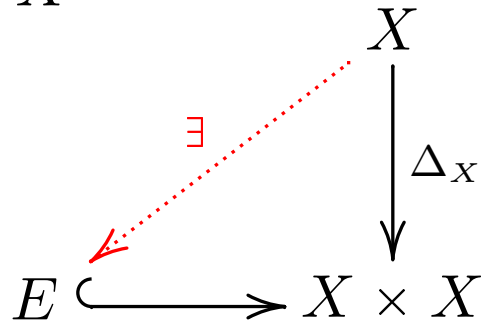
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$$\Delta_X(X) \subseteq E$$

Spaces

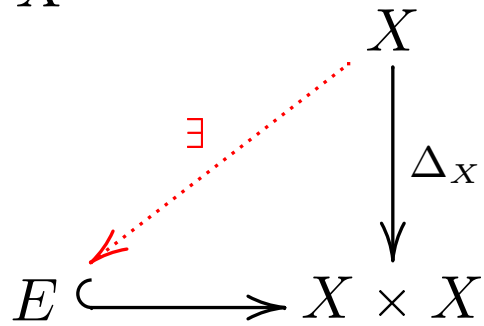
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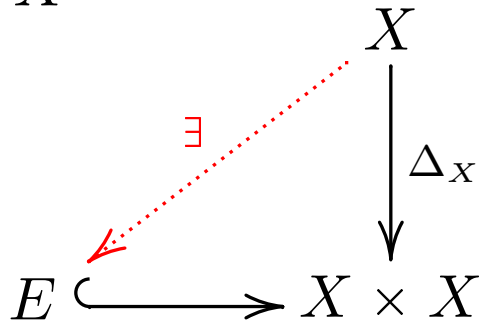


$$E \in \Omega(X \times X)$$

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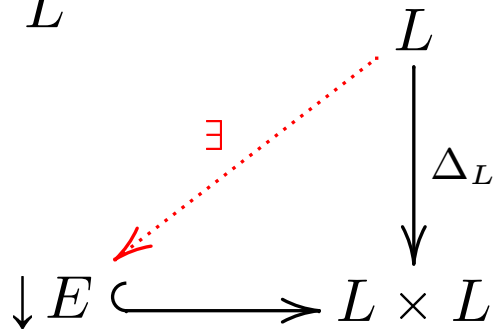


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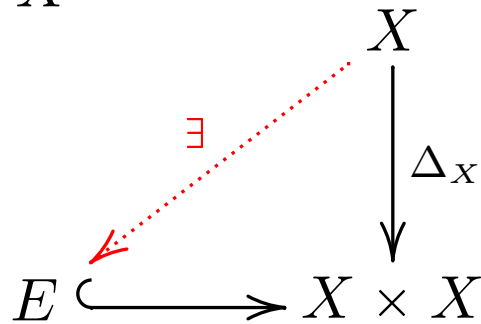
Locales

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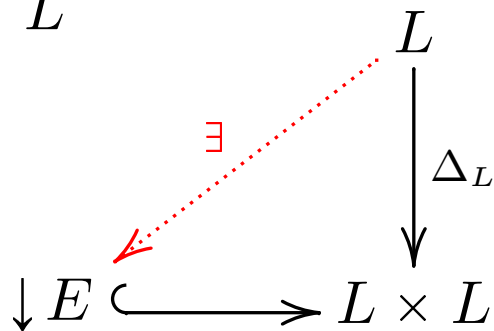
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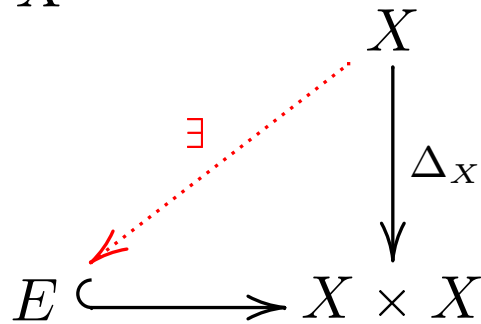
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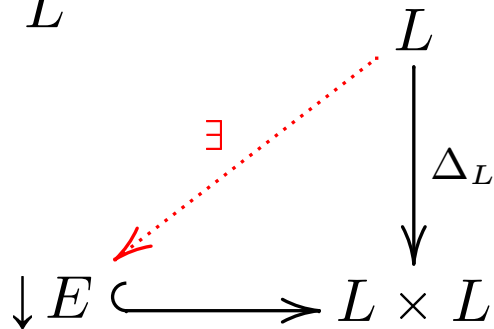
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pointfreely: happens in $\Omega(X) \times \Omega(X)$

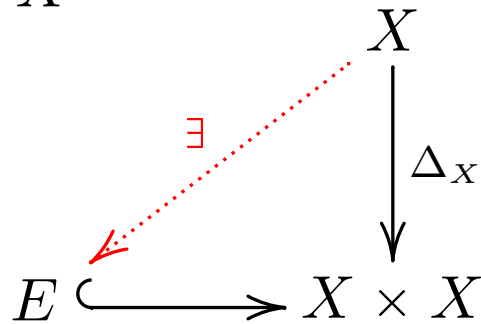
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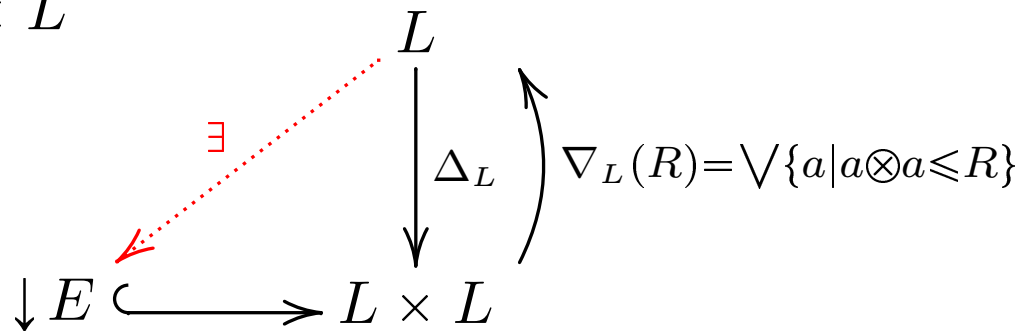
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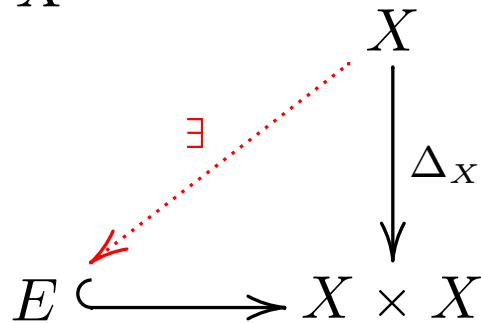
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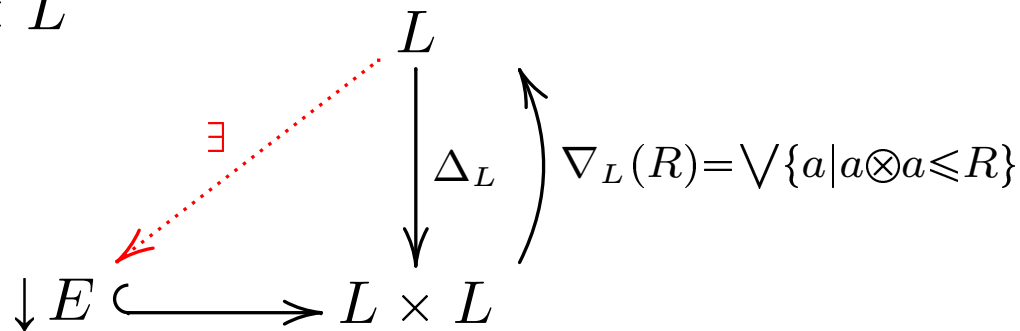
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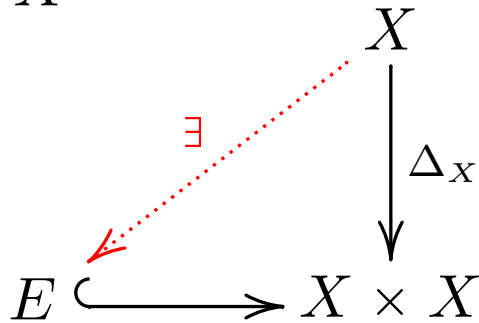
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$$\nabla_L(E) = 1$$

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$$E \subseteq X \times X$$



$$\Delta_X(X) \subseteq E$$

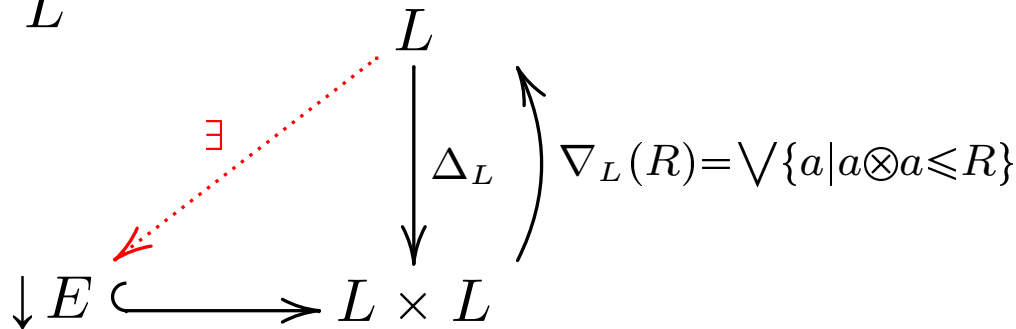
$$E \in \Omega(X \times X)$$

classically: happens in $\Omega(X \times X)$

pointfreely: happens in $\Omega(X) \times \Omega(X)$

Locales

$$E \in L \times L$$



$$\nabla_L(R) = \bigvee \{a \mid a \otimes a \leq R\}$$

$$\bigvee \{a \mid a \otimes a \leq E\}$$

||

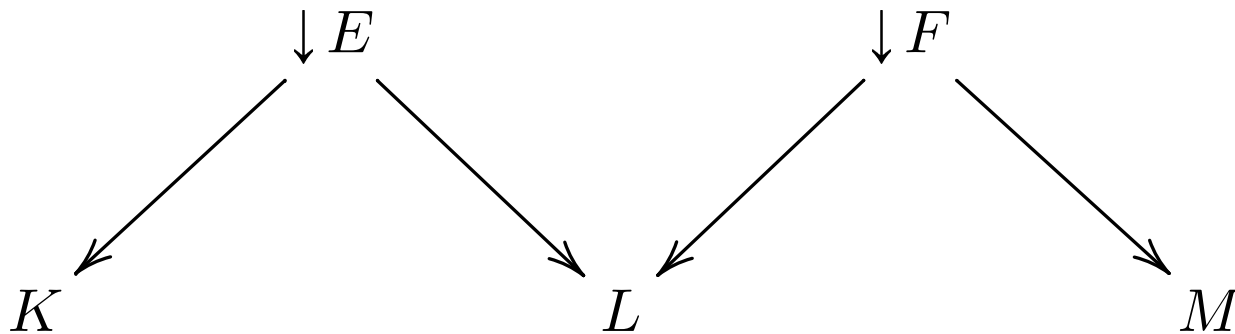
$$\nabla_L(E) = 1$$

UNIFORMITIES (Weil type)

entourages: $E \in L \otimes L$ such that $\bigvee \{a \mid a \otimes a \leq E\} = 1$. $(\text{Ent } L, \leq)$

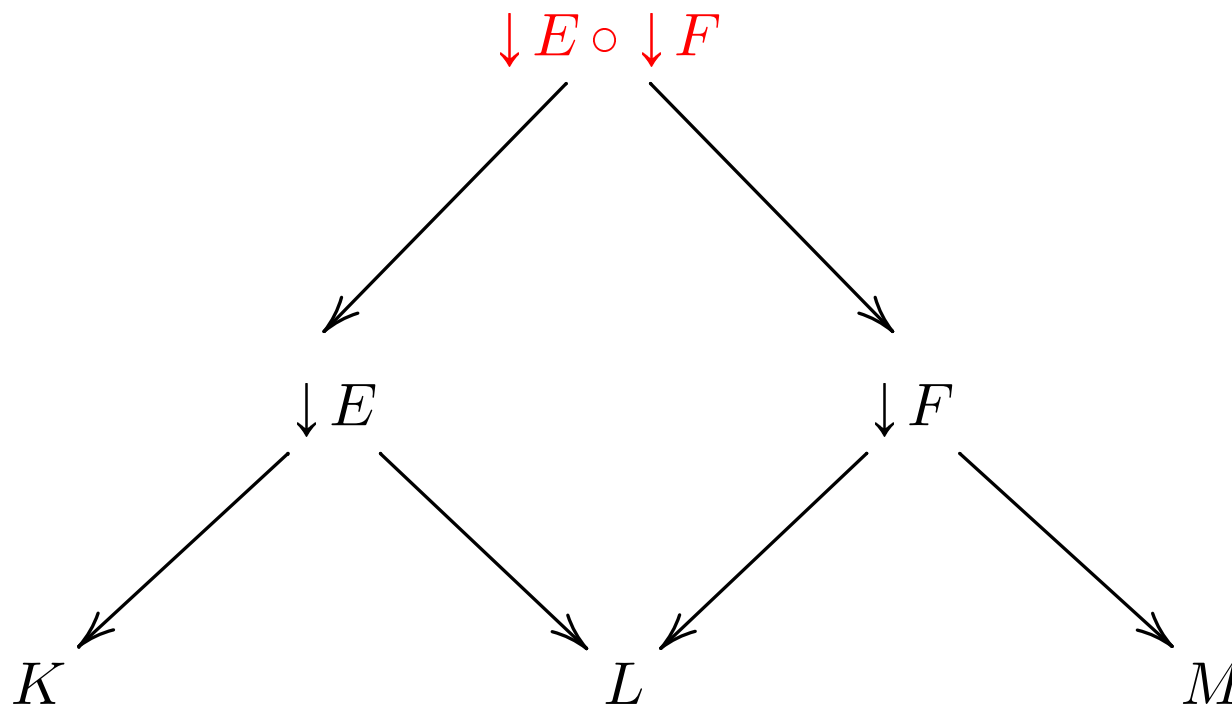
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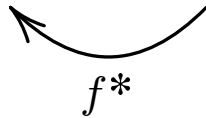
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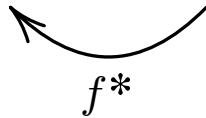
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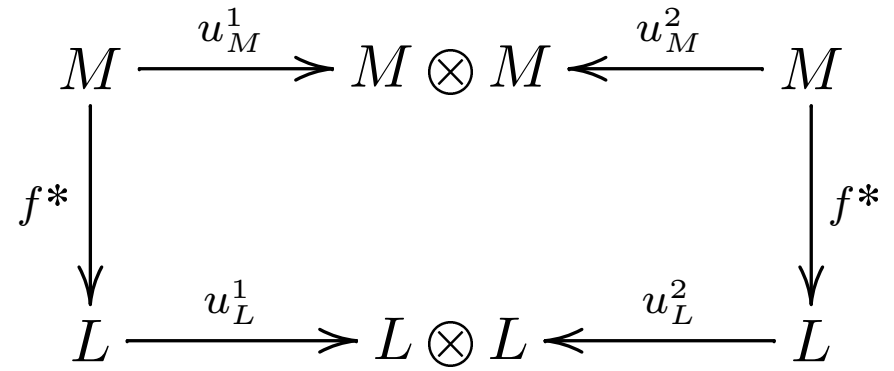


frame homomorphism

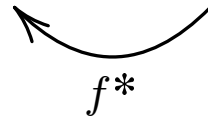
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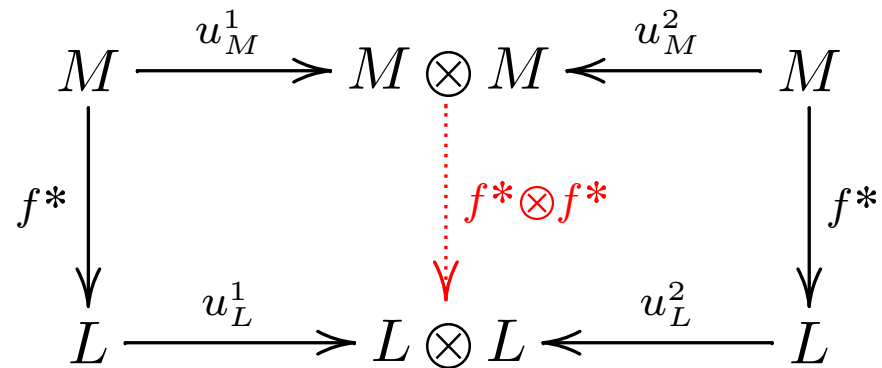
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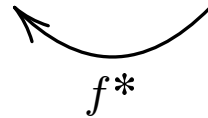
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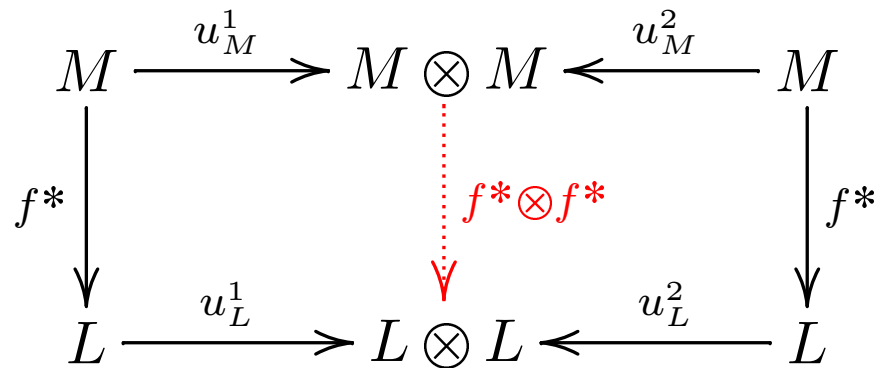


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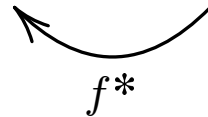


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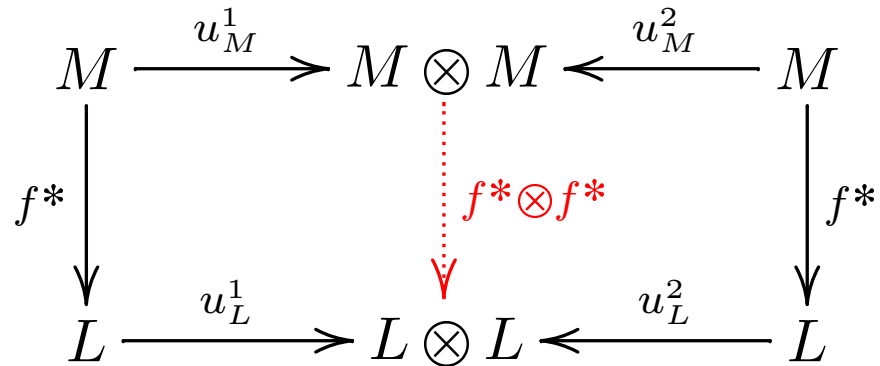


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THEOREM.

The categories U_ELoc and U_CLoc are concretely isomorphic.

(Surprising, since Ω does not preserve products.)

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Nice features of localic products

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① $a \otimes b \leq E_U, b \neq 0 \implies a \leq Ub.$

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Nice features of localic products

- 1 $a \otimes b \leq E_U, b \neq 0 \implies a \leq Ub.$
- 2 $0 \neq a \otimes b \leq \underbrace{E}_{\text{symmetric}} \implies (a \vee b) \otimes (a \vee b) \leq E \circ E.$

Under this isomorphism:

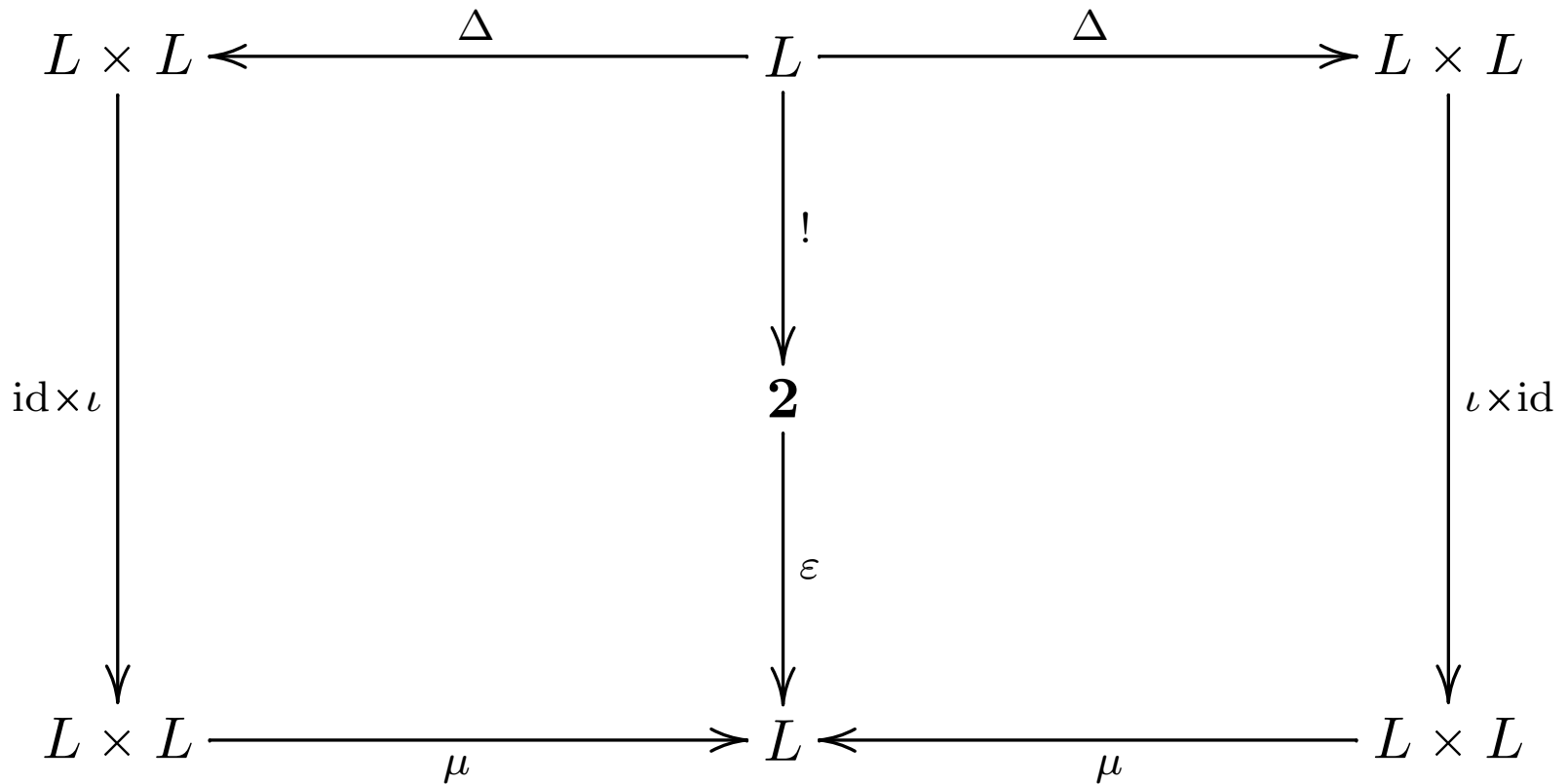
Under this isomorphism:

$$\mathcal{U}_1 \rightsquigarrow \mathcal{E}_1$$

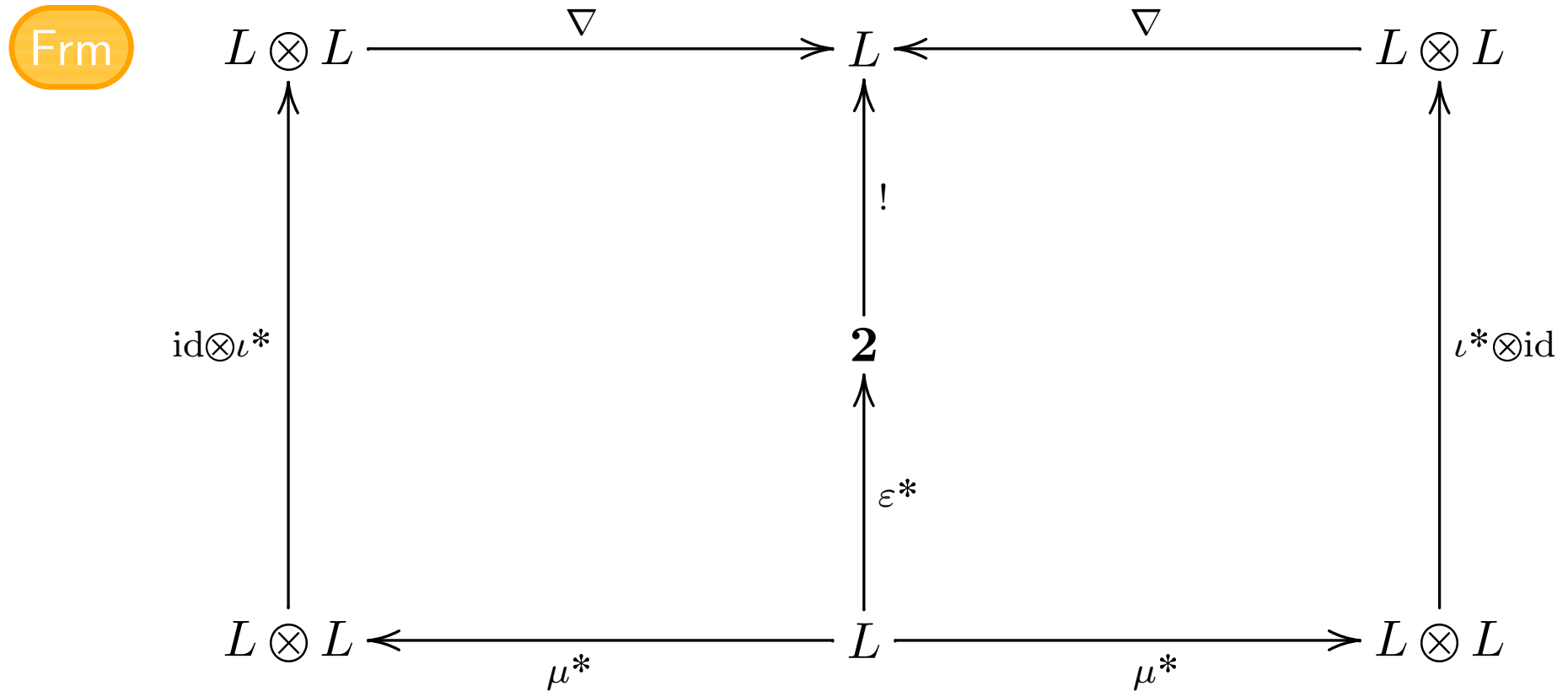
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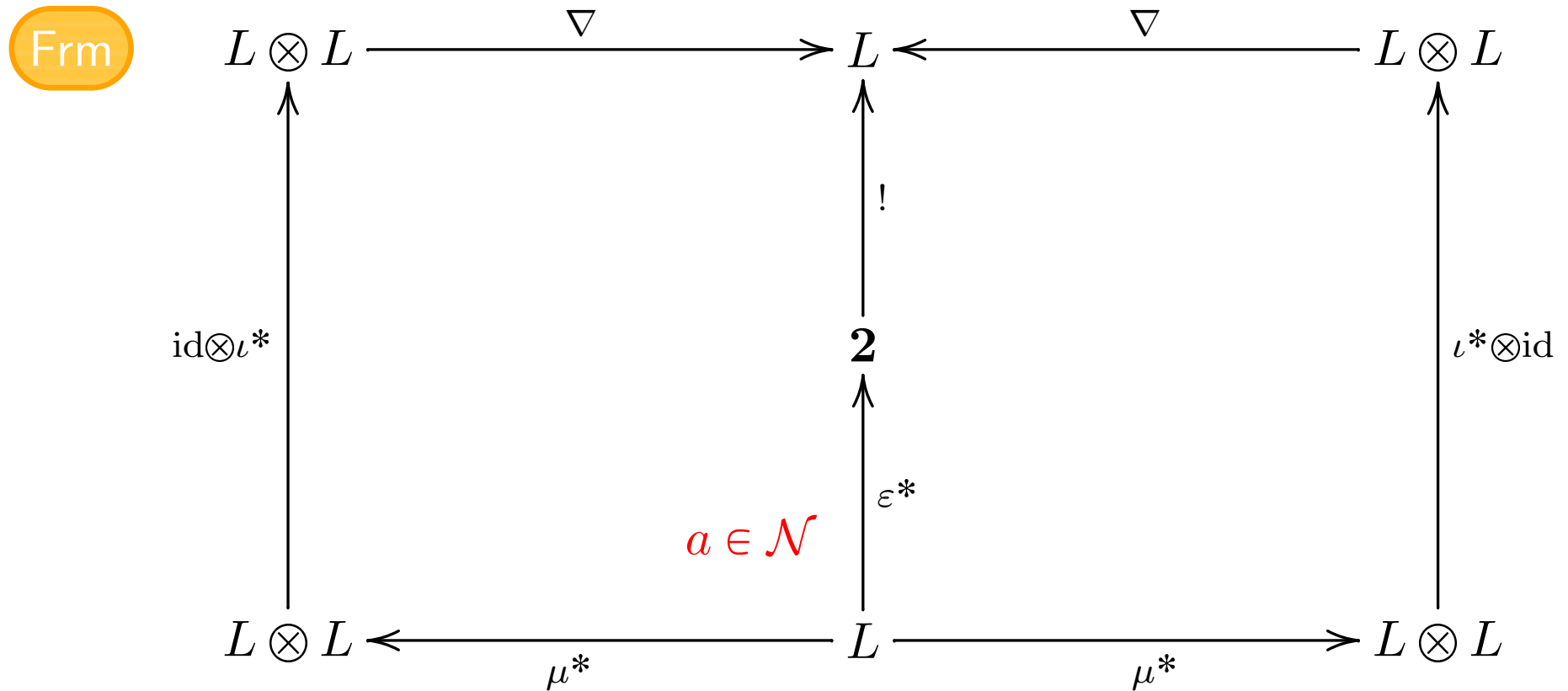
Loc



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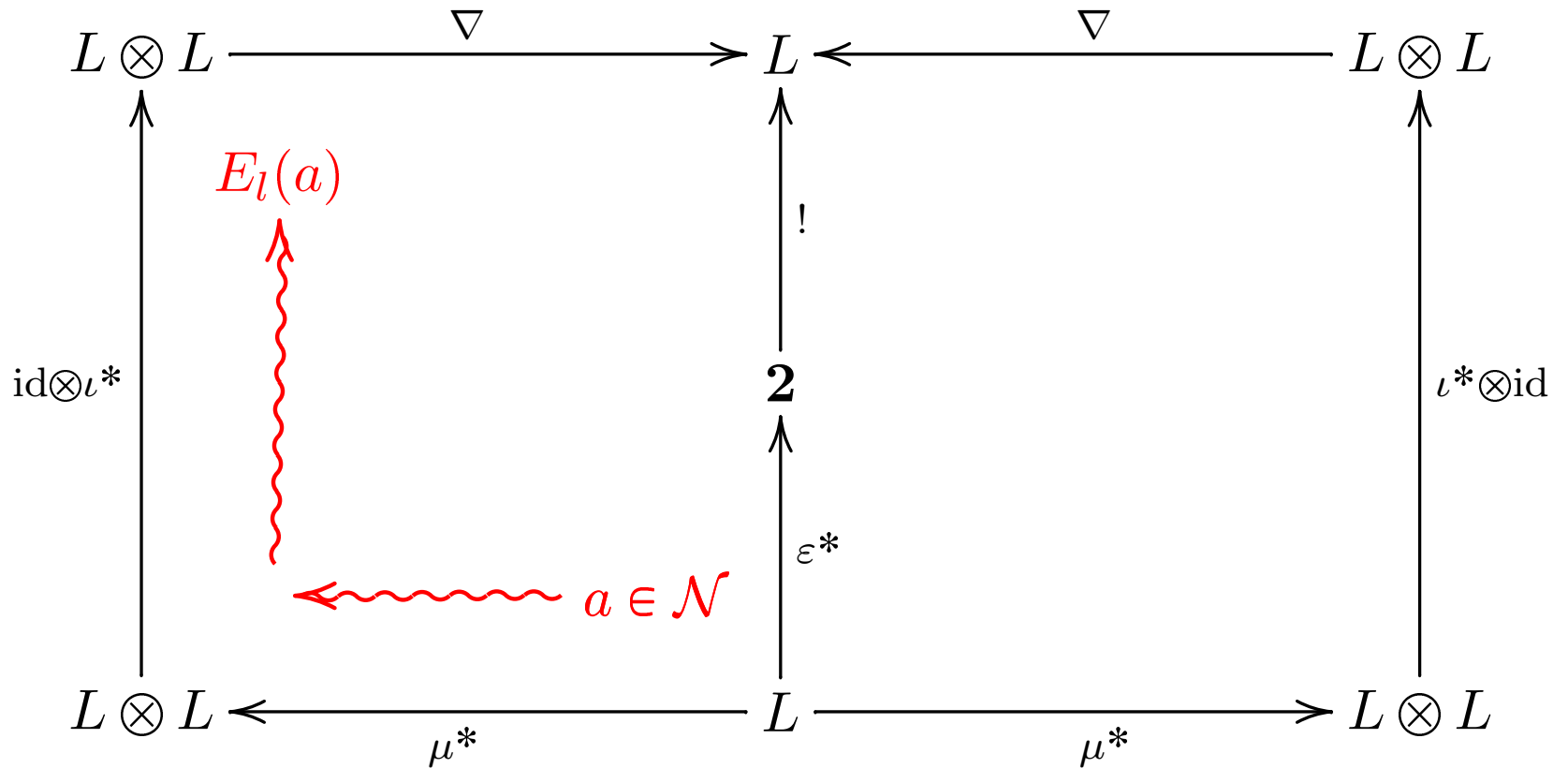


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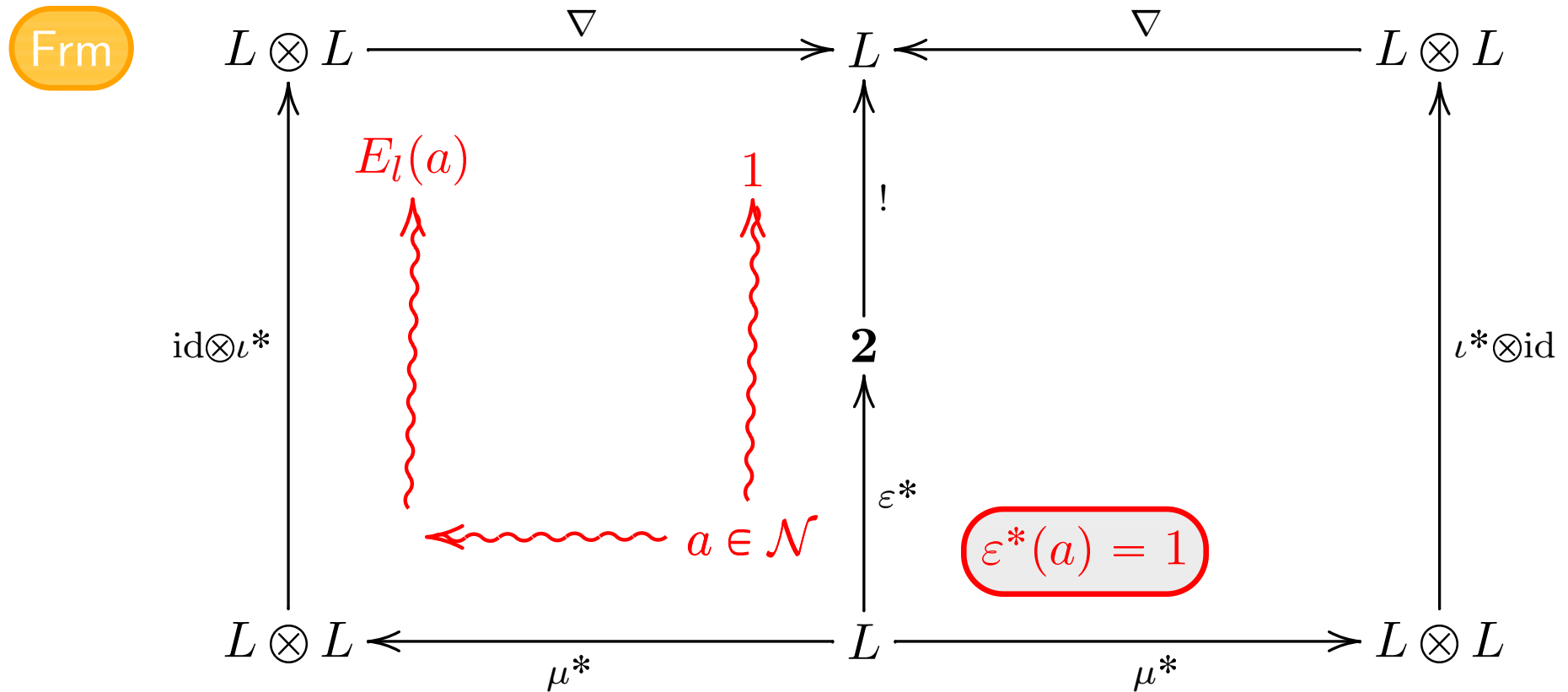


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Frm



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[P.T. Johnstone, 1988]

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Semigroup $(L, *)$ with $(-)^{-1}$

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JP & A. PULTR

Entourages, covers and localic groups, Appl. Categ. Struct., to appear

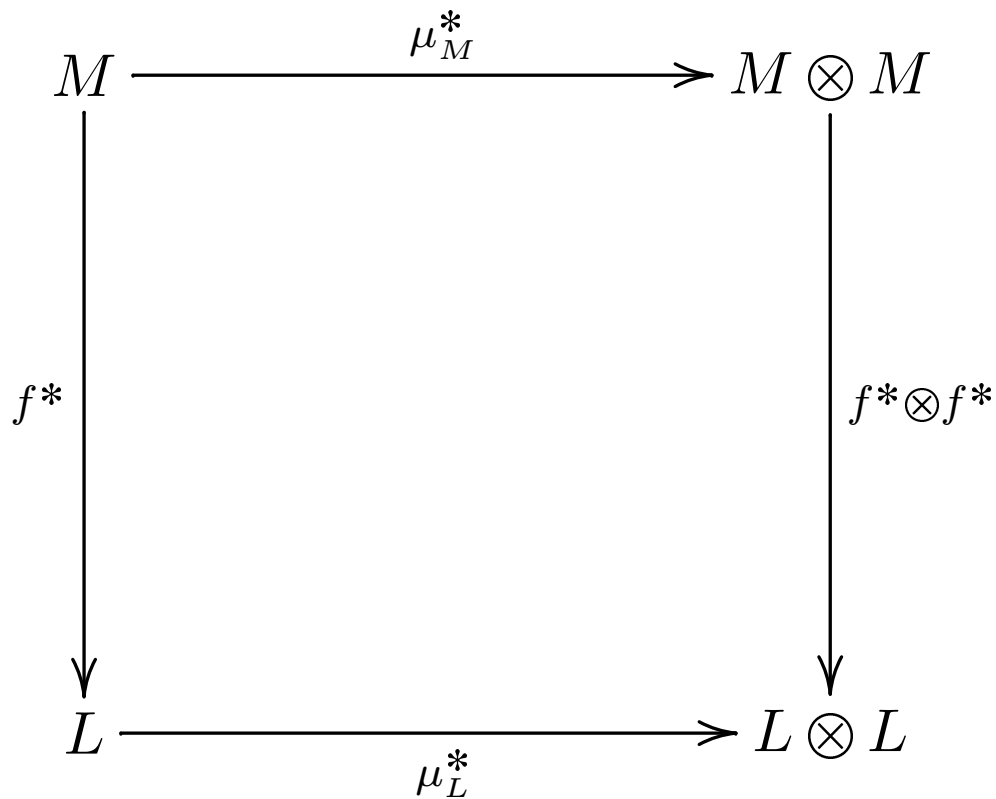
QUESTION ON FUNCTORIALITY

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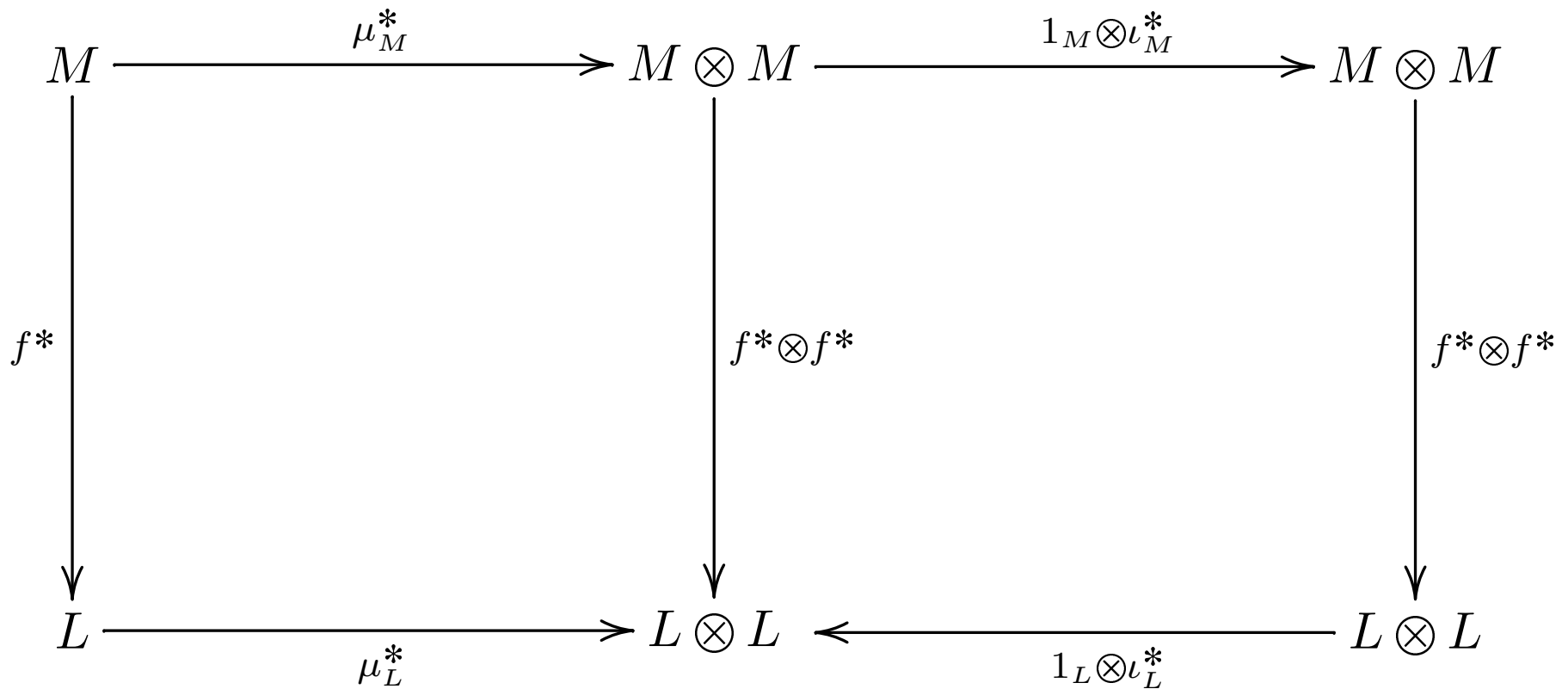
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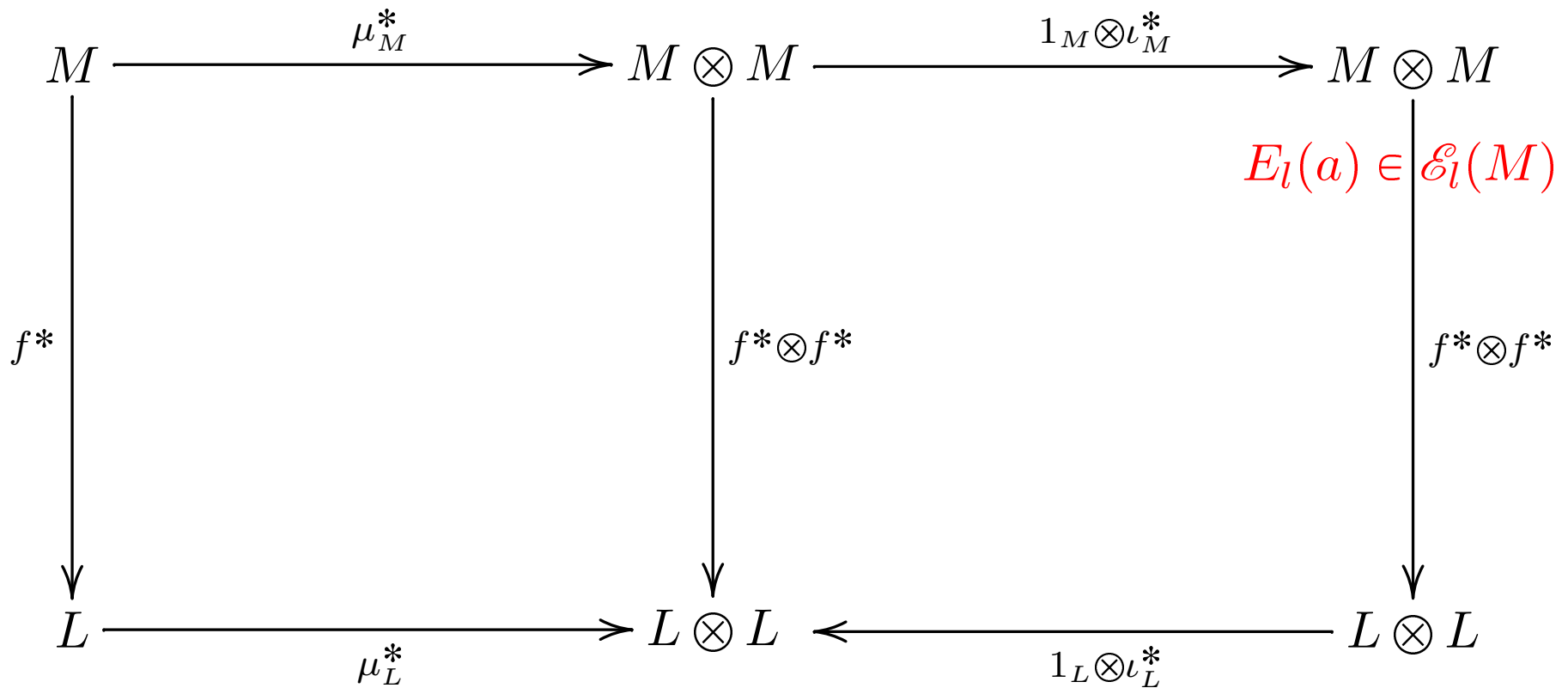
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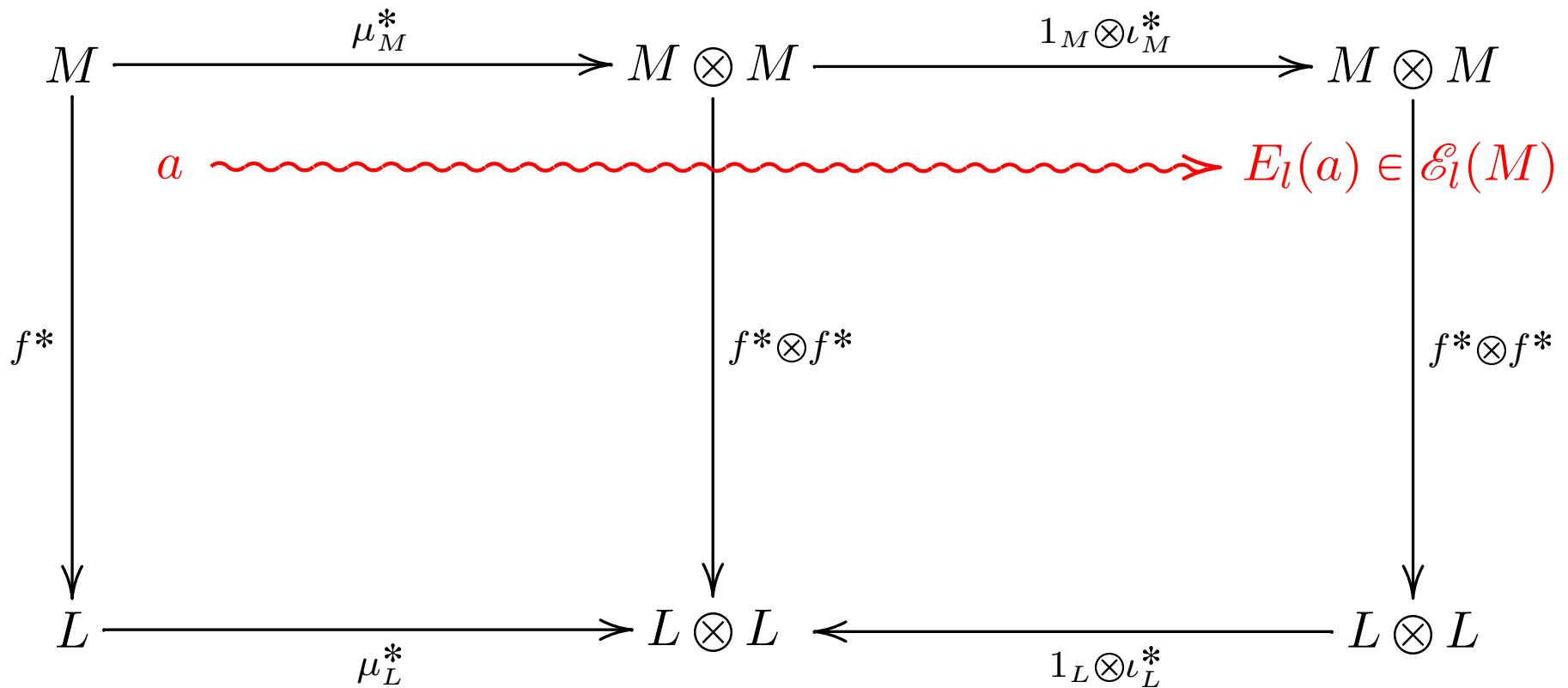
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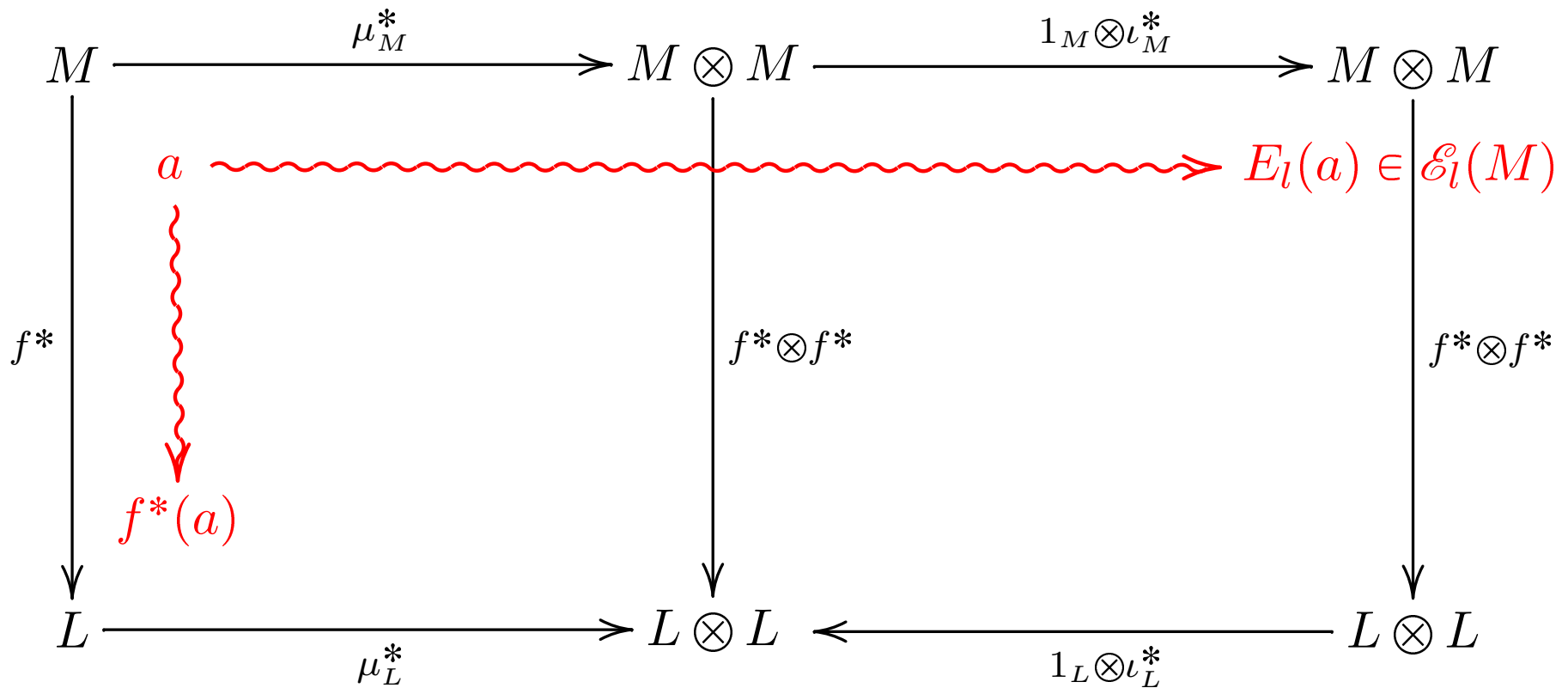
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