Real-valued functions in Pointfree Topology: semicontinuity

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In memory of Professor Wang Guo-Jun

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Localic Katětov-Tong insertion theorem and localic Tietze extension theorem

Li Yong-ming, Wang Guo-jun

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Keywords: frame, locale, lower (upper) continuous chain, normal locale

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Y.-M. Li and Z.-H. Li,
Constructive insertion theorems and extension theorems on extremally disconnected frames,
Algebra Universalis 44 (2000) 271–281.

Continuous real functions

«The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.

(...) the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.»

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HOW CAN WE EXTEND THIS STUDY TO SEMICONTINUOUS REAL FUNCTIONS?

PART I: how to deal with semicontinuity in pointfree topology

Urysohn's Lemma

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F and G, there exists a continuous $h\colon X \to [0,1]$ such that $h[F] = \{0\}$ and $h[G] = \{1\}$.

(3) For every closed set F and open set U such that $F \subseteq U$, there exists a continuous $h\colon X \to \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

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QUESTION

Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be such that $f \in \mathsf{USC}(X)$, $g \in \mathsf{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in \mathsf{C}(X)$ such that $f \leq h \leq g$?

ANSWER

Yes, if X is metric [Hahn, 1917]

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MOTIVATION: Katětov-Tong Insertion Theorem

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Yes, iff X is normal [Katětov 1951, Tong 1952]

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Katětov-Tong Insertion Theorem

TFAE for a topological space X:

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- M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* 38 (1951) 85-91; correction 40 (1953) 203-205.
- H. Tong, Some characterizations of normal and perfectly normal spaces, *Duke Math. J.* 19 (1952) 289-292.

Stone Insertion Theorem

TFAE for a topological space X:

(1) X is extremally disconnected (i.e., any two disjoint open sets in X have disjoint closures).

(2) For every $f \in \mathsf{USC}(X)$ and $g \in \mathsf{LSC}(X)$ with $g \leq f$, there exists a continuous $h \in \mathsf{C}(X)$ such that $g \leq h \leq f$.

- M.H. Stone, Boundedness properties in function-lattices, *Canad. J. Math.* 1 (1949) 176–186.

Dowker Insertion Theorem

TFAE for a topological space X:

(1) X is normal and countably paracompact.

(2) For every $f \in \mathsf{USC}(X)$ and $g \in \mathsf{LSC}(X)$ with $f < g$, there exists a continuous $h \in \mathsf{C}(X)$ such that $f \leq h \leq g$.

- C.H. Dowker, On countably paracompact spaces, *Canad. J. Math.* 3 (1951) 219–224.

Michael Insertion Theorem

TFAE for a topological space X:

(1) X is perfectly normal (i.e., every two disjoint closed sets can be precisely separated by a continuous real valued function).

(2) For every $f \in \mathsf{USC}(X)$ and $g \in \mathsf{LSC}(X)$ with $f \leq g$, there exists a continuous $h \in \mathsf{C}(X)$ such that $f \leq h \leq g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.

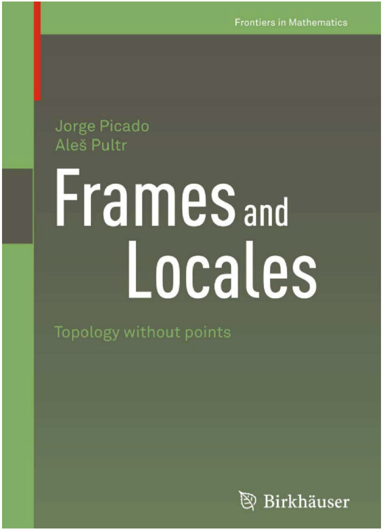
- E. Michael, Continuous selections I, *Ann. Math.* 63 (1956) 361–382.

OUR GOAL: to extend this to the setting of pointfree topology

- It is an approach to topology taking the **lattices of open sets** as the primitive notion.

- The techniques may hide some geometrical intuition, but often offers powerful **algebraic** tools and opens new perspectives.

- Better **categorical** properties. Ramifications in **logic and topos theory**.



Some history

The idea of approaching topology via algebra (lattice theory) goes back to the '30s-40's:

Stone, McKinsey and Tarski, Wallman, ...

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- RAMIFICATIONS: category theory, topos theory, logic and computer science.

MOTIVATION: from topological spaces to frames

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$(X, \mathcal{O}X)$

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$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$

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$$U \wedge V = U \cap V, \quad 1 = X$$

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$$\boxed{U \wedge \bigvee_I V_i = \bigvee_I (U \wedge V_i)}$$

Top

$$\begin{tikzcd}
(X, \mathcal{O}X) \arrow[r, squiggly] \arrow[d, "f"'] & \color{red}(\mathcal{O}X, \subseteq) \\
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- frame homomorphisms: $h \colon M \to L$ that preserve $\bigvee$ and $\wedge$

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The algebraic nature of the objects of Frm is obvious.

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Pseudocomplement: $a^* = a \to 0 = \bigvee\{b \mid b \wedge a = 0\}$.

Example: in $\mathcal{O}X$, $U^* = \operatorname{int}(X \smallsetminus U)$.

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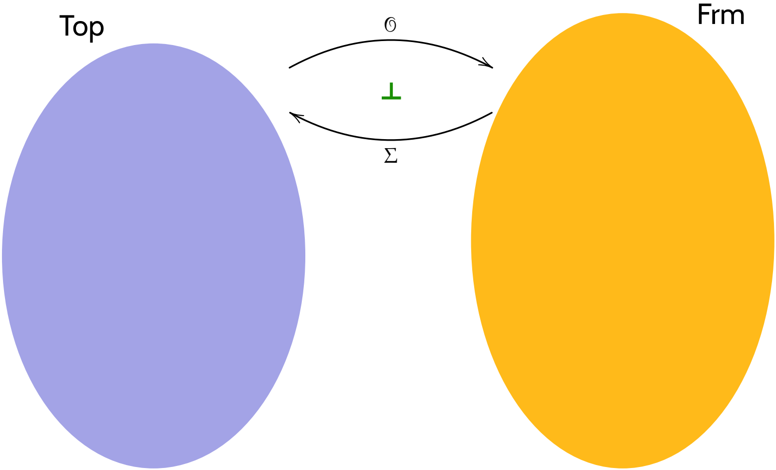

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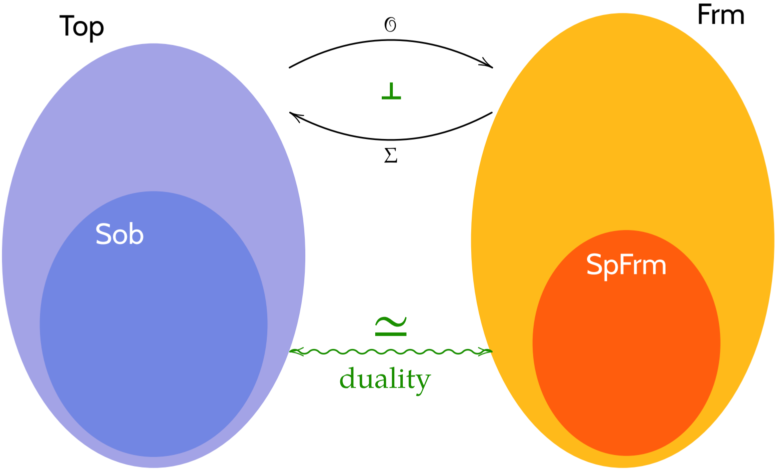

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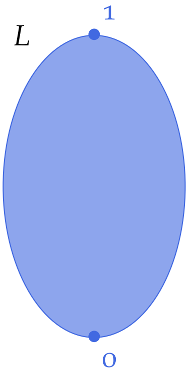

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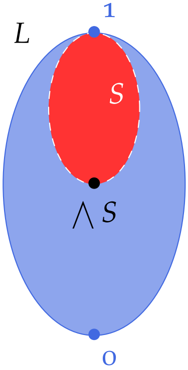

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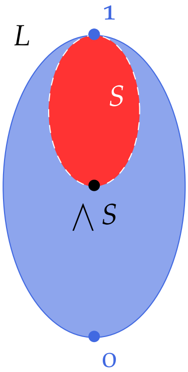

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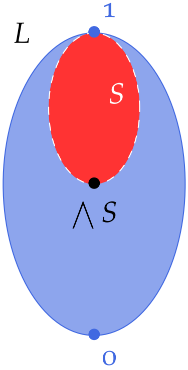

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

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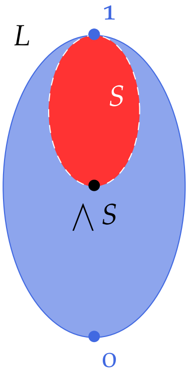

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Proposition

This lattice is a **co-frame** !

Generalized subspaces: SUBLOCALES

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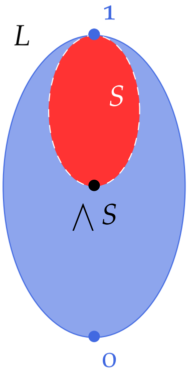

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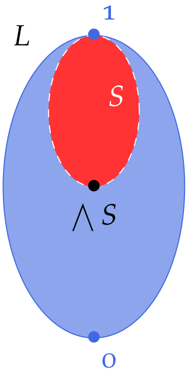

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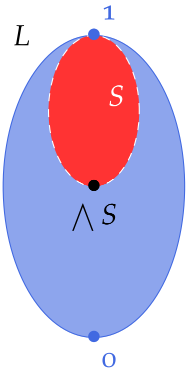

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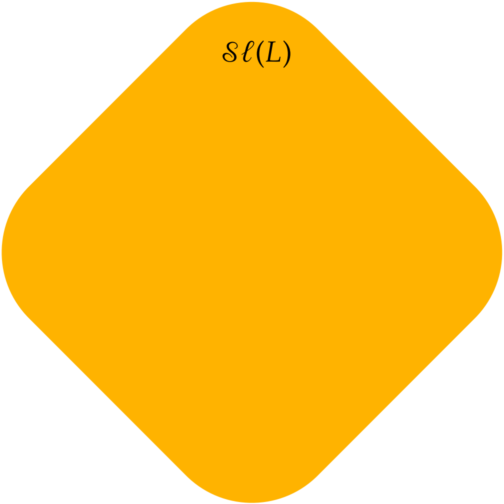

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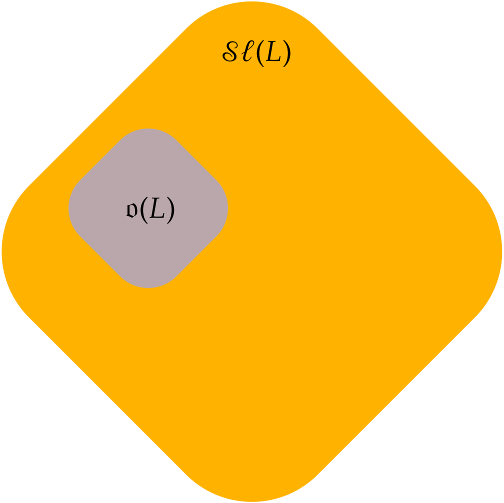

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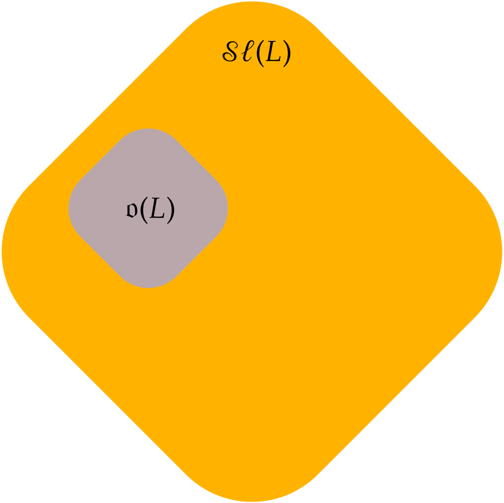

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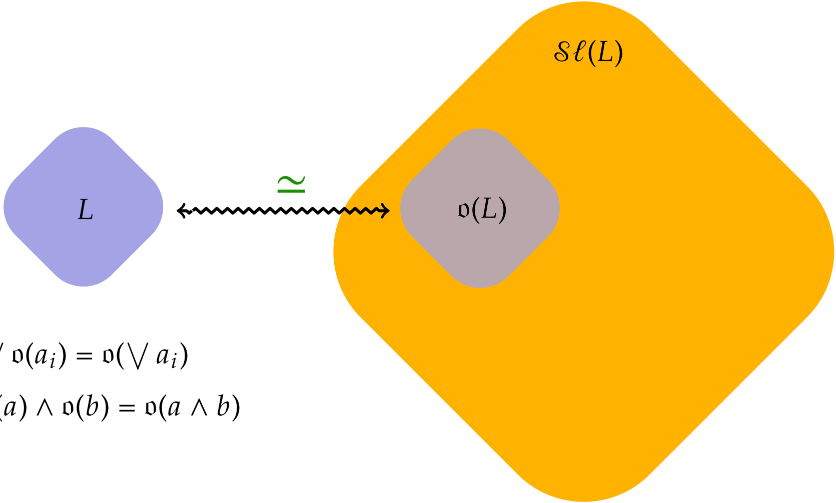

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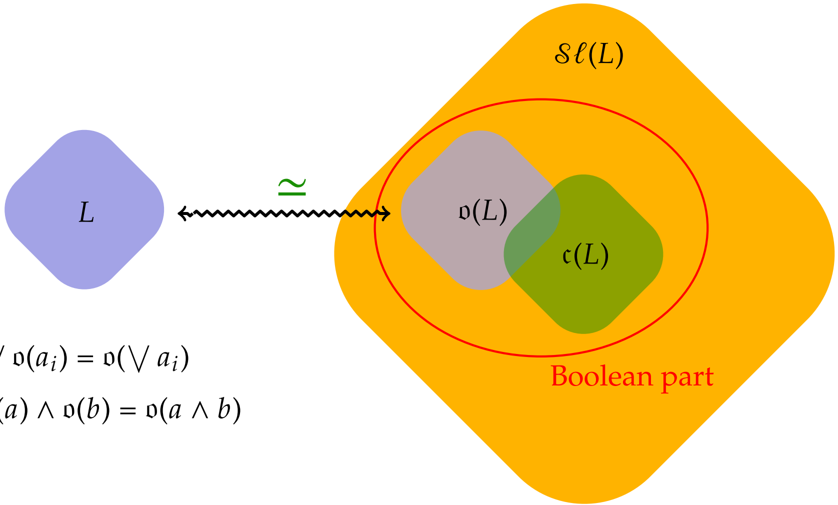

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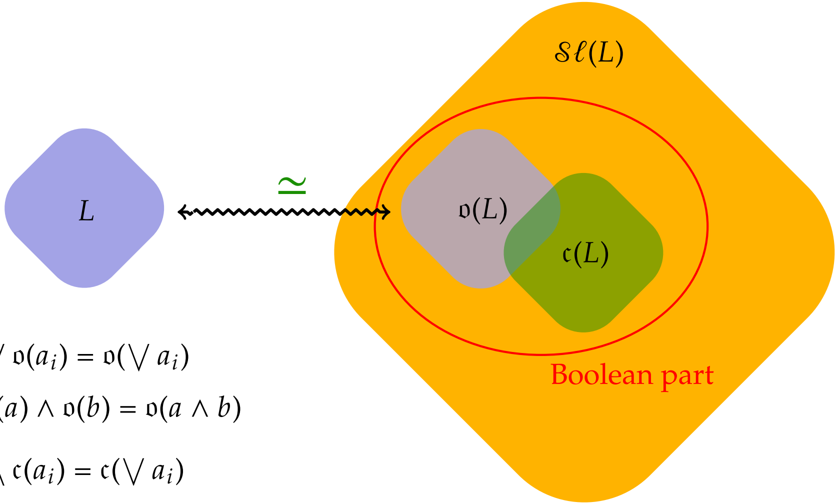

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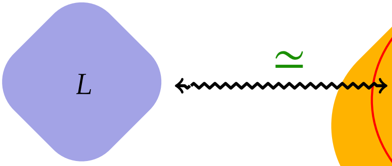

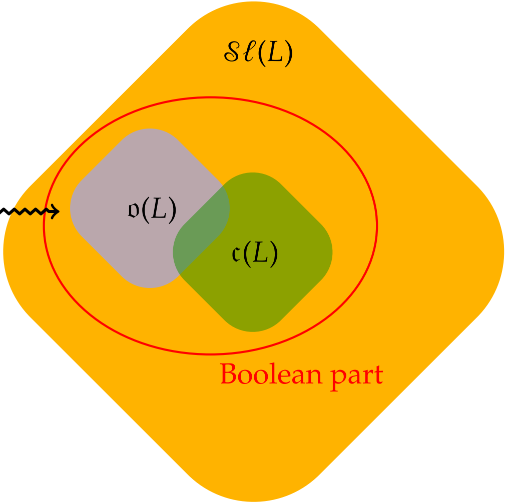

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Starting doing topology: CLOSURE and INTERIOR operators

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INTERIOR: $\operatorname{int} S = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}.$

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EXAMPLE

$\overline{\mathfrak{o}(b)} = \mathfrak{c}(\bigwedge \mathfrak{o}(b)) = \mathfrak{c}(b \to 0) = \mathfrak{c}(b^*)$.

CLOSURE: $\overline{S} = \bigwedge\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \mid a \leq \bigwedge S\}) = \mathfrak{c}(\bigwedge S)$.

$\uparrow a$

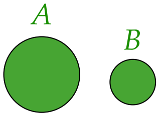
INTERIOR: $\operatorname{int} S = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}$.

EXAMPLE

$\overline{\mathfrak{o}(b)} = \mathfrak{c}(\bigwedge \mathfrak{o}(b)) = \mathfrak{c}(b \to 0) = \mathfrak{c}(b^*)$.

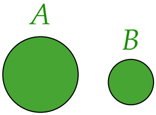
By complementation, $\operatorname{int} \mathfrak{c}(b) = \mathfrak{o}(b^*)$.

Doing topology in Frm: **NORMALITY**



Doing topology in Frm: NORMALITY

$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0$



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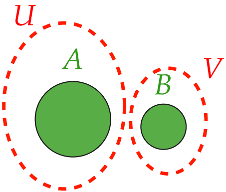

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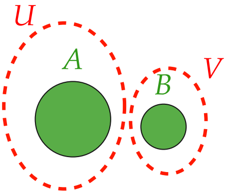
Doing topology in Frm: **NORMALITY**

$$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0$$
$$\Downarrow$$
$$\exists u, v \colon \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 0, \mathfrak{c}(a) \leq \mathfrak{o}(u), \mathfrak{c}(b) \leq \mathfrak{o}(v).$$



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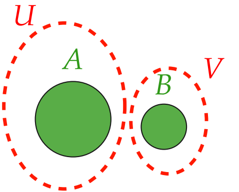

So L is normal iff

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \implies \exists u, v\colon \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 0, \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

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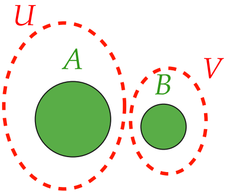

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Internally in L:
(by $\mathfrak{o}L \cong L$)

$$\boxed{a \vee b = 1 \;\Rightarrow \exists u, v\colon u \wedge v = 0, a \vee u = 1 = b \vee v}$$

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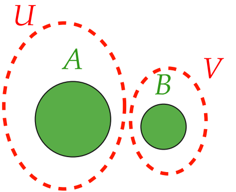

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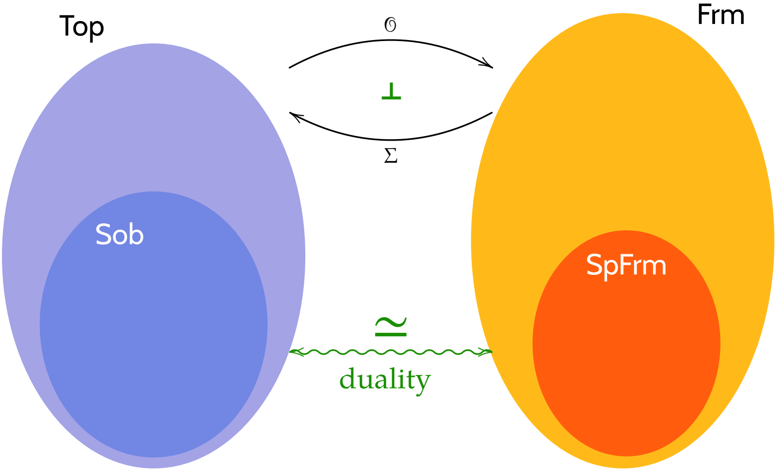
(Conservative extension: X is normal iff the frame $\mathfrak{O}(X)$ is normal.)

There is also a (contravariant) functor in the opposite direction, the spectrum functor, adjoint on the right to the open functor:



There is also a (contravariant) functor in the opposite direction, the spectrum functor, adjoint on the right to the open functor:

This means, in particular, that for every space X and every frame L

$$\boxed{\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathcal{O}X)}$$

A new feature: Frm is an algebraic category

PRESENTATIONS BY GENERATORS AND RELATIONS:

just take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

Frame of reals $\mathfrak{L}(\mathbb{R})$

generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

(R1)

(R2)

(R3)

(R4)

EXAMPLE OF A PRESENTATION

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(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

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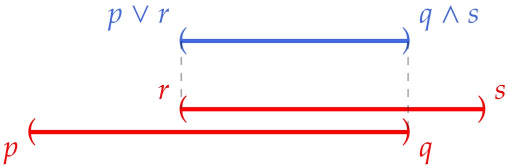

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(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,

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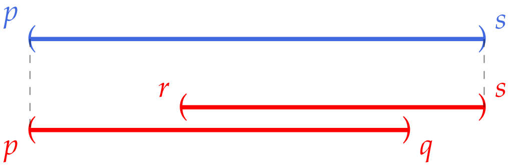

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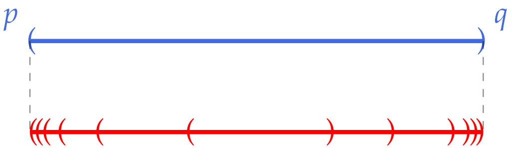

EXAMPLE OF A PRESENTATION

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Continuous real functions

Applying

$$\boxed{\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathcal{O}X)}$$

to $L = \mathfrak{L}(\mathbb{R})$, we get

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$$\mathsf{Top}(X, \underbrace{\Sigma\mathfrak{L}(\mathbb{R})}_{\simeq\ \mathbb{R}}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{O}X)$$

This motivates the following definition:

$$\mathsf{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

Each

$$\boxed{\mathsf{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

is a lattice-ordered ring with unit.

- B. Banaschewski,
  The real numbers in pointfree topology,
  Mathematical Texts, University of Coimbra, 1997.

$\mathfrak{L}(\mathbb{R}) = \mathsf{Frm}\langle\ (-,q),(p,-)(p,q\in\mathbb{Q})\ |\quad (1)\ (-,q)\wedge(p,-)=0 \text{ for } q\leq p,$

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$\boxed{(p,q) := (p,-)\wedge(-,q)}$

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Semicontinuous real functions

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$$f \leq g \equiv f(-,q) \vee g(p,-) = 1 \quad \text{whenever } p < q \in \mathbb{Q}$$

$$g \leq f \equiv g(p,-) \wedge f(-,p) = 0 \;\; \forall p \in \mathbb{Q}$$

Caution: things are not so easy!

In the spatial case, these notions do not correspond exactly to the classical notions.

Semicontinuous real functions

$(\mathbb{R}, \mathcal{T}_l)$ fails to be sober.

(the reals with the lower topology)

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usc(L) frame homomorphisms $f \colon \mathfrak{L}_l(\mathbb{R}) \to L$ satisfying

$$\boxed{\bigwedge_{q \in \mathbb{Q}} \mathfrak{o}(f(-, q)) = 0}$$

Semicontinuous real functions

Top | **Frm**

c $\quad f : X \to (\mathbb{R}, \mathcal{T}_e)$ | $f : \mathfrak{L}(\mathbb{R}) \to L$

Semicontinuous real functions

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c	$f : X \to (\mathbb{R}, \mathcal{T}_e)$	$f : \mathfrak{L}(\mathbb{R}) \to L$
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Semicontinuous real functions

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lsc	$f: X \to (\mathbb{R}, \mathcal{T}_u)$	$f: \mathfrak{L}_u(\mathbb{R}) \to L$ such that $\bigwedge_{p \in \mathbb{Q}} \mathfrak{o}(f(p, -)) = 0$

- J. Gutiérrez García and J.P.,
  On the algebraic representation of semicontinuity,
  J. Pure Appl. Algebra 210 (2007) 299-306.

Katětov-Tong Insertion Theorem for frames

TFAE for a frame L:

(1) L is normal
(i.e., $a \vee b = 1 \Rightarrow \exists u, v \in L\colon u \wedge v = 0, a \vee u = 1 = b \vee v$).

(2) For every $f \in \mathsf{usc}(L)$ and $g \in \mathsf{lsc}(L)$ with $f \leq g$, there exists a continuous $h \in \mathsf{c}(L)$ such that $f \leq h \leq g$.

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Lemma

L is normal iff for any countable $\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}} \subseteq L$

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L is normal iff for any countable $\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}} \subseteq L$

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Construction of the insertion

$$f(-,q) = a_q,\ g(p,-) = b_p$$

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$$\left.\begin{array}{l} L \text{ is normal} \\ f \text{ is usc} \\ g \text{ is lsc} \\ f \leq g \end{array}\right\} \Rightarrow \exists (u_{r_i})_i \in L : \begin{cases} q > r_i \Rightarrow f(-,q) \vee u_{r_i} = 1 \\ p < r_i \Rightarrow g(p,-) \vee u^*_{r_i} = 1 \\ r_{i_1} < r_{i_2} \Rightarrow u_{r_{i_1}} \vee u^*_{r_{i_2}} = 1. \end{cases}$$

$$h(p,-) = \bigvee_{r_i > p} u_{r_i} \quad , \quad h(-,q) = \bigvee_{r_i < q} u^*_{r_i}$$

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Katětov-Tong Insertion Theorem for frames

TFAE for a frame L:

(1) L is normal.

(2) For every $f \in \mathsf{usc}(L)$ and $g \in \mathsf{lsc}(L)$ with $f \leq g$, there exists a continuous $h \in \mathsf{c}(L)$ such that $f \leq h \leq g$.

- J. Gutiérrez García and J.P.,
  On the algebraic representation of semicontinuity,
  J. Pure Appl. Algebra 210 (2007) 299-306.

Stone Insertion Theorem for frames

TFAE for a frame L:

(1) L is extremally disconnected
(i.e., $a \wedge b = 0 \Rightarrow \exists u, v \in L \colon u \vee v = 1, a \wedge u = 0 = b \wedge v$)
(equivalently, $a^* \vee a^{**} = 1 \;\; \forall a \in L$).

(2) For every $f \in \mathsf{usc}(L)$ and $g \in \mathsf{lsc}(L)$ with $g \leq f$, there exists a continuous $h \in \mathsf{c}(L)$ such that $g \leq h \leq f$.

- J. Gutiérrez García, T. Kubiak and J.P.,
Lower and upper regularizations of frame semicontinuous real functions,
Algebra Universalis 60 (2009) 169-184.

Bounded Insertion Theorem for frames

TFAE for a frame L:

(1) L is perfectly normal.

(2) L is normal and for every $g \in \mathsf{lsc}^*(L)$, there exists a continuous $h \in \mathsf{c}(L)$ such that $\mathbf{0} \leq h \leq g$ and $h(0, -) = g(0, -)$.

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$\boxed{\mathbf{0} \leq f}$ means $f(-, 0) = 0$; $\boxed{f \leq \mathbf{1}}$ means $f(1, -) = 0$.

bounded functions: $\mathbf{0} \leq f \leq \mathbf{1}$ $\quad$ $\mathsf{c}^*(L), \mathsf{usc}^*(L), \mathsf{lsc}^*(L)$

- J. Gutiérrez García, T. Kubiak and J.P., Pointfree forms of Dowker and Michael insertion theorems, *J. Pure Appl. Algebra* 213 (2009) 98-108.

Strict Insertion Theorem for frames

TFAE for a frame L:

(1) L is countably paracompact.

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$\boxed{f < g}$ means $f \leq g$ and $\bigvee_{r \in \mathbb{Q}} f(-, r) \wedge g(r, -) = 1$.

$\boxed{\mathbf{0} < g} \Leftrightarrow g(0, -) = 1$.

- J. Gutiérrez García, T. Kubiak and J.P., Pointfree forms of Dowker and Michael insertion theorems, *J. Pure Appl. Algebra* 213 (2009) 98-108.

PART II: how to extend this to the setting of pointfree topology

«The set C(X) of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. (...)

In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty). (...)

Therefore C(X) is a commutative ring, a subring of $\mathbb{R}^X$.»

L. Gillman and M. Jerison
[Rings of Continuous Functions (1960)]