Compatibility on Courant algebroids

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Abstract

We introduce a notion of compatible tensors on Courant algebroids and construct several hierarchies of pairs of compatible tensors. Among other examples the Poisson-Nijenhuis hierarchy is included.

1. Tensors on Courant algebroids

A *Courant algebroid* structure on a vector bundle E equipped with a fiberwise symmetric bilinear form $\langle .,. \rangle$ is a pair $(\rho, [.,.])$, where the *anchor* ρ is a bundle map from E to TM and the *Dorfman bracket* $[.,.]$ is a Leibniz bracket (i.e., a $\mathbb{R}$-bilinear non necessarily skew-symmetric bracket) on $\Gamma(E)$ satisfying the relations

$$\rho(X) \cdot \langle Y, Z \rangle = \langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle$$
$$\langle Z, [X,Y] + [Y,X] \rangle = \rho(Z) \cdot \langle X, Y \rangle$$
$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]],$$

for all $X, Y, Z \in \Gamma(E)$ and $f \in C^\infty(M)$.

Given a Courant algebroid structure on $(E, \langle .,. \rangle)$, there is a unique and natural manner to construct a (sheaf of) graded commutative algebra $\mathcal{F}_E$ and to endow it with a graded Poisson bracket, i.e. a bilinear map:

$$\{\cdot,\cdot\} : \mathcal{F}_E^k \times \mathcal{F}_E^l \mapsto \mathcal{F}_E^{k+l-2},$$

which is a (graded) derivation in each variable and satisfies the graded Jacobi identity. Recall that, locally, this algebra is generated by the coordinates on the base manifold $(x_1, \ldots, x_n)$ (of degree 2), a trivialization $u_1, \ldots, u_{rk(E)}$ of E (thought as variables of degree 1), and variables $p_1, \ldots, p_n$ (of degree 2), and the relations:

$$\{x_i, p_j\} = \delta_i^j, \quad \{u_i, u_j\} = \langle u_i, u_j \rangle,$$

while all the other brackets are 0.

THEOREM 1.1. *There is a 1-1 correspondence between Courant algebroid structures on* $(E, \langle .,. \rangle)$ *and functions* $\Theta \in \mathcal{F}_E^3$ *such that* $\{\Theta, \Theta\} = 0$.

More precisely, the anchor and the Dorfman bracket associated to a given Θ are defined, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by

$$\rho(X) \cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X,Y] = \{\{X, \Theta\}, Y\}.$$

A $(1,1)$-tensor $J : E \to E$ such that $\langle JX, Y \rangle + \langle X, JY \rangle = 0$, for all $X, Y \in \Gamma(E)$, is said *skew-symmetric*.

When $E = A \oplus A^*$ and $\langle .,. \rangle$ is the usual symmetric bilinear form, a skew-symmetric $(1,1)$-tensor $J : E \to E$ is of the type

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}, \text{ with } N : A \to A, \pi \in \Gamma(\textstyle\bigwedge^2 A) \text{ and } \omega \in \Gamma(\textstyle\bigwedge^2 A^*).$$

The deformation of the Dorfman bracket $[.,.]$ by a $(1,1)$-tensor $J : E \to E$ is defined, for all $X, Y \in \Gamma(E)$, by

$$[X,Y]_J = [JX, Y] + [X, JY] - J[X,Y].$$

When the $(1,1)$-tensor $J : E \to E$ is skew-symmetric, the deformed bracket $[.,.]_J$ is given, in supergeometric terms, by $\{J, \Theta\} \in \mathcal{F}_E^3$.

Notation: $\Theta_J = \{J, \Theta\}$; $\quad \Theta_{J,I} = \{I, \{J, \Theta\}\}$; $\quad \Theta_n = \Theta_{\underbrace{I, \ldots, I}_{n}}$.

The torsion of J is defined, for all $X, Y \in \Gamma(E)$, by

$$\mathcal{T}_\Theta J(X,Y) = [JX, JY] - J([X,Y]_J).$$

A $(1,1)$-tensor $J : E \to E$ is a *Nijenhuis* tensor on the Courant algebroid (E, Θ) if its torsion vanishes. As it is well known if J is a Nijenhuis tensor then the deformed function Θ_J is a Courant structure.

DEFINITION 1.2. *Two skew-symmetric* $(1,1)$*-tensors* I *and* J *are* compatible *with respect to* Θ *if*

- *they anticommute, i.e.,* $I \circ J = -J \circ I$
- *their concomitant,* $C_\Theta(I,J) = \Theta_{I,J} + \Theta_{J,I}$*, vanishes.*

When I and J are compatible with respect to Θ, we have the following commutative diagram

$$\begin{tikzcd}
& & \Theta \arrow[dll, "I"'] \arrow[drr, "J"] \arrow[ddl, "I\circ J"] \arrow[ddr, "J\circ I"'] & & \\
\Theta_I \arrow[dr, "J"'] & & & & \Theta_J \arrow[dl, "I"] \\
& \Theta_{I,J} & = & -\Theta_{J,I} &
\end{tikzcd}$$

2. Deforming-Nijenhuis pairs

Poisson-Nijenhuis structures admit the following generalization:

DEFINITION 2.1. *Let* I *and* J *be two skew-symmetric* $(1,1)$*-tensors on the Courant algebroid* (E, Θ)*. The pair* (J, I) *is said to be a* deforming-Nijenhuis pair *for* Θ *if*

- I *and* J *are compatible w.r.t.* Θ*;*
- I *is Nijenhuis for* Θ*;*
- J *is deforming for* Θ*, i.e,* $\Theta_{J,J} = \lambda\Theta$, $\lambda \in \mathbb{R}$.

EXAMPLE 2.2. *When* $\Theta = \mu$ *is a Lie algebroid on* A *and* $\lambda = 0$*, we recover the notion of Poisson-Nijenhuis structure on* A*.*

2.1 Deforming-Nijenhuis pair for the Θ_n hierarchy

Starting from a Nijenhuis tensor I for the Courant algebroid (E, Θ), we construct a hierarchy of Courant structures Θ_n, and we show that, under some conditions, if (J, I) is a deforming-Nijenhuis pair for Θ, it remains a deforming-Nijenhuis pair for the whole hierarchy.

LEMMA 2.3. *Let* I *be a skew-symmetric* $(1,1)$*-tensor on the Courant algebroid* (E, Θ) *and* X, Y *sections of* E*. Then,* $\forall n \in \mathbb{N}$,

$$\mathcal{T}_{\Theta_n} I(X,Y) = \mathcal{T}_{\Theta_{n-1}} I(IX, Y) + \mathcal{T}_{\Theta_{n-1}} I(X, IY) - I(\mathcal{T}_{\Theta_{n-1}} I(X,Y)).$$

The proposition below follows easily from the lemma.

PROPOSITION 2.4. *Let* I *be a Nijenhuis tensor for the Courant algebroid* (E, Θ)*. Then, for all* $n \in \mathbb{N}_0$, Θ_n *is a Courant algebroid structure on* E *and* I *is Nijenhuis for* Θ_n*.*

It makes therefore sense to ask if a deforming-Nijenhuis pair for Θ remains deforming-Nijenhuis for all the Courant structures of the hierarchy.

THEOREM 2.5. *Let* I *and* J *be two skew-symmetric tensors on the Courant algebroid* (E, Θ)*. If* (J, I) *is a deforming-Nijenhuis pair for* Θ *such that* $\{\Theta, \{J, I \circ J\}\} = k\Theta_I$*, for some* $k \in \mathbb{R}$*, then* (J, I) *is a deforming-Nijenhuis pair for the Courant structures* Θ_n*,* $n \in \mathbb{N}$*.*

2.2 Hierarchy of deforming-Nijenhuis pairs for Θ

Starting with a deforming-Nijenhuis pair (J, I) for Θ, our aim is to construct a hierarchy $(J, I^{2n+1})_{n \in \mathbb{N}}$ of deforming-Nijenhuis pairs for Θ. This goes as follows:

THEOREM 2.6. *Let* I *and* J *be two skew-symmetric tensors on the Courant algebroid* (E, Θ)*. If* (J, I) *is a deforming-Nijenhuis pair for* Θ*, then* (J, I^{2n+1}) *is, for all* $n \in \mathbb{N}$*, a deforming-Nijenhuis pair for the Courant structure* Θ*.*

The theorem is an immediate consequence of two technical lemmas:

LEMMA 2.7. *Let* I *be a* $(1,1)$*-tensor on the Courant algebroid* (E, Θ)*. Then, for* $n \geq 2$ *we have, for all sections* X *and* Y *on* E*,*

$$\mathcal{T}_\Theta I^n(X,Y) = \mathcal{T}_\Theta I(I^{n-1}X, I^{n-1}Y) + I(\mathcal{T}_\Theta I^{n-1}(IX,Y)$$
$$+ \mathcal{T}_\Theta I^{n-1}(X, IY)) - I^2(\mathcal{T}_\Theta I^{n-2}(IX, IY)) + I^{2n-2}(\mathcal{T}_\Theta I(X,Y)).$$

LEMMA 2.8. *If* I *is Nijenhuis and is compatible with* J *w.r.t.* Θ*, then* $C_\Theta(I^{2n+1}, J) = 0$ *for all* $n \in \mathbb{N}$*.*

2.3 Hierarchy of Poisson-Nijenhuis pairs for Θ

We introduce the notion of Poisson tensor for a Courant algebroid (E, Θ) and then show that, in the presence of a compatible Nijenhuis tensor I, a Poisson tensor J induces a hierarchy of Poisson tensors $I^n \circ J$ (indeed, we show that $(I^n \circ J, I)_{n \in \mathbb{N}}$ is a hierarchy of Poisson-Nijenhuis pairs).

DEFINITION 2.9. *A skew-symmetric* $(1,1)$*-tensor* J *on a Courant algebroid* (E, Θ) *satisfying* $\Theta_{J,J} = 0$ *is said to be a* Poisson tensor *for* Θ*.*

EXAMPLE 2.10. *When* $\Theta = \mu$ *is a Lie algebroid on* A*, we recover the notion of Poisson bivector on* A*:*

$$\Theta_{\pi,\pi} = 0 \Leftrightarrow \{\pi, \{\pi, \mu\}\} = 0 \Leftrightarrow [\pi, \pi]_\mu = 0.$$

THEOREM 2.11. *Let* I *and* J *be two skew-symmetric* $(1,1)$*-tensors on a Courant algebroid* (E, Θ)*, such that* I *and* J *are compatible w.r.t.* Θ *and* $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$*, for all sections* X *and* Y *on* E*. If* J *is a Poisson tensor for* Θ *and* $\{\Theta, \{J, I \circ J\}\} = 0$*, then* $I^n \circ J$ *is also a Poisson tensor for* Θ*, for all* $n \in \mathbb{N}$*.*

DEFINITION 2.12. *Let* I *and* J *be two skew-symmetric* $(1,1)$*-tensors on a Courant algebroid* (E, Θ)*. The pair* (J, I) *is said to be a* Poisson-Nijenhuis pair *for* Θ *if*

- I *and* J *are compatible w.r.t.* Θ*;*
- I *is Nijenhuis and* J *is Poisson.*

PROPOSITION 2.13. *Let* I *and* J *be two skew-symmetric* $(1,1)$*-tensors on* (E, Θ) *that are compatible with respect to* Θ *and such that* $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$*, for all sections* X *and* Y *of* E*, then*

$$C_\Theta(I, I^n \circ J) = 0, \forall n \in \mathbb{N}_0.$$

COROLLARY 2.14. *Let* (J, I) *be a Poisson-Nijenhuis pair on* (E, Θ) *such that* $\{\Theta, \{J, I \circ J\}\} = 0$*. Then* $(I^n \circ J, I)_{n \in \mathbb{N}}$ *is a hierarchy of Poisson-Nijenhuis pairs for* (E, Θ)*.*

3. Hierarchies of Nijenhuis pairs

Beyond deforming-Nijenhuis pairs, there is an other generalization of usual Poisson-Nijenhuis structures that induces hierarchies:

DEFINITION 3.1. *A pair* (I, J) *of Nijenhuis skew-symmetric* $(1,1)$*-tensors on a Courant algebroid* (E, Θ) *which are compatible w.r.t.* Θ*, is called a* Nijenhuis pair *for* Θ*.*

EXAMPLE 3.2. *Let* (J, I) *be a deforming-Nijenhuis pair. If* $\Theta_{J,J} = \lambda\Theta$ *and* $J^2 = \lambda Id_E$*, for some* $\lambda \in \mathbb{R}$*, then* (J, I) *is a Nijenhuis pair. In particular, if* (J, I) *is a Poisson-Nijenhuis pair, and* $J^2 = 0$*, then* (J, I) *is a Nijenhuis pair.*

THEOREM 3.3. *If* (I, J) *is a Nijenhuis pair for* Θ*, then*
(a) $\Theta_{K_1, K_2, \ldots, K_p}$ *is a Courant structure, where* K_i *is either* I *or* J*, for* $i = 1, \ldots, p$*;*
(b) $(I^{2m+1}, J)_{m \in \mathbb{N}_0}$ *is a hierarchy of Nijenhuis pairs for* Θ *or, more generally, for all the Courant structures of item* (a)*;*
(c) $(I^{2m+1} \circ J^n, J)_{m,n \in \mathbb{N}_0}$ *is a hierarchy of Nijenhuis pairs for* Θ *or, more generally, for all the Courant structures of item* (a)*.*

EXAMPLE 3.4. *Given a Nijenhuis pair* (I, J) *s.t.* $I^2 = J^2 = -Id_E$*, the triple* $(I, J, I \circ J)$ *is a hypercomplex structure. Conversely, for every hypercomplex structure* (I, J, K)*, the pairs* (I, J)*,* (J, K) *and* (K, I) *are Nijenhuis pairs.*

A word on proofs

Several results in this work are based on direct computations involving the definitions of the Nijenhuis torsion of a $(1,1)$-tensor or the concomitant of two $(1,1)$-tensors. As an example we give the proof, for $n = 1$, of Proposition 2.13.

Proof. Using the definitions of $C_\Theta(I, J)$ and $\mathcal{T}_\Theta I$, and taking into account that I and J anti-commute, we have

$$I(C_\Theta(I,J)(X,Y)) = 2I\left([JX, IY] - I[JX, Y] - J[X, IY] + [IX, JY] - I[X, JY] - J[IX, Y]\right),$$

$$2\,\mathcal{T}_\Theta I(JX, Y) = 2([IJX, IY] - I[IJX, Y] - I[JX, IY] + I^2[JX, Y]),$$
$$2\,\mathcal{T}_\Theta I(X, JY) = 2([IX, IJY] - I[IX, JY] - I[X, IJY] + I^2[X, JY]).$$

Summing up the right-hand sides of the three equations gives

$$2([IX, IJY] - IJ[IX, Y] + [IJX, IY] - IJ[X, IY] - I[IJX, Y] - I[X, IJY]) = C_\Theta(I, I \circ J)(X,Y),$$

where we used the fact that I and $I \circ J$ anticommute. Therefore,

$$C_\Theta(I, I \circ J)(X,Y) = I(C_\Theta(I,J)(X,Y)) + 2(\mathcal{T}_\Theta I(JX, Y) + \mathcal{T}_\Theta I(X, JY))$$

and the statement of Proposition 2.13 follows immediately. $\square$

Other results come from routine calculations using the supergeometric tools. As an example, the statement of Theorem 2.11 follows, for $n = 1$, from the formula

$$\Theta_{I \circ J, I \circ J} = \frac{1}{9}\Theta_{J,J,I,I} - \frac{5}{9}\Theta_{\{J, I \circ J\}, I} - \frac{1}{2}\{I \circ J, C_\Theta(I,J)\} - \frac{2}{9}\{I, \{J, C_\Theta(I,J)\}\} + \frac{1}{6}\{J, C_{\Theta_I}(I,J)\},$$

which is obtained by successive applications of the Jacobi identity of $\{.,.\}$.

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