1. Tensors on Courant algebroids

A Courant algebroid structure on a vector bundle E equipped with a fiberwise symmetric bilinear form \( \langle \cdot, \cdot \rangle \) is a pair \((J, \theta)\), where the anchor \( \rho \) is a bundle map from \( E \rightarrow TM \) and the Dorfman bracket \([\cdot, \cdot]\) is a Lie algebroid bracket (i.e., a \( R \)-bilinear non necessarily skew-symmetric function on \( \Gamma(E) \) satisfying the relations:

\[
\rho(\langle X, Y \rangle) = \langle \rho(X), Y \rangle + \langle X, \rho(Y) \rangle, \\
\rho(\rho(X)) = 0, \\
\theta([X, Y]) = \theta(X) \wedge \theta(Y) - \theta(Y) \wedge \theta(X),
\]

for all \( X, Y \in \Gamma(E) \).

Given a Courant algebroid structure \((E, \rho, \theta, \{\cdot, \cdot\})\), there is a unique and natural manner to construct a (sheaf of graded commutative algebra) \( \mathcal{F}_E \) and to endow it with a graded Poisson bracket, i.e., a bigrading map:

\[
\{\cdot, \cdot\} : \mathcal{F}_E \times \mathcal{F}_E \rightarrow \mathcal{F}_E,
\]

which is a (graded) derivation in each variable and satisfies the graded Jacobi identity. Recall that, locally, this algebra is generated by the coor-

dinate functions \( f_1, \ldots, f_n \) for some \( n \), and variables \( y_1, \ldots, y_n, \partial_1, \ldots, \partial_n \), with the grading given by \( \deg(f_i) = 1 \) and \( \deg(y_j) = \deg(\partial_j) = 0 \).

2. Deforming-Nijenhuis pairs

Poisson-Nijenhuis structures admit the following generalization:

**Definition 2.1.** Let \( J_1, J_2 \) be two skew-symmetric (1, 1)-tensors on the Courant algebroid \((E, \rho, \theta)\). The triple \((J_1, J_2)\) is said to be a deforming-Nijenhuis pair for \((E, \rho, \theta)\) if:

1. and \( J_1, J_2 \) are compatible w.r.t. \( J \);
2. \( J_1 \) is Nijenhuis for \( J \);
3. \( J_2 \) is a skew-symmetric tensor for \( J \).

**Example 2.2.** When \( \rho = \theta \) is a Lie bracket on \( \mathfrak{g} \) and \( \lambda = 0 \), we recover the notion of Poisson structure on \( \mathfrak{g} \).

3. Hierarchies of Nijenhuis pairs

Beyond deforming-Nijenhuis pairs, there is another generalization of Poisson-Nijenhuis structures that induces hierarchies:

**Definition 3.1.** A pair \((J_1, J_2)\) is a Nijenhuis-symmetric (1, 1)-tensors on a Courant algebroid \((E, \rho, \theta)\) if and only if \( \theta(J_{1,2}) = 0 \) and such that for all \( X, Y \in \Gamma(E) \):

\[
\theta(X,J_1(Y)) + \theta(Y,J_2(X)) = \theta(J_1(X),Y) + \theta(J_2(Y),X).
\]

**Example 3.4.** Given a Nijenhuis pair \((J, \theta)\) and \( J_1, J_2 \) as in Proposition 3.1, the triple \((J, J_1, J_2)\) is a Poisson-Nijenhuis structure. Conversely, for every Poisson-Nijenhuis structure \((J, J_1, J_2)\), the pairs \((J, J_1), (J, J_2)\) and \((J_1, J_2)\) are Nijenhuis pairs.

A word on proofs

Several results in this work are based on direct computations involving the definitions of the Nijenhuis torsion of a (1, 1)-tensor or the concomitants of two (1, 1)-tensors. As an example we give the proof, for \( n = 1 \), of Proposition 2.3.

Proof. Using the definitions of \( C_{i,j}^{(2)}(J) \) and \( T_{i,j} \), and taking into account that \( I \) and \( J \) anti-commute, we have:

\[
\begin{align*}
T_{i,j}(J)(X)(Y) &= 2(\langle J, J \rangle(X,Y) - \langle J^2, J \rangle(X,Y)) + \langle J, J \rangle(Y,X) - \langle J^2, J \rangle(Y,X)
\end{align*}
\]

Summing up the right-hand sides of the three equations gives:

\[
2(\langle J, J \rangle(X,Y) - \langle J^2, J \rangle(X,Y)) + \langle J, J \rangle(Y,X) - \langle J^2, J \rangle(Y,X) = C_{i,j}^{(2)}(J)(X,Y) - C_{i,j}^{(2)}(J^2)(X,Y),
\]

where we used the fact that \( I = I \) and \( J \) anti-commute. Therefore,

\[
C_{i,j}^{(2)}(J)(X,Y) = C_{i,j}^{(2)}(J^2)(X,Y).
\]

The theorem is then an immediate consequence of Proposition 3.1.

Other results come from routine calculations using the supergeometric tools. As an example, the statement of Theorem 3.1 follows, for \( n = 1 \), from the formula

\[
\varphi_{i,j}(X,Y) = -\varphi_{j,i}(Y,X) + \varphi_{j,i}(X,Y) - \varphi_{i,j}(Y,X) + \varphi_{i,j}(X,Y),
\]

which is obtained by successive applications of the Jacobi identity of \([\cdot, \cdot, \cdot]\).

References


