On the local structure of coisotropic submanifolds of linear Poisson manifolds.*

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Abstract For a Lie algebroid A, the A-tangent bundle to A^*, $\mathcal{T}^A A^*$, is a symplectic Lie algebroid over A^* and the linear Poisson strcucture on A^* may be described in terms of the symplectic section on $\mathcal{T}^A A^*$. We discuss the local nature of a certain type of Lagrangian Lie subalagebroids of $\mathcal{T}^A A^*$ over an affine subbundle C of A^* that turns out to be coisotropic. We expect that these results may be applied in Hamilton-Jacobi theory for Hamiltonian systems on linear Poisson manifolds.

1. Lie Algebroids and linear Poisson structures

Definition 1 [5] A Lie Algebroid is a triple $(\tau_A, [\![\ ,\ ]\!], \rho)$ such that, $\tau_A : A \to M$ is a vector bundle, $[\![\ ,\ ]\!]$ is a Lie algebra structure on the space of sections $\Gamma(A)$ and $\rho : A \to TM$ is a morphism of vector bundles, the anchor map, that induces a Lie algebra homomorphism $\rho : \Gamma(A) \to \mathcal{X}(M)$ satisfying the compatibility condition:

$$[\![X_1, fX_2]\!] = f[\![X_1, X_2]\!] + \rho(X_1)(f)X_2 \text{ for } f \in \mathcal{C}^\infty(M),\ X_1, X_2 \in \Gamma(A)$$

It is possible to define a differential operator on A, $d^A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)$ as follows

$$\begin{aligned} d^A\phi(X_0, \cdots, X_k) = & \sum_{i=0}^{k} (-1)^i \rho(X_i)(\phi(X_0, \cdots, \hat{X}_i, \cdots, X_k)) \\ & + \sum_{i<j} (-1)^{i+j} \phi([\![X_i, X_j]\!], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \end{aligned}$$

for $X_0, \cdots X_k \in \Gamma(A)$ and $\phi \in \Gamma(\Lambda^k A^*)$.
On the dual bundle of a Lie algebroid A^* we have a linear Poisson structure given by

$$\{\tilde{f}, \tilde{g}\} = 0, \qquad \{\tilde{f}, \hat{X}\} = \widetilde{\rho(X) f}, \qquad \{\hat{X}, \hat{Y}\} = -\widehat{[\![X, Y]\!]}$$

where $\tilde{f} = f \circ \tau_{A^*}$ for $f \in \mathcal{C}^\infty(M)$, $\tau_{A^*} : A^* \to M$ is the dual bundle projection and $\hat{X}$ is the linear function associated with $X \in \Gamma(A)$ (see [5]).

Definition 2 [5] Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M and N be a submanifold of M. A Lie subalgebroid of A over N is a vector subbundle B of A over N such that $\rho_B = \rho_{A|B} : B \to TN$ is well defined and given two sections $X, Y \in \Gamma(B)$ and two extensions $\tilde{X}, \tilde{Y} \in \Gamma(A)$ we have that $([\![\tilde{X}, \tilde{Y}]\!]_A)_{|N} \in \Gamma(B)$.

A-tangent bundle of the dual bundle of a Lie algebroid

Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M.

Definition 3 [3] The A-tangent bundle of A^* is the vector bundle over A^* given by

$$\mathcal{T}^A A^* = \bigcup_{\alpha \in A^*} \mathcal{T}^A_\alpha A^* = \bigcup_{\alpha \in A^*} \{(a, v) \in A \times T_\alpha A^* \mid \rho(a) = T_\alpha \tau_{A^*}(v)\}$$

We denote by $\tau^1 : \mathcal{T}^A A^* \to A$ and $\rho^1 : \mathcal{T}^A A^* \to TA^*$ the projections on the first and the second factor.
A section $\eta \in \Gamma(\mathcal{T}^A A^*)$ is projectable if there exists a section $X \in \Gamma(A)$ and a vector field $\bar{X} \in \mathcal{X}(A^*)$, τ_{A^*}-projectable over $\rho(X)$ such that $\eta = (X \circ \tau_{A^*}, \bar{X})$. Given two projectable sections $(X, \bar{X}), (Y, \bar{Y}) \in \Gamma(\mathcal{T}^A A^*)$, we define

$$[\![(X, \bar{X}), (Y, \bar{Y})]\!]_{\mathcal{T}^A A^*} = ([\![X, Y]\!], [\bar{X}, \bar{Y}]).$$

Proposition 1 [3] *$(\mathcal{T}^A A^*, [\![\ ,\ ]\!]_{\mathcal{T}^A A^*}, \rho^1)$ is a Lie algebroid.*

It is possible to define the Liouville section on $\mathcal{T}^A A^*$ as the section λ given by

$$\lambda(\alpha)(X) = \alpha(\tau^1(X)), \quad \forall \alpha \in A^* \text{ and } X \in \mathcal{T}^A A^*$$

and then, introduce the 2-section Ω on $\mathcal{T}^A A^*$ as $\Omega = -d^{\mathcal{T}^A A^*}\lambda$.

Proposition 2 [3] *Ω is a symplectic section on $\mathcal{T}^A A^*$, i.e, Ω is non-degenerate and $d^{\mathcal{T}^A A^*}\Omega = 0$.*

Let $f : A^* \to \mathbb{R}$ be a Hamiltonian function. Since Ω is non-degenrate we may define its Hamiltonian section $\mathcal{H}_f$ by $i_{\mathcal{H}_f}\Omega = d^{\mathcal{T}^A A^*} f$ and one may prove that (see [3])

$$\{f, g\} = \Omega(\mathcal{H}_f, \mathcal{H}_g).$$

Using that Ω is a symplectic section on $\mathcal{T}^A A^*$, we may consider Lagrangian vector subbundles and Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$.

Proposition 3 *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid and L be a Lagrangian Lie subalgebroid of $\mathcal{T}^A A^*$ over C.* **Then, C is coisotropic in A^*.**

We recall that a submanifold C of a Poisson manifold M with Poisson 2-vector Π is said to be coisotropic if $\Pi^\#(T^0 C) \subset TC$ (see [6]).

2. Coisotropic submanifolds of linear Poisson manifolds

Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M.

Local model of coisotropic affine subbundles in A^*.

$$C(B, \phi) = \left\{\alpha \in A^* \mid \alpha_{|B(\tau_{A^*}(\alpha))} = \phi(\tau_{A^*}(\alpha)),\ \tau_{A^*}(\alpha) \in N\right\}$$

where B is a Lie subalgebroid of A over N and $\phi \in \Gamma(B^*)$ is a 1-cocycle, i.e, $d^B\phi = 0$.

In particular, the annihilator B^0 of a Lie subalgebroid B of A is a coisotropic submanifold of A^* (see [7]).

Theorem 1 [1] *Local structure of coisotropic affine subbundles of the dual bundle to a Lie Algebroid Let $(A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M, N be a submanifold of M and B be a vector subbundle over N. If $C \hookrightarrow A^*$ is a coisotropic affine subbundle of A^* modelled over B^0 and $j : A^*_N \to B^*$ is the canonical projection, then for every $\xi \in C$ there exists an open neighbourhood V of ξ and a 1-cocycle $\phi \in \Gamma(B^*)$, $d^B\phi = 0$, such that V is an open set of $j^{-1}(\phi(N))$*

$$\begin{tikzcd} A^*_N \arrow[r, "j"] & B^* \arrow[d, shift left, "\pi_B"] \\ C \arrow[u] \arrow[ur, dotted, "j_C"] \arrow[r, "\pi_{A^*|C}"'] & N \arrow[u, shift left, "\phi"] \end{tikzcd}$$

Example: Coisotropic affine subbundles of an action Lie algebroid

Let $\tau_A : M \times \mathfrak{g} \to M$ be an action Lie algebroid associated with the left infinitesimal action $\Phi : \mathfrak{g} \to \mathcal{X}(M)$. Let N be a submanifold of M and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ acting on $\mathcal{X}(N)$, then $\tau_B : B = N \times \mathfrak{h} \to N$ is an action Lie subalgebroid. Furthermore, if $\alpha \in \mathfrak{h}^*$ is a 1-cocycle for the Lie subalgebra $\mathfrak{h}$, then α induces a 1-cocycle of the Lie subalgebroid B. Thus, if $\tilde{\alpha} \in \mathfrak{g}^*$ is an extension of $\alpha \in \mathfrak{h}^*$, the coisotropic affine subbundle of $\mathcal{T}^A A^*$ associated with $N \times \mathfrak{h}$ and α is

$$C(N \times \mathfrak{h}, \alpha) = \left\{(q, \tilde{\alpha} + \gamma) \in N \times \mathfrak{g}^* \mid \gamma \in \mathfrak{h}^0\right\}.$$

3. Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$

Relations between coisotropic submanifolds of a Poisson manifold P and Lagrangian Lie subalgebroids of T^*P were discussed in [2]. Next, we consider the case when P is the dual bundle A^* of a Lie algebroid and we replace T^*P by the Lie algebroid $\mathcal{T}^A A^*$.

Let B be a Lie subalgebroid over N, $\phi \in \Gamma(B^*)$ be a 1-cocycle on B and denote by $C(B, \phi)$ the corresponding coisotropic affine submanifold of A^*.

Local model of a Lagrangian Lie subalgebroid L of $\mathcal{T}^A A^*$ over $C = C(B, \phi)$ satisfying $\tau^1(L) \subset B$

$$L = \mathcal{T}^B C = \bigcup_{\alpha_q \in C} \left\{(b_q, X_{\alpha_q}) \in B \times T_{\alpha_q} C \mid \rho_B(b_q) = T_{\alpha_q}\tau_{A^*|C}(X_{\alpha_q})\right\}$$

Theorem 2 [1] *Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M. Let L be a Lagrangian Lie subalgebroid of $(\mathcal{T}^A A^*, \Omega)$ over a coisotropic affine subbundle $C(B, \phi)$ of A^* such that $\tau^1(L) \subset B$. Then, $L = \mathcal{T}^B C$ locally.*

Example: Lagrangian Lie subalgebroids of an action Lie algebroid

If $(q, \mu) \in A^* = M \times \mathfrak{g}^*$ it follows that

$$\mathcal{T}^A_{(q,\mu)} A^* = \left\{\left((q, \eta), (X_q, \alpha)\right) \in M \times \mathfrak{g} \times T_q M \times \mathfrak{g}^* \mid \Phi(\eta)(q) = X_q\right\} \cong \mathfrak{g} \times \mathfrak{g}^*.$$

Then, the A-tangent bundle to A^* may be identified with the trivial vector bundle $(M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to M \times \mathfrak{g}^*$. Under this identification the Lagrangian Lie subalgebroid associated with $C(N \times \mathfrak{h}, \alpha)$ is given by

$$\mathcal{T}^B C(N \times \mathfrak{h}, \alpha) = \left\{\left((q, \tilde{\alpha} + \gamma), (\xi, \gamma')\right) \in (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \mid q \in N,\ \gamma \in \mathfrak{h}^0,\ \xi \in \mathfrak{h} \text{ and } \gamma' \in \mathfrak{h}^0\right\}$$

Future work

- Local description of an arbitrary coisotropic submanifold of A^* (see [1]).
- Local desription of Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ over an arbitrary coisotropic submanifold of A^* (see [1]).
- Apply these results to develop a Hamilton-Jacobi theory for Hamiltonian systems on linear Poisson manifolds.

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