q-Poisson algebras as Poisson-Lie groups

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Abstract

The full set of Poisson-Lie (PL) structures on the 3D book group are presented. They are, by construction, invariant under the group multiplication, and define a multiparameter Poisson-Hopf algebra. We show that the two different q*-deformed versions of the* $sl(2,R)$ *Poisson algebra are particular cases of Poisson-Lie book groups.*

1. Motivation

Poisson-Lie groups

- PL groups are **quadratic Poisson-Hopf algebras** on group manifolds that are invariant under the group product (the **coproduct map** Δ).
- **Quantum Groups** are quantizations of PL structures on Lie groups: therefore, PL groups are the **semiclassical limit of quantum groups**.
- PL structures on the group G are in one to one correspondence to Lie bialgebra structures on $g = Lie(G)$ (V.G. Drinfel'd 1986).
- If the Lie bialgebra (g, δ) is a **coboundary** one, the PL bracket is given by the **Sklyanin bracket** associated to a certain classical r-matrix.
- **Lack of complete classifications** for PL structures for non-(semi)simple groups.
- Each PL structure can gives rise to an infinite family of **ND integrable systems** by using the 'coalgebra approach' (A.B., O. Ragnisco 1998).

Classifications: state of the art for 3D real Lie groups

- For **(semi)simple groups** all PL structures are coboundaries. Thus, classification of classical r-matrices (V. Chari, A. Pressley 1994).
- **Heisenberg-Galilei** fully classified and obtained (B. Kupershmidt, JPA 1993; A.B., F.J. Herranz, P. Parashar, JPA 1997).
- **Poincaré-Euclidean** classification has been also completed (P. Stachura, JPA 1998).
- All **coboundary PL structures** obtained for any real group in 3D (A. Rezaei-Aghdam *et al*, JPA 2005).
- Full 3D classification, including **many new non-coboundary ones** (A.B., A. Blasco, F. Musso, 2011).
- 4D: only **oscillator** and **extended Galilei** groups studied (A.B., F.J. Herranz, JPA 1996; A.B., F.J. Herranz, E. Celeghini, JPA 2000).

2. The book Lie algebra and group

- The **book Lie algebra** (J.F. Cariñena, A. Ibort, G. Marmo, A. Perelomov 1994) is defined by the commutation relations:

$$[e_1, e_3] = e_1 \qquad [e_2, e_3] = e_2 \qquad [e_1, e_2] = 0$$

and is isomorphic to the Bianchi Lie algebra of type V.
- Lie algebra representation:

$$\rho(e_1) = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \quad \rho(e_2) = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix} \quad \rho(e_3) = \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$$

- The generic Lie group element M is given by:

$$M = \exp(z\rho(e_1))\exp(y\rho(e_2))\exp(x\rho(e_3)) = \begin{pmatrix} \exp(-x) & 0 & z\\ 0 & \exp(-x) & y\\ 0 & 0 & 1 \end{pmatrix}$$

- New variables $X = \exp(-x)$, $Y = y$, $Z = z$, then $M = \begin{pmatrix} X&0&Z\\0&X&Y\\0&0&1 \end{pmatrix}$
- The **multiplication law** between two group elements $M \otimes 1$ and $1 \otimes M$ can be read as a **coproduct map** Δ:

$$\Delta(X) = X \otimes X$$
$$\Delta(Y) = X \otimes Y + Y \otimes 1$$
$$\Delta(Z) = X \otimes Z + Z \otimes 1$$

- Equivalently, in 'local' coordinates

$$\Delta(e^{-x}) = e^{-x} \otimes e^{-x} \longrightarrow \Delta(x) = 1 \otimes x + x \otimes 1$$
$$\Delta(y) = e^{-x} \otimes y + y \otimes 1$$
$$\Delta(z) = e^{-x} \otimes z + z \otimes 1$$

3. Classification of PL structures on the book group

Theorem (A.B., A. Blasco, F. Musso, 2011)

- The most general Poisson tensor that is **quadratic in the group entries** of M is given by

$$\{X, Y\} = aX^2 - bXY - 2cXZ - aX$$
$$\{X, Z\} = dX^2 + 2eXY + bXZ - dX$$
$$\{Y, Z\} = -fX^2 + eY^2 + bYZ - dY + cZ^2 + aZ + f$$

where a, b, c, d, e, f are free parameters.
- The **Casimir function** for this Poisson algebra is given by

$$\mathcal{C} = \frac{(f(1+X^2) + d(-1+X)Y + eY^2 + aZ(1-X) + Z(bY+cZ))}{X}$$

Coboundary and non-coboundary cases

- **Most general constant classical r-matrix** on the book Lie algebra:

$$r = r^{12}(e_1 \wedge e_2) + r^{13}(e_1 \wedge e_3) + r^{23}(e_2 \wedge e_3)$$

- **Sklyanin bracket** $\{f, g\} = r^{ij}\,(L_i f\, L_j g - R_i f\, R_j g)$

$$\{X, Y\} = r^{23}(X-1)X$$
$$\{X, Z\} = r^{13}(X-1)X$$
$$\{Y, Z\} = -r^{12}(X^2-1) + r^{23}Z - r^{13}Y$$

Thus, **coboundaries** are **particular cases** with $a = r^{23}, b = 0, c = 0, d = r^{13}, e = 0, f = r^{12}$.
- **Non-coboundary PL structures**: either $b \neq 0$ or $c \neq 0$ or $e \neq 0$.

4. q-Poisson algebras are non-coboundary PL structures

The standard q-deformation of $sl(2,R)$

- **Change of variables** $X = e^{-\eta X_0}$ including a 'deformation' parameter η:

$$\{X_0, Y\} = \left[a\,(1 - e^{-\eta X_0}) + b\,Y + 2\,c\,Z\right]/\eta$$
$$\{X_0, Z\} = \left[d\,(1 - e^{-\eta X_0}) - 2\,e\,Y - b\,Z\right]/\eta$$
$$\{Y, Z\} = f\,(1 - e^{-2\eta X_0}) + e\,Y^2 + b\,Y\,Z - d\,Y + c\,Z^2 + a\,Z.$$

- If $b = \eta, f = 1/\eta$ and $a = c = d = e = 0$ (**non-coboundary!**) then

$$\{X_0, Y\} = \;Y$$
$$\{X_0, Z\} = -Z$$
$$\{Y, Z\} = \frac{1 - e^{-2\,\eta\,X_0}}{\eta} + \eta\,Y\,Z$$

which is the **standard q-Poisson $sl(2)$**-algebra ($q = e^\eta$).
- **$sl(2)$** is obtained as the limit $\eta \to 0$ and the deformed Casimir reads:

$$\mathcal{C}_\eta = \frac{1}{\eta}(e^{\eta\,X_0} + e^{-\eta\,X_0}) + \eta\,Y\,Z.$$

The non-standard q-deformation of $sl(2,R)$

- Through a **different change** $X = e^{-\eta\,S_-}$, $Y = S_+$, $Z = S_0$ we get again a **multiparametric deformation** of a 'classical spin algebra'

$$\{S_-, S_+\} = \left[a\,(1 - e^{-\eta\,S_-}) + b\,S_+ + 2\,c\,S_0\right]/\eta$$
$$\{S_-, S_0\} = \left[d\,(1 - e^{-\eta\,S_-}) - 2\,e\,S_+ - b\,S_0\right]/\eta$$
$$\{S_+, S_0\} = f\,(1 - e^{-2\,\eta\,S_-}) + e\,S_+^2 + b\,S_+\,S_0 - d\,S_+ + c\,S_0^2 + a\,S_0$$

- **A non-coboundary case** $d = 1, c = -\eta$ and $a = b = e = f = 0$:

$$\{S_-, S_+\} = -2\,S_0$$
$$\{S_-, S_0\} = \frac{1}{\eta}(1 - e^{-\eta\,S_-})$$
$$\{S_+, S_0\} = -\,S_+ - \eta\,S_0^2$$

which is the Poisson version of the **non-standard $sl_q(2)$ algebra**.
- The Casimir function is: $C = \frac{1}{\eta}(-1 + e^{-\eta\,S_-})\,S_+ - \;e^{\eta\,S_-}\,S_0^2$.