Integrable deformations of Lotka-Volterra systems from Poisson-Lie structures

Ángel Ballesteros, Alfonso Blasco, Fabio Musso

(angelb@ubu.es, ablasco@ubu.es, fmusso@ubu.es)

Physics Department, Universidad de Burgos (Spain)

Abstract

The quadratic Poisson structure underlying the integrability structure of 3D Lotka-Volterra (LV) equations is shown to be a particular Poisson-Lie structure on a three-dimensional group. By considering the most generic Poisson-Lie structure compatible with the coalgebra map Δ *defined through the group multiplication, a two-parametric integrable deformation of the LV systems is explicitly found. Moreover, 3N-dimensional integrable systems containing deformed versions of the 3D Lotka-Volterra equations can be obtained by making use of the Poisson comultiplication map* Δ*.*

1. Hamiltonian structure of the Lotka-Volterra equations

Poisson algebra and Hamiltonian function

Let us consider the quadratic Poisson algebra $\mathcal{P}$:

$$\{X,Y\}=\alpha XY \qquad \{X,Z\}=\beta XZ \qquad \{Y,Z\}=\gamma YZ \qquad \alpha,\beta\neq 0$$

whose Casimir function reads

$$\mathcal{C}=X^{-\gamma}Y^{\beta}Z^{-\alpha}.$$

If we consider the Hamiltonian function (which generalizes [1])

$$\mathcal{H}=a_1\,X+a_2\,Y+a_3\,Z+b_1\log X+b_2\log Y+b_3\log Z$$

we get the LV equations as the integrable dynamical system given by $\dot{F}=\{F,\mathcal{H}\}$. Namely,

$$\begin{aligned}\dot{X}&=X[\alpha a_2\,Y+\beta a_3\,Z+(\alpha b_2+\beta b_3)]\\ \dot{Y}&=Y[-\alpha a_1\,X+\gamma a_3\,Z+(\gamma b_3-\alpha b_1)]\\ \dot{Z}&=Z[-\beta a_1\,X-\gamma a_2\,Y-(\beta b_1+\gamma b_2)]\end{aligned}$$

The LV Poisson algebra as a Poisson-Lie group

Key observation [2]: The map

$$\begin{aligned}\Delta(X)&=X\otimes X &&\rightarrow \Delta^{(2)}(X)=X_2X_1\\ \Delta(Y)&=X^{\frac{\gamma}{\beta}}\otimes Y+Y\otimes 1 &&\rightarrow \Delta^{(2)}(Y)=X_2^{\frac{\gamma}{\beta}}Y_1+Y_2\\ \Delta(Z)&=X^{-\frac{\gamma}{\alpha}}\otimes Z+Z\otimes 1 &&\rightarrow \Delta^{(2)}(Z)=X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2\end{aligned}$$

(where $F_1=\mathcal{I}\otimes F$, $F_2=F\otimes\mathcal{I}$) is a Poisson map with respect to the LV Poisson algebra. Therefore, the LV equations are an integrable Hamiltonian system defined on this Poisson-Hopf algebra.

Remark: The coproduct Δ is just the group law for the Lie group generated by the multiparametric Lie algebra $g_{\alpha,\beta,\gamma}$ given by

$$[x,y]=\frac{\gamma}{\beta}y \qquad [x,z]=-\frac{\gamma}{\alpha}z \qquad [y,z]=0.$$

Therefore, $(\mathcal{P},\Delta)$ is a multiparametric Poisson-Lie (PL) group.

2. The deformed LV Poisson algebra

Theorem [2]

The most generic Poisson structure quadratic in $\{X, X^{\frac{\gamma}{\beta}}, X^{-\frac{\gamma}{\alpha}}, Y, Z, 1\}$ and for which the comultiplication Δ is a Poisson map is given by the following (3+2)-parameter Poisson bracket $\mathcal{P}_{\delta,\epsilon}$:

$$\begin{aligned}\{X,Y\}&=\alpha XY+\delta X(1-X^{\frac{\gamma}{\beta}})\\ \{X,Z\}&=\beta XZ+\epsilon X(1-X^{-\frac{\gamma}{\alpha}})\\ \{Y,Z\}&=\gamma YZ+\frac{\gamma\epsilon}{\beta}Y+\frac{\gamma\delta}{\alpha}Z+\frac{\gamma\delta\epsilon}{\beta\alpha}\left(1-X^{\frac{\gamma}{\beta}-\frac{\gamma}{\alpha}}\right).\end{aligned}$$

Moreover, the Casimir function for $\mathcal{P}_{\delta,\epsilon}$ is given by

$$\mathcal{C}_{\delta,\epsilon}=\left[\delta(1-X^{\frac{\gamma}{\beta}})+\alpha Y\right]^{-\frac{\beta}{\alpha}}\left[\epsilon(X^{\frac{\gamma}{\alpha}}-1)+\beta ZX^{\frac{\gamma}{\alpha}}\right].$$

Corollary: $(\mathcal{P}_{\delta,\epsilon},\Delta)$ is a 5-parametric PL structure on the Lie group $G=\exp(g_{\alpha,\beta,\gamma})$.

A new integrable deformation of LV equations

Now, from $\mathcal{H}=a_1\,X+a_2\,Y+a_3\,Z+b_1\log X+b_2\log Y+b_3\log Z$ but considering that the dynamical variables generate $\mathcal{P}_{\delta,\epsilon}$, we get the following integrable (δ,ϵ)-deformation of the LV equations:

$$\begin{aligned}\dot{X}&=X\,[\alpha a_2Y+\beta a_3Z+(\alpha b_2+\beta b_3)]\\ &\quad+\delta X\left(1-X^{\frac{\gamma}{\beta}}\right)\left(a_2+\frac{b_2}{Y}\right)+\epsilon X\left(1-X^{-\frac{\gamma}{\alpha}}\right)\left(a_3+\frac{b_3}{Z}\right)\\ \dot{Y}&=Y\,[-\alpha a_1X+\gamma a_3Z+(-\alpha b_1+\gamma b_3)]+\delta\left[(X^{\frac{\gamma}{\beta}}-1)(a_1X+b_1)+\frac{\gamma}{\alpha}(a_3Z+b_3)\right]\\ &\quad+\frac{\epsilon\gamma}{\beta}\left[Y\left(a_3+\frac{b_3}{Z}\right)+\frac{\delta}{\alpha}\left(a_3+\frac{b_3}{Z}\right)\left(1-X^{\frac{\gamma}{\beta}-\frac{\gamma}{\alpha}}\right)\right]\\ \dot{Z}&=Z\,[-\beta a_1X-\gamma a_2Y+(-\gamma b_2-\beta b_1)]-\frac{\delta\gamma}{\alpha}Z\left[a_2+\frac{b_2}{Y}\right]\\ &\quad+\epsilon\left[(X^{-\frac{\gamma}{\alpha}}-1)(a_1X+b_1)-\frac{\gamma}{\beta}(a_2\,Y+b_2)\right]+\frac{\delta\epsilon\gamma}{\alpha\beta}\left[(X^{\frac{\gamma}{\beta}-\frac{\gamma}{\alpha}}-1)\left(a_2+\frac{b_2}{Y}\right)\right]\end{aligned}$$

that include both polynomial and rational perturbation terms.

3. A higher dimensional integrable system containing a deformation of LV equations

The existence of the coproduct map Δ allows the definition of a completely integrable 6D Hamiltonian given by [2]

$$H^{(2)}:=\Delta^{(2)}(\mathcal{H})=a_1\,(X_2X_1)+a_2\left(X_2^{\frac{\gamma}{\beta}}Y_1+Y_2\right)+a_3\left(X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2\right)+b_1\log\,(X_2X_1)+b_2\log\left(X_2^{\frac{\gamma}{\beta}}Y_1+Y_2\right)+b_3\log\left(X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2\right).$$

The Hamilton equations for this 6D system, when computed on $\mathcal{P}$, read:

$$\begin{aligned}\dot{X}_2&=X_2(\alpha a_2Y_2+\beta a_3Z_2)+X_2\left[\alpha Y_2\left(\frac{b_2}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right)+\beta Z_2\left(\frac{b_3}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right)\right]\\ \dot{Y}_2&=Y_2(-\alpha a_1X_2X_1+\gamma a_3Z_2)+\gamma Y_2\left[\frac{b_3\,Z_2}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right]\\ &\quad-\alpha Y_2\left[b_1+\frac{\gamma}{\beta}X_2^{\frac{\gamma}{\beta}}Y_1\left(a_2+\frac{b_2}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right)-\frac{\gamma}{\alpha}X_2^{-\frac{\gamma}{\alpha}}Z_1\left(a_3+\frac{b_3}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right)\right]\\ \dot{Z}_2&=Z_2(-\beta a_1X_2X_1-\gamma a_2Y_2)-\gamma Z_2\left[\frac{b_2\,Y_2}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right]\\ &\quad-\beta Z_2\left[b_1+\frac{\gamma}{\beta}X_2^{\frac{\gamma}{\beta}}Y_1\left(a_2+\frac{b_2}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right)-\frac{\gamma}{\alpha}X_2^{-\frac{\gamma}{\alpha}}Z_1\left(a_3+\frac{b_3}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right)\right]\end{aligned}$$

Under the constraint $\dot{X}_1=0$ the equations for $\{X_2,Y_2,Z_2\}$ are just an integrable deformation of LV equations,

and the same happens for the first block $\{X_1,Y_1,Z_1\}$, when $\dot{X}_2=0$:

$$\begin{aligned}\dot{X}_1&=X_1\left(\alpha\,a_2Y_1X_2^{\frac{\gamma}{\beta}}+\beta\,a_3Z_1X_2^{-\frac{\gamma}{\alpha}}\right)+X_1\left[\alpha Y_1\left(\frac{b_2X_2^{\frac{\gamma}{\beta}}}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right)+\beta Z_1\left(\frac{b_3X_2^{-\frac{\gamma}{\alpha}}}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right)\right]\\ \dot{Y}_1&=Y_1\left(-\alpha a_1X_1X_2+\gamma a_3Z_1X_2^{-\frac{\gamma}{\alpha}}\right)+\gamma Y_1\left[\frac{b_3Z_1X_2^{-\frac{\gamma}{\alpha}}}{X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2}\right]-\alpha\,b_1Y_1\\ \dot{Z}_1&=Z_1\left(-\beta\,a_1X_1X_2-\gamma Y_1X_2^{\frac{\gamma}{\beta}}\right)-\gamma\,Z_1\left[\frac{b_2Y_1X_2^{\frac{\gamma}{\beta}}}{X_2^{\frac{\gamma}{\beta}}Y_1+Y_2}\right]-\beta\,b_1Z_1\end{aligned}$$

The integrals of the motion are the Hamiltonian and the Casimir functions

$$\mathcal{C}_1=X_1^{-\gamma}Y_1^{\beta}Z_1^{-\alpha} \qquad \mathcal{C}_2=X_2^{-\gamma}Y_2^{\beta}Z_2^{-\alpha}$$

$$\mathcal{C}^{(2)}:=\Delta^{(2)}(\mathcal{C})=(X_1X_2)^{-\gamma}\left(X_2^{\frac{\gamma}{\beta}}Y_1+Y_2\right)^{\beta}\left(X_2^{-\frac{\gamma}{\alpha}}Z_1+Z_2\right)^{-\alpha}$$

A further deformation is got by using $\mathcal{P}_{\delta,\epsilon}$, and by making use of the N-th coproduct [3], a 3N-dimensional integrable LV system is constructed [2].

Bibliography

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