Geometric Theory of Lie–Hamilton Systems

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Abstract

The theory of Lie systems of differential equations has been shown to be very efficient in dealing with many problems in physics and in mathematics. The usefulness of the existence of additional geometric structures in the manifold where the Lie system is defined, for instance Poisson structures, will be analysed and the theory will be illustrated with several examples as the Smorodinsky–Winternitz oscillator and the second-order Riccati equation

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Lie systems: an introduction

Lie or Lie–Scheffers systems = Non-autonomous systems of first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}, \quad x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and where $k_1, \ldots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for homogeneous linear systems for which $m = n$ and $x = \Phi(x_{(1)}, \ldots, x_{(n)}; k_1, \ldots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}$ but

i) The number m may be different form the dimension n

ii) The function Φ is nonlinear in the more general case

They appear quite often in many different branches of Science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc

One particular example is Riccati equation, of a fundamental importance in physics (for instance factorization of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and mathematics

These systems are related with equations in Lie groups and in general connections in fibre bundles

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported form group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections

Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations)

The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions

Dynamical evolution is described by systems of time-dependent first order differential equations

$$\frac{dx^i(t)}{dt} = X^i(x,t)\ , \qquad i = 1,\dots,n.$$

Autonomous systems, given

$$\frac{dx^i(t)}{dt} = X^i(x)\ , \qquad i = 1,\dots,n.$$

can be considered as those determining the integral curves of a vector field in $\mathbb{R}^n$

$$X(x) = \sum_{i=1}^n X^i(x)\frac{\partial}{\partial x^i}\ .$$

The theory can be generalised to vector fields in a differentiable manifold M instead of $\mathbb{R}^n$, $X \in \mathfrak{X}(M)$, all the above expressions being local coordinate expressions.

Similarly, from the geometrical viewpoint, the previous non-autonomous system determines the integral curves of the time-dependent vector field

$$X(x,t) = \sum_{i=1}^n X^i(x,t)\frac{\partial}{\partial x^i}\ ,$$

i.e. X is a vector field along $\pi : M\times\mathbb{R} \to M$, where M is a n–dimensional manifold.

We recall that a vector field along π is given by a map

$$\begin{tikzcd} & TM \arrow[d, "\tau"] \\ M \times \mathbb{R} \arrow[ur, rightsquigarrow, "X"] \arrow[r, "\pi"] & M \end{tikzcd}$$

and it can be seen as given by a $\mathbb{R}$-linear map $X : \mathcal{F}(M) \to \mathcal{F}(M \times \mathbb{R})$ such that

$$X(fg) = (\pi^* f)Xg + (\pi^* g)Xf.$$

Its coordinate expression is

$$X(x,t) = \sum_{i=1}^{n} X^i(x,t)\frac{\partial}{\partial x^i} \ .$$

A curve in $M \times \mathbb{R}$, $\gamma : \mathbb{R} \to M \times \mathbb{R}$, is said to be an integral curve of X if the restriction of X onto γ coincides with the tangent vector to the curve $\pi \circ \gamma$. Therefore, γ is an integral curve of X if $X \circ \gamma = T\pi \circ \dot{\gamma}$.

The important point is that the vector field X along π can be seen as a one-parameter family of vector fields,

$$X_t = \{X(\cdot, t) \in \mathfrak{X}(M) \mid t \in \mathbb{R}\}$$

and it defines at each point of M a linear subspace (its rank may be non constant), i.e. it determines a 'generalised' distribution.

Lie theorem

Theorem: *Given a non-autonomous system of n first order differential equations*

$$\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t), \quad i = 1 \ldots, n,$$

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, $\quad x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and where $k_1, \ldots, k_n$, are n arbitrary constants, is that the system can be written as

$$\frac{dx^i}{dt} = Z^1(t)\xi^i_1(x) + \cdots + Z^r(t)\xi^i_r(x), \qquad i = 1 \ldots, n,$$

where $Z^1, \ldots, Z^r$, are r functions depending only on t and ξ^i_α, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$X_\alpha \equiv \sum_{i=1}^{n} \xi^i_\alpha(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \ , \qquad \alpha = 1, \ldots, r,$$

close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \ .$$

The number r satisfies $r \leq mn$.

The t-dependent vector field

$$X(t,x) = \sum_{i=1}^{n} X^i(t,x) \frac{\partial}{\partial x^i}$$

can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$, one for each value of t.

Definition: *The minimal Lie algebra of a given a t-dependent vector field X over M is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$.*

Definition: *The vector field associated to a non-autonomous system X allows us to define a generalised distribution $\mathcal{D}^X : x \in M \mapsto \mathcal{D}^X_x \subset TM$, where $\mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_xM$, and X also gives rise to a generalised co-distribution $\mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^*M$, where $\mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}^X_x\}$.*

Remark that the Lie–Scheffers theorem can be reformulated as follows:

Theorem: *A system X admits a superposition rule if and only if the minimal Lie algebra V^X is finite-dimensional.*

Definition: *A function $f : U \subset U^X \to \mathbb{R}$ is a local first integral (or t-independent constant of the motion) for a given t-dependent vector field X over $\mathbb{R}^n$ if $Xf = 0$*

Then f is a first integral if and only if $df \in \mathcal{V}^X|_U$.

One can easily prove that:

Property. *Given a t-dependent vector field X on a n-dimensional manifold M and a point $x \in U^X$ where the rank of $\mathcal{D}^X$ is equal to k, the associated co-distribution $\mathcal{V}^X$ admits, in a neighbourhood of x, a local basis of the form, $df_1, \ldots, df_{n-k}$, where, $f_1, \ldots, f_{n-k}$, is a family of fist integrals of X. Additionally, the space $\mathcal{I}^X|_U$ of first-integrals of the system X over an open U of M, can be put in the form*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{n-k} \to \mathbb{R},\ g = F(f_1, \ldots, f_{n-k})\}.$$

There exist different procedures to derive superposition rules for Lie systems. We can use a method based on the *diagonal prolongation* notion.

Definition: *Given a t-dependent vector field X over M, its diagonal prolongation to M^{m+1} is the t-dependent vector field $\widetilde{X}$ over M^{m+1} such that*

- $\widetilde{X}$ *projects onto X by the map* $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in M^{m+1} \mapsto x_{(0)} \in M$, *that is,* $\mathrm{pr}_* \widetilde{X} = X$.
- $\widetilde{X}$ *is invariant under permutation of the variables $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.*

The procedure to determine superposition rules described is:

i) Take a basis $X_1, \ldots, X_r$ of the Vessiot–Guldberg Lie algebra V associated with the Lie system.

ii) Choose the minimum integer m such that the diagonal prolongations to M^m of the elements of the previous basis are linearly independent at a generic point.

ii) Obtain n common first-integrals for the diagonal prolongations, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, to M^{m+1} (for instance, by means of *the method of characteristics*).

iii) Obtain the expression of the variables of one of the spaces M only in terms of the other variables of M^{m+1} and the above mentioned n first-integrals.

The so obtained expressions give rise to a superposition rule in terms of any generic family of m particular solutions and n constants corresponding to the possible values of the derived first-integrals.

Examples of Lie systems

A) Inhomogeneous linear systems:

For inhomogeneous systems,

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \ , \qquad i = 1, \ldots, n.$$

the time-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \right) \frac{\partial}{\partial x^i} \ ,$$

which is a linear combination with t-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, Y_{ij} + \sum_{i=1}^{n} B^i(t)\, Y_i \ ,$$

of the n^2 vector fields $Y_{ij} = x^j \partial/\partial x^i$ and the n vector fields

$$Y_i = \frac{\partial}{\partial x^i} \ , \qquad i = 1, \ldots, n \, .$$

Now, these last vector fields commute among themselves

$$[Y_i, Y_k] = 0 \ , \qquad \forall\, i, k = 1, \ldots, n \, ,$$

and

$$[Y_{ij}, Y_k] = -\delta_{kj}\, Y_i \ , \qquad \forall\, i, j, k = 1, \ldots, n \, .$$

Therefore, as the set $\{Y_i \mid i = 1, \ldots, n\}$ generates an Abelian ideal and $\{Y_{ij} \mid i, j = 1, \ldots, n\}$ generates a Lie subalgebra, the Lie algebra generated by the vector fields $\{Y_{ij}, Y_k \mid i, j, k = 1, \ldots, n\}$ is a semidirect sum that is isomorphic to the $(n^2 + n)$-dimensional Lie algebra of the affine group.

In this case $r = n^2 + n$ and $m = n + 1$ and the equality $r = m\, n$ also follows.

B) The Riccati equation $(n = 1)$

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t) \ .$$

Now $m = r = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k \ ,$$

or in other words,

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k\, x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k\,(x_1(t) - x_3(t))}\ .$$

The value $k = \infty$ must be accepted, otherwise we do not obtain the solution x_2.

Here the superposition rule involves three different solutions, $m = 3$.

The vector fields $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ are given by

$$Y^{(1)} = \frac{\partial}{\partial x}\,, \quad Y^{(2)} = x\,\frac{\partial}{\partial x}\,, \quad Y^{(3)} = x^2\,\frac{\partial}{\partial x}\,,$$

that close on a three-dimensional real Lie algebra, i.e. $r = 3$, with defining relations

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}\,, \quad [Y^{(1)}, Y^{(3)}] = 2Y^{(2)}\,, \quad [Y^{(2)}, Y^{(3)}] = Y^{(3)}\,,$$

i.e. the $\mathfrak{sl}(2,\mathbb{R})$ algebra.

C) Lie–Scheffers systems on Lie groups

A prototypical example is when M is a Lie group G and the vector fields X_α in G are either left–invariant or right–invariant as corresponding to the Lie algebra $\mathfrak{g}$ of G or the opposite algebra.

Let us choose a basis $\{a_1, \ldots, a_r\}$ for the tangent space T_eG at the neutral element $e \in G$. If X^R_α denotes the right invariant vector field in G such that $X^R_\alpha(e) = a_\alpha$, the Lie–Scheffers system will be written

$$\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X^R_\alpha(g(t)) \ .$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides of the equation we will obtain the differential equation on T_eG

$$(R_{g(t)^{-1}})_{*g(t)}(\dot{g}(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ . \qquad (**)$$

This equation is usually written with a slight abuse of notation:

$$(\dot{g}\, g^{-1})(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ .$$

If $\bar{g}(t)$ is a solution of (**) with initial condition $\bar{g}(0) = e$, the solution with initial conditions $g(0) = h$ is given by $\bar{g}(t)h$.

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. Then, G can be seen as a principal bundle over G/H: $(G, \tau, G/H)$.

It is known that the right–invariant vector fields X_α^R are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M.

$$\tau_{*g} X_\alpha^R(g) = -X_\alpha(gH) \ ,$$

and we will have an associated Lie–Scheffers system on M:

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \ .$$

Therefore, a solution of this last system starting from x_0 will be:

$$x(t) = \Phi(g(t), x_0) \ ,$$

with $g(t)$ being a solution of (**).

The converse property is true: Given a Lie Scheffers system defined by complete vector fields with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action given by integrating the vector fields.

The restriction to an orbit will provide a homogeneous space of the above type.

The choice of a point x_0 in the homogeneous space allows us to identify the homogeneous space M with G/H, where H is the stability group of x_0. Different choices for x_0 will lead to conjugate subgroups.

D) As another example consider the differential equation of an n-dimensional Winternitz–Smorodinsky oscillator of the form

$$\left\{\begin{array}{rcl} \dot{x}_i & = & p_i, \\ \dot{p}_i & = & -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{array}\right. \qquad i = 1, \ldots, n.$$

which describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n$

$$X_t = \sum_{i=1}^{n} \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3}\right)\frac{\partial}{\partial p_i}\right],$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with X_1, X_2 and $X_3 = -[X_1, X_2]$ being given by

$$X_1 = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^{n}\left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3}\frac{\partial}{\partial p_i}\right), \quad X_3 = \sum_{i=1}^{n} \frac{1}{2}\left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right).$$

Note that X_t is a Lie system, because X_1, X_2 and X_3 close on a $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$[X_1, X_2] = -2X_3, \qquad [X_1, X_3] = -X_1, \qquad [X_2, X_3] = X_2.$$

Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i$ with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{k}{x_i^2}\right), \qquad h_3 = frac12\sum_{i=1}^{n} x_i p_i,$$

which obey that

$$\{h_1, h_2\} = 2h_3, \qquad \{h_1, h_3\} = h_1, \qquad \{h_2, h_3\} = -h_2.$$

Consequently, every curve h_t that takes values in the Lie algebra $(W, \{\cdot,\cdot\})$ spanned by h_1, h_2 and h_3 gives rise to a Lie system which is Hamiltonian in $\mathrm{T}^*\mathbb{R}^n$ with respect to the symplectic structure ω_0 in such a way that the t-dependent vector field is given by

$$X_t = X_2 + \omega^2(t)X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t)dh_1),$$

i.e. the Hamiltonian is $h_t = h_2 + \omega^2(t)h_1$.

This suggests us to consider Lie systems admitting a similar Hamiltonian description.

Lie systems admitting symplectic structures

An interesting case is when (M, Ω) is a symplectic manifold and the vector fields arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the group G on (M, Ω).

The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = dh_\alpha = -df_a$, do not close on the same Lie algebra when the Poisson bracket is considered, but we can only say that

$$d\left(\{f_\alpha, f_\beta\} - f_{[a_\alpha, a_\beta]}\right) = 0 \ ,$$

and then they span a Lie algebra extension of the original one.

More explicitly, we first recall that:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{-\widehat{\Omega}^{-1}\circ d} \mathfrak{X}_H(M, \Omega) \longrightarrow 0$$

is an exact sequence of Lie algebras.

A strongly symplectic action of a Lie group on (M, Ω) defines a Lie algebra morphism $X : \mathfrak{g} \to \mathfrak{X}_H(M, \Omega)$, and choosing a Hamiltonian function h_a for each fundamental vector field X_a, i.e. $i(X_a)\Omega = dh_a$ we obtain a map from $\mathfrak{g}$ to $C^\infty(M)$ and the property of $-\widehat{\Omega}^{-1} \circ d$ being a Lie algebra homomorphism implies that $d\left(\{f_\alpha, f_\beta\} - f_{[a_\alpha, a_\beta]}\right) = 0$, with $h_\alpha = -f_a$. In fact,

$$d\{f_a, f_b\} = -\widehat{\Omega}([X_a, X_b]) = -\widehat{\Omega}(X_{[a,b]}) = df_{[a,b]}$$

and then $\{f_a, f_b\} - f_{[a,b]}$ is a constant function, giving rise to a skewsymmetric map $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,

$$\lambda(a, b) = \{f_a, f_b\} - f_{[a,b]}.$$

If we replace f_a by a different $f'_a = f_a + r(a)$, the function λ becomes $\lambda'(a, b) = \lambda(a, b) - r([a, b])$ and sometimes we can use this ambiguity to obtain a comomentum map that is Lie algebra homomorphism, when there exists a linear map $r : \mathfrak{g} \to \mathbb{R}$ such that $\lambda(a, b) = r([a, b])$. The action is then said to be Hamiltonian

The important fact is that we can define a t-dependent Hamiltonian

$$h_t = \sum_\alpha b_\alpha(t)\, h_\alpha,$$

with the functions $h_\alpha = -f_a$ closing a Lie algebra, in such a wat hat $i(X_t)\Omega = -dh_t$.

The situation in Quantum Mechanics is quite similar:

The Hilbert space $\mathcal{H}$ can be seen as a real manifold with a global chart. The tangent space $T_\phi\mathcal{H}$ at any point $\phi \in \mathcal{H}$ can be identified with $\mathcal{H}$ itself: the isomorphism associates $\psi \in \mathcal{H}$ with the vector $\dot{\psi} \in T_\phi\mathcal{H}$ given by:

$$\dot{\psi} f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0} \ , \qquad \forall f \in C^\infty(\mathcal{H}) \ .$$

It is endowed with a symplectic 2-form Ω:

$$\Omega_\phi(\dot{\psi}, \dot{\psi}') = 2\,\mathrm{Imag}\,\langle\dot{\psi}|\dot{\psi}'\rangle \ .$$

A vector field is just a map $A\colon \mathcal{H} \to \mathcal{H}$; therefore a linear operator A on $\mathcal{H}$ is a special kind of vector field.

Given a smooth function $a\colon \mathcal{H} \to \mathbb{R}$, its differential da_ϕ at $\phi \in \mathcal{H}$ is an element of the (real) dual $\mathcal{H}'$ given by:

$$\langle da_\phi, \psi\rangle := \left(\frac{d}{dt} a(\phi + t\psi)\right)_{|t=0} \ .$$

Actually, the skew-Hermitian linear operators in $\mathcal{H}$ define Hamiltonian vector fields, the Hamiltonian function of $-i\,A$ for a self-adjoint operator A being $a(\phi) = \frac{1}{2}\langle\phi, A\phi\rangle$.

The Schrödinger equation plays the rôle of Hamilton equations because it determines the integral curves of the vector field $-i\,H$.

Lie system theory applies when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.

An example: t-dependent linear potential

Let us consider the classical system described by a classical Hamiltonian

$$H_c = \frac{p^2}{2m} + f(t)\, x \ ,$$

and the corresponding quantum Hamiltonian

$$H_q = \frac{P^2}{2m} + f(t)\, X \ ,$$

describing, for instance when $f(t) = q\, E_0 + q\, E\, \cos \omega t$, the motion of a particle of electric charge q and mass m driven by a monochromatic electric field.

E_0 is the strength of the constant confining electric field and E that of the time-dependent electric field that drives the system with a frequency $\omega/2\pi$.

We will study in parallel the classical and the quantum problem by reduction of both problems to similar equations and using the Wei–Norman method to solve such an equation. The only difference is that the Lie algebra arising in the quantum problem is not the same one as in the classical one, but a central extension.

The classical Hamilton equations of motion are

$$\left\{ \begin{array}{rcl} \dot{x} & = & \dfrac{p}{m}\,, \\ \dot{p} & = & -f(t)\,, \end{array} \right.$$

and therefore, the motion is given by

$$\begin{array}{rcl} x(t) & = & x_0 + \dfrac{p_0\, t}{m} - \dfrac{1}{m}\displaystyle\int_0^t dt' \int_0^{t'} f(t'')\, dt'' \ , \\ p(t) & = & p_0 - \displaystyle\int_0^t f(t')\, dt' \end{array}$$

The t-dependent vector field describing the time evolution is

$$X = \frac{p}{m}\,\frac{\partial}{\partial x} - f(t)\,\frac{\partial}{\partial p} \ .$$

This vector field can be written as a linear combination

$$X = \frac{1}{m}\, X_1 - f(t)\, X_2 \ ,$$

with

$$X_1 = p\,\frac{\partial}{\partial x} \ , \qquad X_2 = \frac{\partial}{\partial p} \ ,$$

being vector fields closing on a 3-dimensional Lie algebra with

$$X_3 = \frac{\partial}{\partial x} \ ,$$

isomorphic to the Heisenberg algebra, namely,

$$[X_1, X_2] = -X_3 \ , \qquad [X_1, X_3] = 0 \ , \qquad [X_2, X_3] = 0 \ .$$

The flow of these vector fields is given, respectively, by

$$\begin{aligned} \phi_1(t, (x_0, p_0)) &= (x_0 + p_0\, t, p_0)\,, \\ \phi_2(t, (x_0, p_0)) &= (x_0, p_0 + t)\,, \\ \phi_3(t, (x_0, p_0)) &= (x_0 + t, p_0)\,. \end{aligned}$$

X_1, X_2 and X_3 are Hamiltonian vector fields w.r.t. the usual symplectic structure, $\Omega = dx \wedge dp$, the corresponding Hamiltonian functions h_i such that $i(X_i)\Omega = -dh_i$ being

$$h_1 = -\frac{p^2}{2}\,, \qquad h_2 = x\,, \qquad h_3 = -p \ ,$$

therefore

$$\{h_1, h_2\} = -h_3\,, \quad \{h_1, h_3\} = 0\,, \quad \{h_2, h_3\} = -1\,,$$

which close on a four-dimensional Lie algebra with $h_4 = 1$, that is, a central extension of the previous one.

If $\{a_1,\, a_2,\, a_3\}$ be a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = -a_3$, the corresponding equation in the group is

$$\dot g\, g^{-1} = -\frac{1}{m} a_1 + f(t)\, a_2\ .$$

Using the the Wei–Norman formula with the factorization $g = \exp(-u_3\, a_3)\, \exp(-u_2\, a_2)$ ex
we will arrive to the system of differential equations

$$\dot u_1 = \frac{1}{m}\ , \qquad \dot u_2 = -f(t)\ , \qquad \dot u_3 - \dot u_1\, u_2 = 0\ ,$$

together with the initial conditions

$$u_1(0) = u_2(0) = u_3(0) = 0\ ,$$

with solution

$$u_1 = \frac{t}{m}\ , \quad u_2 = -\int_0^t f(t')\, dt'\ , \quad u_3 = -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\, dt''\ .$$

Therefore the motion will be given by

$$\begin{pmatrix} x \\ p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{m} & -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\, dt'' \\ 0 & 1 & -\int_0^t f(t')\, dt' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \\ 1 \end{pmatrix}\ .$$

We can recover the constants of motion

$$
\begin{aligned}
I_1 &= p(t) + \int_0^t f(t')\,dt' \ , \\
I_2 &= x(t) - \tfrac{1}{m}\left(p(t) + \int_0^t f(t')\,dt'\right) t + \tfrac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\,dt'' \ ,
\end{aligned}
$$

As far as the quantum problem is concerned, notice that the quantum Hamiltonian H_q may be written as a sum

$$
H_q = \frac{1}{m}\,H_1 - f(t)\,H_2 \ ,
$$

with

$$
H_1 = \frac{P^2}{2} \ , \qquad H_2 = -X \ .
$$

Note that $-i\,H_1$ and $-i\,H_2$ close on a four-dimensional Lie algebra with $-i\,H_3 = -i\,P$, and $-iH_4 = i\,I$, isomorphic to the above mentioned extension of the Heisenberg Lie algebra,

$$
[-iH_1, -iH_2] = -iH_3 \, , \ [-iH_1, -iH_3] = 0 \, , \ [-iH_2, -iH_3] = -iH_4 \, .
$$

The Schrödinger equation for the Hamiltonian H_q is like that of a Lie system. This Hamiltonian is time–dependent and such systems are seldom studied, because it is generally difficult to find the time evolution of such systems. However, this system is a Lie system and therefore we can find the time-evolution operator by applying the reduction of the problem to an equation on the Lie group and using the Wei–Norman method.

Let $\{a_1,\, a_2,\, a_3,\, a_4\}$ be a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = a_3$ and $[a_2, a_3] = a_4$.

The equation in the group to be considered is now

$$\dot g\, g^{-1} = -\frac{1}{m}\, a_1 + f(t)\, a_2\ .$$

Using $g = \exp(-u_4\, a_4) \exp(-u_3\, a_3)\, \exp(-u_2\, a_2)\, \exp(-u_1\, a_1)$ the Wei–Norman method provides the following equations:

$$\begin{array}{ll} \dot u_1 = \frac{1}{m}\,, & \dot u_2 = -f(t)\,, \\ \dot u_3 + u_2\, \dot u_1 = 0\,, & \dot u_4 + u_3\, \dot u_2 - \frac{1}{2}\, u_2^2\, \dot u_1 = 0\,, \end{array}$$

and written in normal form

$$\begin{array}{ll} \dot u_1 = \frac{1}{m}\,, & \dot u_2 = -f(t)\,, \\ \dot u_3 = -\frac{1}{m}\, u_2\,, & \dot u_4 = f(t)\, u_3 + \frac{1}{2m} u_2^2\,, \end{array}$$

together with the initial conditions $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 0$, whose solution is

$$u_1(t) = \frac{t}{m}\,, \quad u_2(t) = -\int_0^t f(t')\,dt'\,,$$
$$u_3(t) = \frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\,dt''\,,$$

and

$$u_4 = \frac{1}{m}\int_0^t dt' f(t') \int_0^{t'} dt'' \int_0^{t''} f(t''')\,dt''' + \frac{1}{2m}\int_0^t dt' \left(\int_0^{t'} dt'' f(t'')\right)^2 .$$

These functions provide the explicit form of the time-evolution operator:

$$U(t,0) = \exp(-iu_4(t))\exp(iu_3(t)P)\exp(-iu_2(t)X)\exp(iu_1(t)P^2/2) .$$

Note that in order to find the expression of the wave-function in a simple way, it is advantageous to use the factorization

$$g = \exp(-v_4\, a_4)\exp(-v_2\, a_2)\, \exp(-v_3\, a_3)\, \exp(-v_1\, a_1)\,.$$

In such a case, the Wei–Norman method gives the system

$$\begin{array}{ll} \dot v_1 = \frac{1}{m}\,, & \dot v_2 = -f(t)\,, \\ \dot v_3 = -\frac{1}{m}\, v_2\,, & \dot v_4 = -\frac{1}{2m} v_2^2\,, \end{array}$$

jointly with the initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$. The solution is

$$
\begin{aligned}
v_1(t) &= \frac{t}{m}\,, \quad v_2(t) = -\int_0^t dt'\, f(t')\,, \\
v_3(t) &= \frac{1}{m}\int_0^t dt' \int_0^{t'} dt'' f(t'')\,, \\
v_4(t) &= -\frac{1}{2m}\int_0^t dt' \left(\int_0^{t'} dt'' f(t'')\right)^2 .
\end{aligned}
$$

Then, applying the evolution operator onto the initial wave-function $\phi(p,0)$, which is assumed to be written in momentum representation, we have

$$
\begin{aligned}
& \phi(p,t) = U(t,0)\phi(p,0) \\
= \;& \exp(-iv_4(t))\exp(-iv_2(t)X)\exp(iv_3(t)P)\exp(iv_1(t)P^2/2)\phi(p,0) \\
= \;& \exp(-iv_4(t))\exp(-iv_2(t)X)e^{i(v_3(t)p+v_1(t)p^2/2)}\phi(p,0) \\
= \;& \exp(-iv_4(t))e^{i(v_3(t)(p+v_2(t))+v_1(t)(p+v_2(t))^2/2)}\phi(p+v_2(t),0)\,,
\end{aligned}
$$

where the functions $v_i(t)$ are given by the preceding equations.

Second order Riccati differential equation

The usual Riccati equation comes from reduction of a linear differential equation by taking into account the invariance under dilations of such equations.

Starting from

$$A_3\,\dddot{y} + A_2\,\ddot{y} + A_1\,\dot{y} + A_0\,y = 0$$

where we can assume that $A_3(t) > 0$, and writing $y = e^u$, with $x = \dot{u}$ we arrive to

$$A_3(\ddot{x} + 3\,x\dot{x} + x^3) + A_2(\dot{x} + x^2) + A_1\,x + A_0 = 0,$$

and if we change the independent variable t to a new variable τ, then $d/dt = \dot{\tau}\,d/d\tau$, and if we denote $x' = dx/d\tau$, $x'' = d^2x/d\tau^2$, we obtain

$$\dot{x} = \dot{\tau}\,x', \quad \ddot{x} = \dot{\tau}\frac{d}{d\tau}\left(\dot{\tau}\,\frac{dx}{d\tau}\right) = \dot{\tau}^2\,x'' + \frac{\ddot{\tau}}{\dot{\tau}}\,x'$$

and therefore the original equation reduces to

$$A_3\left(\dot{\tau}^2\,x'' + \frac{\ddot{\tau}}{\dot{\tau}}\,x' + 3\,x\dot{\tau}\,x' + x^3\right) + A_2\left(\dot{\tau}x' + x^2\right) + A_1\,x + A_0 = 0.$$

If we choose τ such that $A_3\,\dot\tau^2=1$, and therefore

$$\dot\tau=A_3^{-1/2}\Longrightarrow\ddot\tau=-\frac{1}{2}A_3^{-3/2}\dot A_3,\qquad \frac{\ddot\tau}{\dot\tau}=-\frac{1}{2}A_3^{-1}\,\dot A_3,$$

we find the equation

$$x''-\frac{1}{2}A_3^{-1}\dot A_3\,x'+3A_3^{-1/2}x\,x'+A_3\,x^3+A_2A_3^{-1}\,x'+A_2\,x^2+A_1\,x+A_0=0,$$

which can be rewritten in he form:

$$\ddot x+(b_0(t)+b_1(t)x)\dot x+c_0(t)+c_1(t)x+c_2(t)x^2+c_3(t)x^3=0,$$

with

$$b_1(t)=3\sqrt{A_3(t)},\qquad b_0(t)=\frac{A_2(t)}{\sqrt{A_3(t)}}-\frac{\dot A_3(t)}{2A_3(t)},$$

and is considered as the most general second orden Riccati equation.

It has recently been shown (JFC+ MF Rañada+M Santander, JMP **46**, 062703 (2005)) that such a second-order Riccati equations admit a Lagrangian of the form:

$$L(t,x,v)=\frac{1}{v+U(t,x)},$$

with $U(t,x)=a_0(t)+a_1(t)x+a_2(t)x^2$.

The corresponding t-dependent Hamiltonian obtained from the Legendre transformation

$$p = \frac{\partial L}{\partial v} = -\frac{1}{(v+U(t,x))^2} \Longrightarrow v = \frac{1}{\sqrt{-p}} - U(t,x),$$

i.e. the image is the open submanifold $\mathcal{O} = \{(x,p) \in \mathrm{T}^*_x\mathbb{R} \mid p < 0\}$ and we can define in $\mathcal{O}$ the Hamiltonian

$$h(t,x,p) = p\left(\frac{1}{\sqrt{-p}} - U(t,x)\right) - \sqrt{-p} = -2\sqrt{-p} - p\,U(t,x).$$

Consequently, the Hamilton equations for h are

$$\begin{cases} \dot{x} &= \dfrac{\partial h}{\partial p} = \dfrac{1}{\sqrt{-p}} - U(t,x), \\ \dot{p} &= -\dfrac{\partial h}{\partial x} = p\dfrac{\partial U}{\partial x}(t,x). \end{cases}$$

which, taking into account the form of $U(t,x)$ turn out to be

$$\begin{cases} \dot{x} &= \dfrac{\partial h}{\partial p} = \dfrac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\ \dot{p} &= -\dfrac{\partial h}{\partial x} = p(a_1(t) + 2a_2(t)x). \end{cases}$$

This is a Lie system: In fact, consider the set of vector fields

$$X_1 = \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p},$$
$$X_4 = x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p}, \quad X_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p}.$$

The time-dependent vector field describing the system is

$$X(t,x) = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4,$$

and the vector fields close on the commutation relations

$$\begin{aligned}
&[X_1, X_2] = 0, && [X_1, X_3] = \frac{1}{2}X_1, && [X_1, X_4] = X_5, && [X_1, X_5] = 0,\\
&[X_2, X_3] = X_2, && [X_2, X_4] = 2X_3, && [X_2, X_5] = X_1, &&\\
&[X_3, X_4] = X_4, && [X_3, X_5] = \frac{1}{2}X_5, && &&\\
&[X_4, X_5] = 0. && && &&
\end{aligned}$$

and then we see that it is a Lie system related to a Vessiot-Guldberg Lie algebra of vector fields V.

More specifically, the vector fields $X_1, \ldots, X_5$ span a five dimensional Lie algebra of vector fields V that is not solvable because $[V,V] = V$.

Moreover, V is not a semisimple algebra. It admits an Abelian solvable ideal $V_1 = \langle X_1, X_5 \rangle$, and $V_2 = \langle X_2, X_3, X_4 \rangle$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Therefore V is a semidirect sum $V_1 \oplus_s V_2$.

Consequently, the Lie algebra V gives rise to a Lie group of the form $G = \mathbb{R}^2 \lhd SL(2,\mathbb{R})$, where $\lhd$ denotes the semidirect product of $SL(2,\mathbb{R})$ by $\mathbb{R}^2$, and a Lie group action $\Phi : G \times \mathcal{O} \to \mathcal{O}$ whose fundamental vector fields are exactly those of V.

Indeed, it is a long, but straightforward computation, to show that

$$\Phi\left(\left((\lambda_1, \lambda_2), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right), (x,p)\right) = \left(\frac{\bar{x} - \lambda_1\sqrt{-\bar{p}}}{1 + \lambda_2(-\bar{p})^{-1/2}}, -(\sqrt{-\bar{p}} + \lambda_2)^2\right),$$

where

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \bar{p} = p\,(\delta + \gamma x)^2 \,.$$

This action enables us to put the general solution $\xi(t)$ of the system of Hamilton equations in the form $\xi(t) = \Phi(g(t), \xi_0)$, where $g(t)$ is the solution of the equation

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{5} b_\alpha(t) X_\alpha^R(g), \qquad g(0) = e,$$

on G, with the X_α^R being a family of right-invariant vector fields over G such that the $X_\alpha^R(e) \in T_e G$ close the same commutation relations as the X_α.

To be remarked that the vector fields X_i here considered are Hamiltonian with respect to the usual symplectic form in $\mathrm{T}^*\mathbb{R}$, their Hamiltonians being respectively given by:

$$h_1 = 2\sqrt{-p}, \quad h_2 = -p, \quad h_3 = -xp, \quad h_4 = -x^2 p,$$

and it turns out that their nonvanishing Poisson brackets are

$$\{h_1, h_3\} = \tfrac{1}{2} h_1, \quad \{h_1, h_4\} = h_5, \quad \{h_1, h_5\} = 2, \quad \{h_2, h_3\} = h_2,$$

$$\{h_2, h_4\} = 2h_3, \quad \{h_2, h_5\} = h_1, \quad \{h_3, h_4\} = h_4, \quad \{h_3, h_5\} = \tfrac{1}{2} h_5$$

with $h_5 = 2x\sqrt{-p}$. They close on a six-dimensional real Lie algebra with the function $h_6 = 1$. Moreover, it can be seen that the t-dependent system can be put into the form $\widehat{\Lambda}(-dh_t)$, where h_t is a t-parametrized family of functions over $\mathcal{O}$ of the form

$h_t = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2 h_4$ and therefore the Lie system we are considering is Hamiltonian

Finally, a superposition rule for the second order Riccati equation can be obtained through the common first-integrals for the appropriated diagonal prolongations $\widehat{X}_1, \widehat{X}_2, \widehat{X}_3, \widehat{X}_4, \widehat{X}_5$ on a certain $\mathcal{O}^{(m)} \subset \mathrm{T}^*\mathbb{R}^{(m)}$ (i.e. such that their projections $\pi_*(\widehat{X}_\alpha)$, with $\alpha = 1, \ldots, 5$, are linearly independent at a generic point of $\mathrm{T}^*\mathbb{R}^m$). In our case, it can be easily verified that $m = 4$. The resulting first-integrals, turn out to be

$$\Delta_1 = (x_{(2)} - x_{(3)})\sqrt{p_{(2)}p_{(3)}} + (x_{(3)} - x_{(1)})\sqrt{p_{(3)}p_{(1)}} + (x_{(1)} - x_{(2)})\sqrt{p_{(2)}p_{(1)}},$$
$$\Delta_2 = (x_{(1)} - x_{(2)})\sqrt{p_{(1)}p_{(2)}} + (x_{(2)} - x_{(0)})\sqrt{p_{(2)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(1)}p_{(0)}},$$
$$\Delta_3 = (x_{(1)} - x_{(3)})\sqrt{p_{(1)}p_{(3)}} + (x_{(3)} - x_{(0)})\sqrt{p_{(3)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(1)}p_{(0)}}.$$

In order to obtain a superposition principle, we just need to obtain the value of $p_{(0)}$ in terms of the remaining variables from one of the above integrals, e.g. Δ_2, to get

$$\sqrt{-p_{(0)}} = \frac{\Delta_2 + (x_{(2)} - x_{(1)})\sqrt{p_{(1)}p_{(2)}}}{(x_{(2)} - x_{(0)})\sqrt{-p_{(2)}} + (x_{(0)} - x_{(1)})\sqrt{-p_{(1)}}},$$

and to plug this value in one of the others variables, e.g. Δ_3, to have

$$
\begin{aligned}
x_{(0)} &= \frac{\Delta_3(\sqrt{-p_{(2)}}x_{(2)}-\sqrt{-p_{(1)}}x_{(1)})+\Delta_2(\sqrt{-p_{(1)}}x_{(1)}-\sqrt{-p_{(3)}}x_{(3)})-\Delta_1 x_{(1)}\sqrt{-p_{(1)}}}{\Delta_3(\sqrt{-p_{(2)}}-\sqrt{-p_{(1)}})+\Delta_2(\sqrt{-p_{(1)}}-\sqrt{-p_{(3)}})-\sqrt{-p_{(1)}}\Delta_1}, \\
p_{(0)} &= -\left[\frac{k_2(\sqrt{-p_{(1)}}-\sqrt{-p_{(3)}})-k_3(\sqrt{-p_{(1)}}-\sqrt{-p_{(2)}})-\Delta_1\sqrt{-p_{(1)}}}{(\Delta_1\sqrt{p_{(1)}p_{(2)}}(x_{(1)}-x_{(2)})-\Delta_1 k_2)(k_2+\sqrt{p_{(1)}p_{(2)}}(x_{(2)}-x_{(1)}))^{-1}}\right]^2 .
\end{aligned}
$$

The above expression gives us a superposition rule for second order Riccati differential equation.

In addition, as its general solution, $(x_{(0)}(t), p_{(0)}(t))$, satisfies that $x_{(0)}(t)$ is the general solution, the first part of the above expressions gives us the solution of second-order Riccati equations in terms of three particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$, their associated moments $p_{(1)}(t), p_{(2)}(t), p_{(3)}(t)$, and two constants Δ_1, Δ_2

Note that once a family of particular solutions is chosen the constant Δ_1 gets fixed.

Lie systems admitting Poisson structures

Note that some of the above examples of Lie systems are Hamiltonian dynamical systems. We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds.

Definition: *A Poisson manifold is a pair (M, Λ) where Λ is a bivector in the differentiable manifold M in such a way that the Schouten bracket $[\cdot, \cdot]_{\mathrm{S.B.}} = 0$. The bivector field gives by contraction a map denoted $\widehat{\Lambda}$ such that*

$$\widehat{\Lambda}(\alpha)(\beta) = \Lambda(\alpha, \beta)$$

In particular, if $f_1, f_2 \in C^\infty(M)$, we define the Poisson bracket $\{f_1, f_2\}$ by

$$\{f_1, f_2\} = \Lambda(df_1, df_2),$$

and this Poisson bracket satisfies Jacobi identity because of the vanishing of the Schouten bracket condition

The Lie bracket over $C^\infty(M)$ holds the Leibnitz rule

$$\{fg, h\} = \{f, h\}g + \{g, h\}f, \qquad \forall f, g, h \in C^\infty(M).$$

Consequently, the above Lie bracket becomes a derivation in each entry and, hence, given a function $f \in C^\infty(M)$, there exists a vector field X_f over M such that $X_f g = \{g, f\}$ for each $g \in C^\infty(M)$, i.e. $X_f = \widehat{\Lambda}(-df)$. The vector field X_f is called the *Hamiltonian vector field* associated with f. The Jacobi identity for the Poisson structure entails that

$$X_{\{f,g\}} = -[X_f, X_g], \qquad \forall f, g \in C^\infty(M).$$

In other words, the mapping $f \mapsto X_f$ is a Lie algebra anti-homomorphism between the Lie algebras $(C^\infty(M), \{\cdot,\cdot\})$ and $(\Gamma(\tau_M), [\cdot,\cdot])$.

Equivalently, we can say that $\widehat{\Lambda} \circ d : C^\infty(M) \to \mathfrak{X}_H(M, \Lambda)$ is a Lie algebra homomorphism.

Definition: *The elements of the kernel of the previous homomorphism are called Casimir functions. The set of such Casimir functions will be denoted $\mathcal{C}$, We call Casimir codistribution of the Poisson manifold (M, Λ) the codistribution given by $\mathcal{C}^\Lambda = \ker \widehat{\Lambda}$.*

This can be summarising by saying that the following sequence is exact:

$$0 \longrightarrow \mathcal{C} \longrightarrow C^\infty(M) \xrightarrow{\widehat{\Lambda}\circ d} \mathfrak{X}_H(M,\Lambda) \longrightarrow 0$$

Definition. *A Lie–Hamiltonian structure is a triple (M,Λ,h), where (M,Λ) is a Poisson manifold and h is a t-parametrised family of functions $h_t : M \to \mathbb{R}$ such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ is a finite-dimensional real Lie algebra.*

Definition: *A t-dependent system X on a manifold M is said to be a Lie–Hamilton system if there exists a Lie–Hamiltonian structure (M,Λ,h) such that $X_t \in \widehat{\Lambda}(-dh_t)$, for every $t \in \mathbb{R}$.*

As an example, the previously studied cases of Winternitz–Smorodinsky oscillators and second-order Riccati equations were Lie–Hamilton system with respect to non-degenerated Poisson structures. For instance, in the latter case if Λ denotes the Poisson bivector associated with the natural symplectic structure in $\mathcal{O} \subset \mathrm{T}^*\mathbb{R}$ and $h_t = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2 h_4$, then, $X_t = -\widehat{\Lambda}(-dh_t)$ is a Lie–Hamilton system with Lie–Hamiltonian structure $(\mathcal{O} \subset \mathrm{T}^*\mathbb{R}, \Lambda, h)$.

The dual $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is endowed with a natural structure of a Poisson manifold.

The differential $(df)_u$ of a function $f \in C^\infty(\mathfrak{g}^*)$ at a point $u \in \mathfrak{g}^*$ is a linear map $(df)_u : \mathrm{T}_u\mathfrak{g}^* \to \mathbb{R}$.

On the other side as $\mathfrak{g}^*$ is a linear space, when considered as a differentiable manifold, we can identify in a natural way the linear space $\mathfrak{g}^*$ with the tangent space $\mathrm{T}_u\mathfrak{g}^*$, at each point and then $(df)_u : \mathrm{T}_u\mathfrak{g}^* \to \mathbb{R}$ can be seen as an element of $(\mathfrak{g}^*)^*$, i.e. of $\mathfrak{g}$.

Denoting such element as $\delta_u f$, i.e.

$$\langle y, \delta_u f\rangle = (df)_u(y) = \frac{d}{dt} f(u + ty)\bigg|_{t=0}, \qquad y \in \mathfrak{g}^*,$$

the Poisson structure in $\mathfrak{g}^*$ is given by

$$\{f, g\}(u) = \big\langle u, [\delta_u f, \delta_u g]\big\rangle.$$

In particular Jacobi identity for $\{\cdot,\cdot\}$ is a consequence of Jacobi identity for $[\cdot,\cdot]$.

If $a \in \mathfrak{g}$, let ξ_a denotes the linear function $\xi_a \in C^\infty(\mathfrak{g}^*)$ given by

$$\xi_a(u) = \langle u, a\rangle,$$

and then $\delta_u \xi_a \in \mathfrak{g}$ is such that $\delta_u \xi_a = a$, because

$$(d\xi_a)_u(y) = \frac{d}{dt}\xi_a(u+ty)\bigg|_{t=0} = \langle y, a\rangle.$$

Consequently, for any pair, a, b of elements in $\mathfrak{g}$,

$$\{\xi_a, \xi_b\}(u) = \big\langle u, [a,b]\big\rangle = \xi_{[a,b]}(u).$$

Given a basis of $\mathfrak{g}$, $\{a_1, \ldots, a_n\}$, and $\{\alpha^1, \ldots, \alpha^n\}$ the dual basis of $\mathfrak{g}^*$, and using the shorter notation ξ_i for ξ_{a_i}, we see that if $u \in \mathfrak{g}^*$ is $u = \sum_{i=1}^n u_i \alpha^i$, then

$$\langle u, a_k\rangle = u_k = \xi_k(u),$$

and therefore ξ_i are coordinate functions in $\mathfrak{g}^*$, corresponding to the dual basis. The expression of the Poisson structure is the one given by

$$\{f, g\}(u) = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j} c_{ij}{}^k \xi_k,$$

where $c_{ij}{}^k$ are the structure constants of $\mathfrak{g}$ relative to the basis $\{a_1, \ldots, a_n\}$ of $\mathfrak{g}$, i.e.

$$[a_i, a_j] = \sum_{k=1}^r c_{ij}{}^k\, a_k.$$

The Hamiltonian vector fields corresponding to the coordinate functions ξ_i are given by

$$X_{\xi_i} = \sum_{j=1}^{n} \{\xi_j, \xi_i\} \frac{\partial}{\partial \xi_j} = - \sum_{j,k=1}^{n} c_{ij}{}^{k} \xi_k \frac{\partial}{\partial \xi_j} = - \sum_{j,k=1}^{n} [\mathrm{ad}(a_i)]^k{}_j \xi_k \frac{\partial}{\partial \xi_j},$$

and therefore, if $a = \sum_{i=1}^{n} y^i \, a_i$,

$$X_a = \sum_{j,k=1}^{n} \mathrm{coad} \left(\sum_{i=1}^{n} y^i \, a_i \right)_j{}^k \, \xi_k \frac{\partial}{\partial \xi^j},$$

that can be written as

$$X_a = \mathrm{coad}(a)$$

One can see that these vector fields turn out to be the fundamental vector fields corresponding to the coadjoint action of a group G with Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$.

Therefore, the vector fields $X_i(\theta) = [\mathrm{coad}(a_i)](\theta) \in \mathrm{T}_\theta\mathfrak{g}^*$, with $\alpha = 1, \ldots, r$, span a Vessiot–Guldberg Lie algebra for a system in $\mathfrak{g}^*$. Indeed, they generate the Lie algebra of fundamental vector fields of the coadjoint action of the Lie group G corresponding to $\mathfrak{g}$ over $\mathfrak{g}^*$ and moreover these vector fields are Hamiltonian with respect to $\widehat{\Lambda}_{\mathfrak{g}^*}$ with Hamiltonian functions $h_i = -\xi_i$.

The functions h_i themselves are a basis for a finite-dimensional real Lie algebra $(W, \{\cdot,\cdot\}_{\mathfrak{g}^*})$ of functions:

$$\{h_i, h_j\} = -\sum_{k=1}^{r} c_{ij}{}^k\, h_k.$$

Therefore, every curve h_t in the Poisson manifold $(\mathfrak{g}^*, \{\cdot,\cdot\}_{\mathfrak{g}^*})$ gives rise to a Lie–Hamiltonian $(\mathfrak{g}^*, \Lambda_{\mathfrak{g}^*}, h)$.

As an example, let us consider, for example, Euler equations on the dual $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, i.e.

$$\frac{d\theta}{dt} = \mathrm{coad}_{\phi(t)}\theta, \qquad \theta \in \mathfrak{g}^*, \tag{1}$$

where $\phi(t)$ is a curve in $\mathfrak{g}$ and $\mathrm{coad}_{\phi(t)}\theta = -\theta \circ \mathrm{ad}_{\phi(t)} \in \mathfrak{g}^*$. These equations can be found, for instance, in the study of geometric phases for classical systems.

This system describes the integral curves of the t-dependent vector field $X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$, where we have assumed $\phi(t) = \sum_{\alpha=1}^{r} b_\alpha(t) e_\alpha$.

Since X_α are Hamiltonian vector fields with respect to $\Lambda^*_{\mathfrak{g}}$, it follows that

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) \widehat{\Lambda}(-dh_\alpha) = \widehat{\Lambda}\left[-d\left(\sum_{\alpha=1}^{r} b_\alpha(t) h_\alpha\right)\right].$$

In other words, $h_t = \sum_{\alpha=1}^{r} b_\alpha(t) h_\alpha$ allows us to build up a Lie–Hamiltonian structure $(\mathfrak{g}^*, \Lambda_{\mathfrak{g}^*}, h)$ for X.

Lemma *For each Lie–Hamilton system X admitting a Lie–Hamiltonian structure (M, Λ, h), the mapping $\widehat{\Lambda} \circ d : \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}) \to V^X$* is a Lie algebra epimorphism. In consequence, as the functions h_i are closing in a real Lie algebra the restriction of $\widehat{\Lambda} \circ d$ to

$$V^X \simeq \frac{\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})}{\ker\left(\widehat{\Lambda} \circ d\right)}.$$

Proof.- It suffices to consider the restriction of $\widehat{\Lambda} \circ d$ to the subalgebra $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ and observe that the image of such a map is V^X.

If $(\mathcal{A}, *)$ is an unital associative algebra there are linear maps $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : \mathbb{K} \to \mathcal{A}$ such that $* = m \circ \pi$, where $\pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the natural projection, and

$$m \circ (m \otimes \mathrm{id}\,) = m \circ (\mathrm{id}\, \otimes m)\,, \qquad m \circ (\eta \otimes \mathrm{id}\,) = \mathrm{id}\, = m \circ (\mathrm{id}\, \otimes \eta)\,, \tag{2}$$

where m and η are given by

$$m(a_1 \otimes a_2) = a_1 * a_2\,, \qquad \eta(1) = 1_{\mathcal{A}}\,,$$

with $1_{\mathcal{A}}$ being the unity in $\mathcal{A}$.

Both relations are equivalent to the commutativity of he diagrams

$$
\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[dl, "m\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes m"] & \\
\mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[dl, "m"] \\
& \mathcal{A} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{K}\otimes\mathcal{A} \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] & \mathcal{A}\otimes\mathbb{K} \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl] \\
& \mathcal{A} &
\end{tikzcd}
$$

where use has been made of the natural isomorphisms

$$(\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{A} \approx \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \approx \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{A})\,.$$

In order to define commutative algebra we van introduce the interchange map $\tau : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ defined by

$$\tau(a_1\otimes a_2) = a_2\otimes a_1\,.$$

Then the algebra $(\mathcal{A}, *)$ is commutative if $m \circ \tau = m$.

Definition: *If* $(\mathcal{A}_1, m_1, \eta_1)$ *and* $(\mathcal{A}_2, m_2, \eta_2)$ *are algebras over the field* $\mathbb{K}$*, then the tensorial product is given by* $(\mathcal{A}_1 \otimes \mathcal{A}_2, (m_1 \otimes m_2) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}), \eta_1 \otimes \eta_2)$*, namely,*

$$m_{\mathcal{A}_1 \otimes \mathcal{A}_2} = (m_1 \otimes m_2) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \,.$$

With this notation morphism of the algebra $(\mathcal{A}_1, m_1, \eta_1)$ in the algebra $(\mathcal{A}_2, m_2, \eta_2)$ is a linear map $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $m_2 \circ (f \otimes f) = f \circ m_1$, i.e. the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}_1 \otimes \mathcal{A}_1 \arrow[r, "f \otimes f"] \arrow[d, "m_1"'] & \mathcal{A}_2 \otimes \mathcal{A}_2 \arrow[d, "m_2"] \\
\mathcal{A}_1 \arrow[r, "f"'] & \mathcal{A}_2
\end{tikzcd}$$

The coalgebra concept appears when using duality in the definition of algebra using themaps m and η. The dual maps

$$\Delta = m^* : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^* \,, \qquad \varepsilon = \eta^* : \mathcal{A}^* \to \mathbb{K}^*$$

enjoys the dual properties:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}\,) \circ \Delta\,, \qquad (\varepsilon \otimes \mathrm{id}\,) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta,$$

which can be expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
& \mathcal{A}^* \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
\mathcal{A}^* \otimes \mathcal{A}^* \arrow[dr, "\Delta\circ\mathrm{id}"'] & & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[dl, "\mathrm{id}\otimes\Delta"] \\
& \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^* &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{A}^* \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\
\mathbb{K} \otimes \mathcal{A}^* & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[l, "\varepsilon\otimes\mathrm{id}"] \arrow[r, "\mathrm{id}\otimes\varepsilon"'] & \mathcal{A}^* \otimes \mathbb{K}
\end{tikzcd}$$

This leads to introduce the following concept:

Definition: *A coalgebra is a triple* $(\mathcal{C}, \Delta, \varepsilon)$*, where* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ *and* $\varepsilon : \mathcal{C} \to \mathbb{K}$ *are linear maps que called comultiplication and counity (or augmentation), such that the following diagrams are commutative:*

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
\mathcal{C} \otimes \mathcal{C} \arrow[dr, "\Delta \otimes \mathrm{id}"'] & & \mathcal{C} \otimes \mathcal{C} \arrow[dl, "\mathrm{id} \otimes \Delta"] \\
 & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\
\mathbb{K} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\varepsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \varepsilon"'] & \mathcal{C} \otimes \mathbb{K}
\end{tikzcd}$$

Therefore, $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

Definition: *A coalgebra* $(\mathcal{C}, \Delta, \varepsilon)$ *is cocommutative if* $\tau \circ \Delta = \Delta$.

Definition: *A linear map from the coalgebra* $(\mathcal{C}, \Delta, \varepsilon)$ *in the coalgebra* $(\mathcal{C}', \Delta', \varepsilon')$ *is a morphism of coalgebras when*

$$\Delta' \circ f = (f \otimes f) \circ \Delta \,, \qquad \varepsilon' \circ f = \varepsilon \,,$$

which can be expressed by the commutativity of the diagrams:

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{C} \arrow[r, "f \otimes f"] & \mathcal{C}' \otimes \mathcal{C}' \\ \mathcal{C} \arrow[u, "\Delta"] \arrow[r, "f"] & \mathcal{C}' \arrow[u, "\Delta'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & \mathcal{C}' \arrow[d, "\varepsilon'"] \\ & \mathbb{K} \end{tikzcd}$$

Recall that $(\mathcal{A}, \cdot, [\cdot, \cdot])$ is a Poisson algebra if $(\mathcal{A}, \cdot)$ is a commutative algebra and $(\mathcal{A}, [\cdot \cdot])$ is a Lie algebra such that

$$[a_3, a_2 \cdot a_1] = [a_3, a_2] \cdot a_1 + a_2 \cdot [a_3, a_1]$$

for arbitrary elements $a_i \in \mathcal{A}$, i.e. $[a, \cdot] : \mathcal{A} \to \mathcal{A}$ is a derivation of the algebra $(\mathcal{A}, \cdot)$.

As an example, if (M, Λ) is a Poisson manifold, then $(C^\infty(M), \{\cdot, \cdot\})$ is a Poisson algebra where $\cdot$ is the usual product of functions and $\{\cdot, \cdot\}$ denotes the Poisson bracket defined by Λ.

Moreover, if $(\mathcal{A}, \cdot, [\cdot,\cdot])$ is a Poisson algebra we can define a new Poisson algebra structure in $\mathcal{A}\otimes\mathcal{A}$ by means of $(a_2\otimes b_2)\star(a_1\otimes b_1)) = (a_2\cdot a_1)\otimes(b_2\cdot b_1)$ and

$$\{a_4\otimes a_3, a_2\otimes a_1\}_{\mathcal{A}\otimes\mathcal{A}} = \{a_4, a_2\}_{\mathcal{A}}\otimes(a_3\cdot a_1) + (a_4\cdot a_2)\otimes\{a_3, a_1\}_{\mathcal{A}},\ a_i\in\mathcal{A}.$$

Definition: *A Poisson coalgebra is given by $(\mathcal{A}, \cdot, \{\cdot,\cdot\}, \Delta)$ where $(\mathcal{A}, \cdot, \{\cdot,\cdot\})$ is a Poisson algebra and the coproduct $\Delta : \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is a Poisson algebra morphism between $\mathcal{A}$ and $\mathcal{A}\otimes\mathcal{A}$, i.e.*

$$\Delta(a_2\cdot a_1) = \Delta(a_2)\cdot\Delta(a_1), \qquad \Delta(\{a_2, a_1\}) = \{\Delta(a_2), \Delta(a_1)\}.$$

Let $\mathfrak{g}$ be a Lie algebra and $T(\mathfrak{g})$ its associated tensor algebra, the adjoint representation of $\mathfrak{g}$, i.e. the Lie algebra morphism $\mathrm{ad} : v\in\mathfrak{g}\mapsto \mathrm{ad}_v\in\mathrm{Der}(\mathfrak{g})$, can be extended to a representation of $\mathfrak{g}$ on $T(\mathfrak{g})$ by derivations. In other words, there exists a Lie algebra homomorphism $\mathrm{ad} : v\in\mathfrak{g}\to\mathrm{ad}_v\in\mathrm{Der}(T(\mathfrak{g}))$.

Definition: *The symmetric algebra of $\mathfrak{g}$ is the free commutative unital associative algebra $(S(\mathfrak{g}), \cdot)$ given by the quotient space $T(\mathfrak{g})/\mathcal{R}$, where $\mathcal{R}$ is the linear subspace of $T(\mathfrak{g})$ spanned by the elements*

$$P \otimes (v \otimes w - w \otimes v) \otimes Q, \qquad v, w \in \mathfrak{g}, \quad P, Q \in T(\mathfrak{g}),$$

endowed with the commutative product $P \cdot Q = \bar{\pi}(P \otimes Q)$, where $\bar{\pi} : T(\mathfrak{g}) \to S(\mathfrak{g})$ is the quotient map.

As $\mathcal{R}$ is invariant through the action of the derivations ad_v, the adjoint action can be extended to a Lie algebra morphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(S(\mathfrak{g}))$. Furthermore, since $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$, every element of $\mathfrak{g}$ can be regarded as a linear function over $\mathfrak{g}^*$. As the space $S(\mathfrak{g})$ is isomorphic to the unital commutative associated ring of polynomials in the elements of a basis of $\mathfrak{g}$, we can consider $S(\mathfrak{g})$ as the commutative unital associative ring of polynomials in a basis of linear functions over $\mathfrak{g}^*$. In addition, such a space can be equipped with a unique Poisson bracket $\{,\} : S(\mathfrak{g}) \times S(\mathfrak{g}) \to S(\mathfrak{g})$ satisfying that $\{v, w\} = \mathrm{ad}_v w$ for every $v, w \in \mathfrak{g}$. In addition $S(\mathfrak{g})$ becomes a coalgebra by defining $\Delta(w) = w \otimes 1 + 1 \otimes w$ for every $w \in \mathfrak{g} \subset S(\mathfrak{g})$ and extending conveniently Δ to an homomorphism from $S(\mathfrak{g})$ to $S(\mathfrak{g}) \times S(\mathfrak{g})$.

Definition: *The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the unital associative algebra obtained by taking the quotient of $T(\mathfrak{g})$ by the linear subspace $\mathcal{L} \subset T(\mathfrak{g})$ spanned by the elements*

$$P \otimes (v \otimes w - w \otimes v - [v, w]) \otimes Q, \qquad v, w \in \mathfrak{g}, \quad P, Q \in T(\mathfrak{g}),$$

and defining $\bar{\pi}(P) \otimes \bar{\pi}(Q) = \bar{\pi}(P \otimes Q)$, where $\bar{\pi} : T(\mathfrak{g}) \to U(\mathfrak{g})$ is the quotient map and $P, Q \in T(\mathfrak{g})$.

As previously, the adjoint action extends to a well-defined action $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(U(\mathfrak{g}))$. To us the following interpretation takes special relevance. The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ can be endowed with a structure of coalgebra by defining

$$\Delta(v_i) = 1 \otimes v_i + v_i \otimes 1,$$

over a basis $v_1, \ldots, v_r$ of $\mathfrak{g}$ and extending the value of the above morphisms over $U(\mathfrak{g})$ by linearity.

It is worth noting that Friederichs theorem ensures that the only *primitive elements* of $U(\mathfrak{g})$, i.e. those $Y \in U(\mathfrak{g})$ obeying that $\Delta(v) = 1 \otimes v + v \otimes 1$, are the generators v_i.

The so-called *symmetrized map*, i.e. the linear mapping $\lambda : S(\mathfrak{g}) \to U(\mathfrak{g})$ of the form

$$\lambda(v_{i_1} \cdot \ldots \cdot v_{i_l}) = \frac{1}{l!} \sum_{s \in \Pi_p} \lambda(v_{s(i_1)}) \ldots \lambda(v_{s(i_l)}), \qquad \forall v_{i_1}, \ldots, v_{i_r} \in \mathfrak{g},$$

establishes a linear isomorphism between $U(\mathfrak{g})$ and $S(\mathfrak{g})$. In addition, we have that

$$\lambda(\mathrm{ad}_v P) = v \otimes \lambda(P) - \lambda(P) \otimes v = \mathrm{ad}_v \lambda(P).$$

The above relation relates the so-called *Casimirs of the Lie algebra* $\mathfrak{g}$, namely those elements $\mathcal{C}$ of $U(\mathfrak{g})$ such that $\mathrm{ad}_v \mathcal{C} = 0$ for all $v \in \mathfrak{g}$, with the Casimir elements of the Poisson algebra $S(\mathfrak{g})$. More specifically, if $\mathcal{C}$ is a Casimir for $\mathfrak{g}$, then $C = \lambda^{-1}(\mathcal{C})$ obeys $\mathrm{ad}_v C = 0$, for every $v \in \mathfrak{g}$ and C Poisson commutes with all the elements of $S(\mathfrak{g})$.

Given a Lie algebra $\mathfrak{g}$, we know that $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. In other words, the elements of $\mathfrak{g}$ can be understood as linear functions over $\mathfrak{g}^*$. Moreover, if $v_1, \ldots, v_r$ is a basis for $\mathfrak{g}$, these elements can be considered as a coordinate system over $\mathfrak{g}^*$. Consequently, if we assume $[v_i, v_j] = \sum_{k=1}^r c_{ijk} v_k$ for $i, j = 1, \ldots, r$, the Poisson bivector $\Lambda_{\mathfrak{g}^*}$ over

$\mathfrak{g}^*$ reads

$$\Lambda = \sum_{i,j,k=1}^{r} c_{ijk} v_k \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j}.$$

The space A of polynomials in these variables becomes a Poisson algebra with the commutative product of functions and the restriction of the above Poisson bracket over A. Indeed, this space is the same as the Poisson algebra $S(\mathfrak{g})$. In fact, $\{v^i, v^j\} = \mathrm{ad}_{v_i} v^j$ and hence every element $C = \lambda^{-1}(\mathcal{C})$ for a Casimir $\mathcal{C}$ of $\mathfrak{g}$ commutes with all the elements of A, i.e. it is a *Casimir element* of A. This is a very relevant fact, as it allows us to build up Casimir elements for A from Casimirs of the initial Lie algebra $\mathfrak{g}$. Similarly to $S(\mathfrak{g})$, the space A is a Poisson coalgebra.

If we denote $\Delta^{(2)} = \Delta$, we can define the algebra morphisms $\Delta^{(N)} : A \to A^{\otimes^N}$ of the form

$$\Delta^{(N)} = (\mathrm{id} \otimes \mathrm{id} \otimes \overset{N-2}{\ldots} \otimes \mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(N-1)}.$$

Obviously, we have that

$$\{\Delta^{(N)}(v_i), \Delta^{(N)}(v_j)\}_{A^{\otimes N}} = \Delta^{(N)}\{v_i, v_j\}_A.$$

Take now a representation $D : \mathfrak{g} \to C^\infty(\mathrm{T}^*\mathbb{R}^n)$. It is straightforward to check that the space $(\bar{A}, \cdot, \{,\})$ of polynomials in the coordinate functions $D(v_1), \ldots, D(v_r)$ is a Poisson algebra with respect to the product of functions and the Lie bracket over $\bar{A}$ defined by the restriction over $\bar{A}$ of the Poisson bracket $\{,\}$ over $C^\infty(\mathrm{T}^*\mathbb{R}^n)$. In addition, $\bar{A}$ is also a Poisson coalgebra with the coproduct $\bar{\Delta} : \bar{A} \to \bar{A} \otimes \bar{A}$, taking $\Delta(D(v_i)) = D(v_i) \otimes 1 + 1 \otimes D(v_i)$ and extending to the whole $\bar{A}$. Summing up, A and $\bar{A}$ are isomorphic Poisson coalgebras.

The Lie algebra representation D can be extended to a Lie algebra morphism $D : C^\infty(\mathfrak{g}^*) \to C^\infty(\mathrm{T}^*\mathbb{R}^n)$ by defining

$$D(H(v_1, \ldots, v_r)) = H(D(v_1), \ldots, D(v_r)).$$

or to the Poisson algebra morphism $D : A \otimes \ldots \otimes A \to C^\infty(\mathrm{T}^*\mathbb{R}^n) \otimes \ldots \otimes C^\infty(\mathrm{T}^*\mathbb{R}^n)$ induced by the relations

$$D(v_{i_1} \otimes \ldots \otimes v_{i_r}) = D(v_{i_1}) \otimes \ldots \otimes D(v_{i_r}).$$

More even, D can be defined over the functions of the form $\Delta^{(N)} H(v_1, \ldots, v_r)$ by setting

$$D\Delta^{(N)}(C(v_1, \ldots, v_r)) = (C(D\Delta^{(N)}(v_1), \ldots, D\Delta^{(N)}(v_r))).$$

From here, it easily turns out that

$$\{D\Delta^{(N)}(C), D\Delta^{(N)}(v_j)\} = D\Delta^{(N)}\{C, v_j\}_A = 0.$$

Theorem 1 (Ballesteros, Corsetti & Ragnisco 1996) *Let $\mathfrak{g}$ be a Lie algebra with a Casimir element C and D a realization of $\mathfrak{g}$ in $C^\infty(\mathrm{T}^*\mathbb{R}^n)$. The Hamiltonian*

$$H^{(s)}(q_1,\ldots,q_s,p_1,\ldots,p_s) = D[C(\Delta^{(s)}(v_1),\ldots,\Delta^{(s)}(v_r))], \quad (q_i,p_i) \in \mathrm{T}^*\mathbb{R}^n,$$

where $v_1,\ldots,v_r$ is a basis of $\mathfrak{g}$, defines a dynamical system in which the $H^{(m)}$ functions with $m = 1,\ldots,s$ are s constants of the motion in involution with respect to the canonical Poisson bracket over $\mathrm{T}^*\mathbb{R}^{sn}$.

Let us apply the above formalism to describe superposition rules for Lie-Hamilton systems through a certain Lie algebra morphism $D : \mathfrak{g} \to C^\infty(\mathrm{T}^*\mathbb{R}^n)$.

If we chose a basis $v_1,\ldots,v_r$ of $\mathfrak{g}$, then $X_t = -\widehat{\Lambda}(dh_t)$ takes values in the Vessiot–Guldberg Lie algebra V generated by the vector fields $X_\alpha = -\widehat{\Lambda}(dD(v_\alpha))$, with $\alpha = 1,\ldots,r$, over $\mathrm{T}^*\mathbb{R}^n$.

Recall that determining a superposition rule for X_t relies on obtaining a family of common first-integrals for all the prolongations $\widehat{X}_1,\ldots,\widehat{X}_r$ to a certain $(\mathrm{T}^*\mathbb{R}^n)^{m+1}$ of the vector fields $X_1,\ldots,X_r$.

As these vector fields are Hamiltonian vector fields with Hamiltonian functions $h_\alpha = D(v_\alpha)$, their prolongations $\widehat{X}_\alpha$ to $(T^*N)^{m+1}$ are Hamiltonian vector fields with respect to the Poisson bivector Λ^{m+1} induced by the canonic symplectic structure on T^*N^{m+1} with Hamiltonian functions

$$\Delta^{(m+1)}h_\alpha = h_\alpha \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes h_\alpha \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes h_\alpha = D\Delta^{(m+1)}v_\alpha.$$

Therefore, if C is a Casimir function, then we can construct functions of the form

$$C^{(m+1)} \equiv D\Delta^{(m+1)}(C) \in C^\infty((\mathrm{T}^*\mathbb{R}^n)^{m+1}).$$

Such functions are common first-integrals for the vector fields $\widehat{X}_\alpha$, namely

$$\widehat{X}_\alpha(C^{(m+1)}) = D\{\Delta^{(m+1)}C, \Delta^{(m+1)}v_\alpha\} = -D\Delta^{(m+1)}\{C, v_\alpha\} = 0.$$

Moreover, recall that

$$\{C^{(s)}, C^{(k)}\} = 0, \qquad s \leq k = 1, \ldots, m+1.$$

As an example, consider the Lie algebra morphism $D : \mathfrak{sl}(2,\mathbb{R}) \to C^\infty(T^*\mathbb{R})$

$$D(J_-) = q_1^2, \qquad D(J_+) = p_1^2 + \frac{1}{x^2}, \qquad D(J_3) = q_1 p_1.$$

where J_-, J_+ and J_3 are a basis for $\mathfrak{sl}(2,\mathbb{R})$ closing on the commutation relations

$$[J_3, J_+] = 2J_+, \qquad [J_3, J_-] = -2J_-, \qquad [J_-, J_+] = 4J_3.$$

In this case A becomes the polynomials in J_-, J_+ and J_3. Now, we define

$$\Delta(J_-) = J_-\otimes 1 + 1\otimes J_-, \qquad \Delta(J_+) = J_+\otimes 1 + 1\otimes J_+, \qquad \Delta(J_3) = J_3\otimes 1 + 1\otimes J_3.$$

As $C = J_- J_+ - J_3^2$ is a Casimir element, we obtain that

$$\{D\Delta^{(s)}(C), D\Delta^{(m+1)} J_i\} = 0, \quad i = +, -, 3.$$

Consequently, the functions $D\Delta^{(s)}(C)$ are first integrals of the prolongations of the vector fields to $\mathrm{T}^*\mathbb{R}^{(m+1)}$. For $m = 2$, they lead to the first-integrals used in the determination of superposition rules for Milne–Pinney equations, i.e.

$$I_{ij} = (x_i p_i - x_j p_i)^2 + \left(\frac{x_j^2}{x_i^2} + \frac{x_i^2}{x_j^2}\right), \quad i < j = 1, 2, 3.$$

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