On the linearization of proper Lie groupoids

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Theorem (rough version)

If $\mathcal{G}$ is a proper Lie groupoid over a manifold M, $\mathcal{O} \subset M$ is an orbit of $\mathcal{G}$ then, around $\mathcal{O}$, $\mathcal{G}$ is isomorphic to its linearization $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$.

Starting point: Conn's linearization theorem

- conjectured by A. Weinstein (in his JDG paper)
- proved by J. Conn (Annals, 1985).

Conn's proof: based on "hard analysis", with no geometric insight ...

Weinstein's question: find a geometric proof for Conn's theorem, like the similar ones from group actions, etc.

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The geometric approach to Conn's theorem

Weinstein's, based on his discovery of symplectic groupoids:

- W1 Integrability.
- W2 Linearization of proper groupoids.
- W3 Take care of the symplectic form.

Our geometric solution (M.C., Rui Loja Fernandes):

- CF1 : Integrability.
- CF2 : standard methods (Moser, averaging, etc), in the context of Lie groupoids instead of Lie groups.

This also allowed for generalizing Conn's theorem around arbitrary symplectic leaves (joint with I.Marcut).
... which, in turn, indicate that W2 itself could be handeled by similar methods (Moser deformation arguments and averaging).

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- 2002: Weinstein shows that the statement can be reduced to the case of fixed points.
- 2004: Zung proves it for fixed points (... Zung's theorem). Still uses analytic arguments, but not so "hard".
- 2011: with I. Struchiner:
    - indeed, Zung's theorem can be proved directly by Moser arguments, averaging.
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Submanifolds

Setting: $\mathcal{O}$ a submanifold of a manifold M.

The local (linear) model for M around $\mathcal{O}$: the normal bundle:

$$\mathcal{N}_{\mathcal{O}} = T_{\mathcal{O}}M/T\mathcal{O}.$$

The linearization theorem:

Theorem (the standard tubular neighborhood theorem)

If $\mathcal{O}$ is embedded then, around $\mathcal{O}$, M is diffeomorphic to $\mathcal{N}_{\mathcal{O}}$.

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Setting: $\mathcal{O}$- an orbit of a Lie group G action on M.

The local (linear) model for M around $\mathcal{O}$: still $\mathcal{N}_{\mathcal{O}}$ (with the induced "linear" action). Standard description:

$$\mathcal{N}_{\mathcal{O}} = G \times_{G_x} \mathcal{N}_x$$

(uses a point $x \in \mathcal{O}$, G_x- the isotropy group at x, $\mathcal{N}_x$ the normal space of $\mathcal{O}$ at x, with the linear isotropy G_x-action).

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If G-compact then, around $\mathcal{O}$, M is G-equivariantly diffeomorphic to $G \times_{G_x} \mathcal{N}_x$.

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Setting: $\mathcal{O}$- a leaf of a foliation $\mathcal{F}$ of M.

The local (linear) model for M around $\mathcal{O}$: still $\mathcal{N}_{\mathcal{O}}$ (with the induced "linear" foliation). Standard description:

$$\mathcal{N}_{\mathcal{O}} = \widetilde{\mathcal{O}} \times_{\Gamma_x} \mathcal{N}_x$$

(uses $x \in \mathcal{O}$, the universal cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, the fundamental group $\Gamma_x = \pi(\mathcal{O}, x)$, and the linear holonomy $\rho : \Gamma_x \longrightarrow GL(\mathcal{N}_x)$).

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Conn's theorem

Setting: $\mathcal{O} = \{x\}$ a singular point of a Poisson manifold (M, π).

The local (linear) model for M around x: still $T_x M$, endowed with the linearization of π at x. Equivalently: use the isotropy Lie algebra $\mathfrak{g}_x$ at x, and

$$T_x M = (\mathfrak{g}_x)^*$$

with the linear Poisson structure.

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If $\mathfrak{g}_x$ is semi-simple of compact type then, around x, M and $\mathfrak{g}_x^*$ are Poisson diffeomorphic.

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Theorem (Conn's theorem)

If $\mathfrak{g}_x$ is semi-simple of compact type then, around x, M and $\mathfrak{g}_x^*$ are Poisson diffeomorphic.

Local forms around symplectic leaves (with I.Marcut)

Setting: $\mathcal{O}$ a symplectic leaf of a Poisson manifold (M, π).

The local (linear) model for M around x: still $\mathcal{N}_O$, endowed with "the linearization of π along $\mathcal{O}$". Equivalently:

$$\mathcal{N}_O = P_x \times_{G_x} \mathfrak{g}_x^*$$

uses $x \in \mathcal{O}$, the Poisson homotopy group G_x, and the Poisson homotopy cover $P_x \longrightarrow \mathcal{O}_x$.

The linearization theorem:

Theorem (Crainic-Marcut, 2011)

If P_x is compact and 2-connected then, around $\mathcal{O}$, M and $P_x \times_{G_x} \mathfrak{g}_x^*$ are Poisson diffeomorphic.

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A Lie groupoid

$$\mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} M \ .$$

An induced partition of M:

$$x \sim y \quad \text{iff} \quad \exists\, g \in \mathcal{G} \text{ with } s(g) = x, t(g) = y.$$

Orbits of $\mathcal{G}$: the members of this partition.
Examples: orbits of group actions, leaves of foliations, symplectic leaves, etc.

Problem: linear local form for $\mathcal{G}$ around an orbit $\mathcal{O}$.

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The local model: a groupoid $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ over $\mathcal{N}_{\mathcal{O}}$

Remark 1: we look not only at $\mathcal{O}$, but also at the induced:

$$\mathcal{G}_{\mathcal{O}} = \{g \in \mathcal{G} : s(g), t(g) \in \mathcal{O}\}.$$

Remark 2: $T\mathcal{G}$ is a groupoid over TM:

$$T\mathcal{G} \underset{dt}{\overset{ds}{\rightrightarrows}} TM. \tag{1}$$

Conclusion: $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ is the normal bundle of $\mathcal{G}_{\mathcal{O}}$ in $\mathcal{G}$.

$$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) := T\mathcal{G}/T\mathcal{G}_{\mathcal{O}} \underset{dt}{\overset{ds}{\rightrightarrows}} \mathcal{N}_{\mathcal{O}} := TM/T\mathcal{O} \tag{2}$$

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Choosing $x \in \mathcal{O}$, we have

- $G_x = s^{-1}(x) \cap t^{-1}(x)$- the isotropy group at x.
- $P_x = s^{-1}(x)$- a principal G_x-bundle over $\mathcal{O}$.
- A linear action of G_x on $\mathcal{N}_x = T_xM/T_x\mathcal{O}$.

Using these,

$$\mathcal{N}_{\mathcal{O}} \cong P_x \times_{G_x} \mathcal{N}_x$$

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Recall: a continuous map $f : X \longrightarrow Y$ is called proper if: for any compact $K \subset Y$, $f^{-1}(K)$ is compact.
It is called proper at $y \in Y$ if any sequence (x_n) with $f(x_n) \to y$ has a convergent sub-sequence.

Definition

Given a Lie groupoid $\mathcal{G}$ over M, $x \in M$, we say that

- $\mathcal{G}$ is s-proper (at x) if $s : \mathcal{G} \longrightarrow M$ is proper (at x).
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Theorem

$\mathcal{G}$ is a Lie groupoid over M, $\mathcal{O}$-the orbit through $x \in M$.If $\mathcal{G}$ is proper at x, then $\mathcal{G}$ is linearizable at x, i.e. there exists neighborhoods U and V of $\mathcal{O}$ in M and $\mathcal{N}_{\mathcal{O}}$, such that

$$\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})|_V .$$

Corollary

If $\mathcal{G}$ is s-proper at x, then $\mathcal{G}$ is inv-linearizable at x, i.e. there exists an invariant neighbrohood U of $\mathcal{O}$ in M such that

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Morita equivalence: a well-behaved notion of “isomorphism” in the world of groupoids, which reflects “the transversal geometry”. Given groupoids $\mathcal{G}$ over M and $\mathcal{H}$ over N, a Morita equivalence between them is given by a principal $\mathcal{G}$-$\mathcal{H}$ bibundle P

$$M \leftarrow P \rightarrow N.$$

Say that $x \in M$ and $y \in N$ are P-related if there is $p \in P$ mapping into them.

Proposition

If x and y are P-related, then

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Morita equivalence: a well-behaved notion of “isomorphism” in the world of groupoids, which reflects “the transversal geometry”. Given groupoids $\mathcal{G}$ over M and $\mathcal{H}$ over N, a Morita equivalence between them is given by a principal $\mathcal{G}$-$\mathcal{H}$ bibundle P

$$M \leftarrow P \rightarrow N.$$

Say that $x \in M$ and $y \in N$ are P-related if there is $p \in P$ mapping into them.

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If x and y are P-related, then

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Slices

Given $\mathcal{G}$ over M, $x \in M$, a **slice at** x is any embedded submanifold $\Sigma \subset M$ s.t.:

- Σ is transversal to every orbit that it meets.
- Σ is of dimension complementary to the dimension of $\mathcal{O}_x$ and $\Sigma \cap \mathcal{O}_x = \{x\}$.

Remark: properness at x implies:

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Using slices

Given $\mathcal{G}$ over M, $x \in M$ and a slice Σ through x:

- $\mathcal{G}|_\Sigma$ is is a Lie groupoid over Σ, which has x as a fixed point.
- The saturation $U \subset M$ of Σ is open, and $\mathcal{G}|_U$ is Morita equivalent to $\mathcal{G}|_\Sigma$.

Hence: the linearization theorem for $\mathcal{G}$ at x is equivalent to the one for $\mathcal{G}|_\Sigma$ at x.

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Hence we may assume that $\mathcal{O} = \{x\}$ (fixed point). The idea: construct a family $\{\mathcal{G}_\epsilon\}$ of groupoids with $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_0 =$ the local model; then use (flows) of multiplicative vector fields to relate the different $\mathcal{G}_\epsilon$.

Small step: by passing from M to a neighborhood of x, we may furthermore assume that:

- $\mathcal{G}$ is proper.
- $M = \mathbb{R}^n$.
- $\mathcal{G}$ sits openly inside $G_x \times \mathbb{R}^n$:

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Deforming $\mathcal{G}$ into its linearization

Deform $\mathcal{G}$ into the local model:

$$\mathcal{G}_{\varepsilon} = \{g \in E : \varepsilon g \in \mathcal{G}\} \subset E = G_x \times \mathbb{R}^n,$$

sitting over $\mathbb{R}^n$, with structure maps

$$s_{\varepsilon}(g) = s(g), t_{\varepsilon}(g) = \frac{1}{\varepsilon}t(\varepsilon g), m_{\varepsilon}(g,h) = \frac{1}{\varepsilon}m(\varepsilon g, \varepsilon h).$$

Useful: put all of these into a (proper!) Lie groupoid over $M \times \mathbb{R}$:

$$\tilde{\mathcal{G}} = \{(g,\varepsilon) \in E \times \mathbb{R} : \varepsilon g \in \mathcal{G}\},$$

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Roughly speaking, it is $\frac{d}{d\epsilon} m_{\epsilon}$.

Small problem: the domain of m_{ϵ} varries with respect to ϵ.
Small sollution: for a groupoid $\mathcal{H}$ over N, instead of using the multiplication map

$$m(g,h) = gh \ \text{ defined on } \mathcal{H}^{(2)} = \{(g,h) \in \mathcal{H} \times \mathcal{H} : s(g) = t(h)\},$$

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Consider $\bar{m}_\epsilon$ instead of m_ϵ; "the deformation cocycle" ξ_λ (at λ):

$$\mathcal{G}_\lambda^{[2]} \ni (p,q) \mapsto \xi_\lambda(p,q) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=\lambda}\bar{m}_\varepsilon(p,q) \in T_{\bar{m}_\lambda(p,q)}\mathcal{G}_\lambda.$$

The cocycle equation: $\frac{d}{d\epsilon}$ of the associativity equation for $\bar{m}_\epsilon$:

Lemma

For any $u,v,k \in \mathcal{G}_\lambda$ such that $(u,k),(v,k) \in \mathcal{G}_\lambda^{[2]}$,

$$(d\bar{m}_\lambda)(\xi_\lambda(u,k),\xi_\lambda(v,k)) = \xi_\lambda(u,v) - \xi_\lambda(\bar{m}_\lambda(u,k),\bar{m}_\lambda(v,k)).$$

(also important, but easier to handle: $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}t_\epsilon$).

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(also important, but easier to handle: $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}t_\epsilon$).

The deformation cocycle

Consider $\bar{m}_\epsilon$ instead of m_ϵ; "the deformation cocycle" ξ_λ (at λ):

$$\mathcal{G}_\lambda^{[2]} \ni (p,q) \mapsto \xi_\lambda(p,q) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=\lambda}\bar{m}_\varepsilon(p,q) \in T_{\bar{m}_\lambda(p,q)}\mathcal{G}_\lambda .$$

The cocycle equation: $\frac{d}{d\epsilon}$ of the associativity equation for $\bar{m}_\epsilon$:

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Using multiplicative vector field

Look for multiplicative vector fields $\tilde{X}$ on $\tilde{\mathcal{G}}$ with second component ∂_{ε}

$$\tilde{X}_{p,\lambda} = X^{\lambda}_{p} + \partial_{\lambda}$$

(each X^{λ} is a vector field on $\mathcal{G}_{\lambda}$!).

Lemma

$\tilde{X}$ is a multiplicative vector field if and only if(each X^{λ} is "compatible with s and u" and)

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Conclusion

Hence we have $\mathcal{G}_\lambda^{[2]} \ni (p,q) \mapsto \xi_\lambda(p,q) \in T_{\bar{m}_\lambda(p,q)}\mathcal{G}_\lambda$ satisfying the cocycle condition

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