## On the linearization of proper Lie groupoids

## Marius Crainic

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Figueira da Foz, June 2011, Poisson Geometry and Applications

Marius Crainic On the linearization of proper Lie groupoids

## Theorem (rough version)

If  $\mathcal{G}$  is a proper Lie groupoid over a manifold  $M, \mathcal{O} \subset M$  is an orbit of  $\mathcal{G}$  then, around  $\mathcal{O}, \mathcal{G}$  is isomorphic to its linearization  $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ .

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Starting point: Conn's linearization theorem

- conjectured by A. Weinstein (in his JDG paper)
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Conn's proof: based on "hard analysis", with no geometric insight ...

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The geometric approach to Conn's theorem

Weinstein's, based on his discovery of symplectic groupoids:

W1 Integrability.

## W2 Linearization of proper groupoids.

W3 Take care of the symplectic form.

Our geometric solution (M.C., Rui Loja Fernandes):

- CF1 : Integrability.
- CF2 : standard methods (Moser, averaging, etc), in the context of Lie groupoids instead of Lie groups.

This also allowed for generalizing Conn's theorem around arbitrary symplectic leaves (joint with I.Marcut).

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- 2002: Weinstein shows that the statement can be reduced to the case of fixed points.
- 2004: Zung proves it for fixed points (... Zung's theorem). Still uses analytic arguments, but not so "hard".
- 2011: with I. Struchiner:
  - indeed, Zung's theorem can be proved directly by Moser arguments, averaging.
  - Weinstein's reduction to the fixed point-case is just a manifastation of Morita invariance.
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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

Submanifolds

Setting:  $\mathcal{O}$  a submanifold of a manifold M.

The local (linear) model for M around  $\mathcal{O}$ : the normal bundle:

 $\mathcal{N}_{\mathcal{O}} = T_{\mathcal{O}}M/T\mathcal{O}.$ 

The linearization theorem:

Theorem (the standard tubular neighborhood theorem)

If  $\mathcal{O}$  is embedded then, around  $\mathcal{O}$ , M is diffeomorphic to  $\mathcal{N}_{\mathcal{O}}$ .

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Theorem (the standard local Reeb stability)

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The local (linear) model for *M* around  $\mathcal{O}$ : still  $\mathcal{N}_{\mathcal{O}}$  (with the induced "linear" foliation). Standard description:

$$\mathcal{N}_{\mathcal{O}} = \widetilde{\mathcal{O}} \times_{\Gamma_{\boldsymbol{X}}} \mathcal{N}_{\boldsymbol{X}}$$

(uses  $x \in \mathcal{O}$ , the universal cover  $\widetilde{\mathcal{O}}$  of  $\mathcal{O}$ , the fundamental group  $\Gamma_x = \pi(\mathcal{O}, x)$ , and the linear holonomy  $\rho : \Gamma_x \longrightarrow GL(\mathcal{N}_x)$ ).

### The linearization theorem:

Theorem (the standard local Reeb stability)

If  $\widetilde{\mathcal{O}}$  is compact then, around  $\mathcal{O}$ , M is diffeomorphic (as a foliated manifold) to  $\widetilde{\mathcal{O}} \times_{\Gamma_x} \mathcal{N}_x$ .

Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

Conn's theorem

Setting:  $\mathcal{O} = \{x\}$  a singular point of a Poisson manifold  $(M, \pi)$ .

The local (linear) model for *M* around *x*: still  $T_xM$ , endowed with the linearization of  $\pi$  at *x*. Equivalently: use the isotropy Lie algebra  $g_x$  at *x*, and

$$T_X M = (\mathfrak{g}_X)^*$$

with the linear Poisson structure.

The linearization theorem:

Theorem (Conn's theorem)

If  $\mathfrak{g}_x$  is semi-simple of compact type then, around x, M and  $\mathfrak{g}_x^*$  are Poisson diffeomorphic.

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

Local forms around symplectic leaves (with I.Marcut)

Setting:  $\mathcal{O}$  a symplectic leaf of a Poisson manifold  $(M, \pi)$ .

The local (linear) model for *M* around *x*: still  $N_0$ , endowed with "the linearization of  $\pi$  along O". Equivalently:

$$\mathcal{N}_{\mathsf{O}} = \mathcal{P}_{\mathsf{X}} imes_{\mathsf{G}_{\mathsf{X}}} \mathfrak{g}_{\mathsf{X}}^{*}$$

uses  $x \in \mathcal{O}$ , the Poisson homotopy group  $G_x$ , and the Poisson homotopy cover  $P_x \longrightarrow \mathcal{O}_x$ .

The linearization theorem:

Theorem (Crainic-Marcut, 2011)

If  $P_x$  is compact and 2-connected then, around  $\mathcal{O}$ , M and  $P_x \times_{G_x} \mathfrak{g}_x^*$  are Poisson diffeomorphic.

Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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Submanifolds Orbits of group actions Leaves of foliations Conn's theorem Symplectic leaves

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The setting The local model Using the isotropy data The hypothesis The statement Examples

A Lie groupoid



An induced partition of *M*:

$$x \sim y$$
 iff  $\exists g \in \mathcal{G}$  with  $s(g) = x, t(g) = y$ .

Orbits of G: the members of this partition. Examples: orbits of group actions, leaves of foliations, symplectic leaves, etc.

Problem: linear local form for  $\mathcal{G}$  around an orbit  $\mathcal{O}$ .

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting **The local model** Using the isotropy data The hypothesis The statement Examples

The local model: a groupoid  $\mathcal{N}_\mathcal{O}(\mathcal{G})$  over  $\mathcal{N}_\mathcal{O}$ 

Remark 1: we look not only at  $\mathcal{O}$ , but also at the induced:  $\mathcal{G}_{\mathcal{O}} = \{g \in \mathcal{G} : s(g), t(g) \in \mathcal{O}\}.$ 

Remark 2: TG is a groupoid over TM:

$$T\mathcal{G} \xrightarrow[dt]{ds} TM.$$
 (1)

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Conclusion:  $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$  is the normal bundle of  $\mathcal{G}_{\mathcal{O}}$  in  $\mathcal{G}$ .

$$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) := T\mathcal{G}/T\mathcal{G}_{\mathcal{O}} \xrightarrow{ds} \mathcal{N}_{\mathcal{O}} := TM/T\mathcal{O}$$
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The setting The local model Using the isotropy data The hypothesis The statement Examples

# Choosing $x \in \mathcal{O}$ , we have

- $G_x = s^{-1}(x) \cap t^{-1}(x)$  the isotropy group at x.
- $P_x = s^{-1}(x)$  a principal  $G_x$ -bundle over  $\mathcal{O}$ .
- A linear action of  $G_x$  on  $\mathcal{N}_x = T_x M / T_x \mathcal{O}$ .

Using these,

 $\mathcal{N}_{\mathcal{O}} \cong \mathcal{P}_{x} \times_{\mathcal{G}_{x}} \mathcal{N}_{x}$ 

and

$$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) \cong (P_x \times P_x) \times_{G_x} \mathcal{N}_x.$$

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3

The linearization of proper groupoids: some history Some linearization theorems (= linear normal forms) Linearization of proper Lie groupoids About the proof The local model Using the isotropy data The hypothesis The statement Examples

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The linearization of proper groupoids: some history
Some linearization theorems (= linear normal forms)
Linearization of proper Lie groupoids
About the proof
The seturing
The local model
Using the isotropy data
The hypothesis
The statement
Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

Recall: a continuous map  $f : X \longrightarrow Y$  is called proper if: for any compact  $K \subset Y$ ,  $f^{-1}(K)$  is compact.

It is called proper at  $y \in Y$  if any sequence  $(x_n)$  with  $f(x_n) \rightarrow y$  has a convergent sub-sequence.

### Definition

Given a Lie groupoid  $\mathcal{G}$  over M,  $x \in M$ , we say that

- $\blacksquare \mathcal{G} \text{ is } s \text{-proper (at } x) \text{ if } s : \mathcal{G} \longrightarrow M \text{ is proper (at } x).$
- G is proper (at x) if (s, t) :  $G \longrightarrow M \times M$  is proper (at (x, x)).

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The linearization of proper groupoids: some history
Some linearization of proper Lie groupoids
Linearization of proper Lie groupoids
About the proof
About the proof
Examples
In the setting
The local model
Using the isotropy data
The hypothesis
The statement
Examples

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イロト 不得 とくほと くほとう

The linearization of proper groupoids: some history
Some linearization theorems (= linear normal forms)
Linearization of proper Lie groupoids
About the proof
About the proof
Examples
The setting
The

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The linearization of proper groupoids: some history
Some linearization theorems (= linear normal forms)
Linearization of proper Lie groupoids
About the proof
About the proof
The setting
The setting
Using the isotropy data
The hypothesis
The statement
Examples

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 $\blacksquare \mathcal{G} \text{ is } s \text{-proper (at } x) \text{ if } s : \mathcal{G} \longrightarrow M \text{ is proper (at } x).$ 

■  $\mathcal{G}$  is proper (at x) if  $(s, t) : \mathcal{G} \longrightarrow M \times M$  is proper (at (x, x)).

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The linearization of proper groupoids: some history
Some linearization theorems (= linear normal forms)
Linearization of proper Lie groupoids
About the proof
About the proof
The setting
The setting
Using the isotropy data
The hypothesis
The statement
Examples

Recall: a continuous map  $f : X \longrightarrow Y$  is called proper if: for any compact  $K \subset Y$ ,  $f^{-1}(K)$  is compact.

It is called proper at  $y \in Y$  if any sequence  $(x_n)$  with  $f(x_n) \rightarrow y$  has a convergent sub-sequence.

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Linearization of proper Lie groupoids
About the proof
About the proof
The setting
The setting
Using the isotropy data
The hypothesis
The statement
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The setting The local model Using the isotropy data The hypothesis The statement Examples

#### Theorem

 $\mathcal{G}$  is a Lie groupoid over M,  $\mathcal{O}$ -the orbit through  $x \in M$ . If  $\mathcal{G}$  is proper at x, then  $\mathcal{G}$  is linearizable at x, i.e. there exists neighborhoods U and V of  $\mathcal{O}$  in M and  $\mathcal{N}_{\mathcal{O}}$ , such that

 $\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})|_V.$ 

#### Corollary

If G is s-proper at x, then G is inv-linearizable at x, i.e. there exists an invariant neighbrohood U of O in M such that

$$\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G}).$$

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The setting The local model Using the isotropy data The hypothesis **The statement** Examples

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Marius Crainic On the linearization of proper Lie groupoids

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis **The statement** Examples

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The setting The local model Using the isotropy data The hypothesis **The statement** Examples

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- **2** For a foliated manifold  $(M, \mathcal{F})$ , take  $\mathcal{G}$  = the foliated fundamental groupoid ... the local Reeb stability.
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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The setting The local model Using the isotropy data The hypothesis The statement Examples

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The linearization of proper groupoids: some history
Some linearization theorems (= linear normal forms)
Linearization of proper Lie groupoids
About the proof
About the proof
Examples
The setting
The setting
The local model
Using the isotropy data
The hypothesis
The statement
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The linearization of proper groupoids: some history Some linearization theorems (= linear normal forms) Linearization of proper Lie groupoids About the proof	s setting e local model ing the isotropy data e hypothesis e statement amples
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The linearization of proper groupoids: some history Some linearization theorems (= linear normal forms) Linearization of proper Lie groupoids About the proof	The setting The local model Using the isotropy data The hypothesis The statement <b>Examples</b>
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The linearization of proper groupoids: some history Some linearization theorems (= linear normal forms) Linearization of proper Lie groupoids About the proof	The setting The local model Using the isotropy data The hypothesis The statement <b>Examples</b>
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The linearization of proper groupoids: some history Some linearization theorems (= linear normal forms) Linearization of proper Lie groupoids About the proof	The setting The local model Using the isotropy data The hypothesis The statement <b>Examples</b>
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Reduction to the fixed point case The fixed point case

Morita equivalence: a well-behaved notion of "isomorphism" in the world of groupoids, which reflects "the transversal geometry". Given groupoids  $\mathcal{G}$  over M and  $\mathcal{H}$  over N, a Morita equivalence between them is given by a principal  $\mathcal{G}$ - $\mathcal{H}$  bibundle P

 $M \leftarrow P \rightarrow N.$ 

Say that  $x \in M$  and  $y \in N$  are *P*-related if there is  $p \in P$  mapping into them.

#### Proposition

- $\mathcal{G}$  is proper at x iff  $\mathcal{H}$  is proper at y.
- $\blacksquare$  *G* is linearizable at x iff *H* is linearizable at y.

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Slices

Given G overM,  $x \in M$ , a **slice at** x is any embedded submanifold  $\Sigma \subset M$  s.t.:

- $\square$   $\Sigma$  is transversal to every orbit that it meets.
- $\Sigma$  is of dimension complementary to the dimension of  $\mathcal{O}_x$ and  $\Sigma \cap \mathcal{O}_x = \{x\}$ .

Remark: properness at x implies:

- $\square$   $\mathcal{O}_{X}$  embedded submanifold.
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Given  $\mathcal{G}$  overM,  $x \in M$  and a slice  $\Sigma$  through x:

- $\mathcal{G}|_{\Sigma}$  is is a Lie groupoid over  $\Sigma$ , which has *x* as a fixed point.
- The saturation  $U \subset M$  of  $\Sigma$  is open, and  $\mathcal{G}|_U$  is Morita equivalent to  $\mathcal{G}|_{\Sigma}$ .

Hence: the linearization theorem for  $\mathcal{G}$  at x is equivalent to the one for  $\mathcal{G}|_{\Sigma}$  at x.

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Hence we may assume that  $\mathcal{O} = \{x\}$  (fixed point). The idea: construct a family  $\{\mathcal{G}_{\epsilon}\}$  of groupoids with  $\mathcal{G}_1 = \mathcal{G}$  and  $\mathcal{G}_0 =$  the local model; then use (flows) of multiplicative vector fields to relate the different  $\mathcal{G}_{\epsilon}$ .

Small step: by passing from M to a neighborhood of x, we may furthermore assume that:

- $\blacksquare \mathcal{G}$  is proper.
- $\blacksquare M = \mathbb{R}^n.$

**G** sits openly inside  $G_X \times \mathbb{R}^n$ :

$$\mathcal{G} \hookrightarrow \boldsymbol{E} := \boldsymbol{G}_{\boldsymbol{X}} \times \mathbb{R}^n,$$

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Reduction to the fixed point case The fixed point case

Deforming  $\mathcal{G}$  into its linearization

Deform  $\mathcal{G}$  into the local model:

$$\mathcal{G}_{\varepsilon} = \{ g \in E : \varepsilon g \in \mathcal{G} \} \subset E = G_x \times \mathbb{R}^n,$$

sitting over  $\mathbb{R}^n$ , with structure maps

$$s_{\varepsilon}(g) = s(g), t_{\varepsilon}(g) = rac{1}{\varepsilon}t(\varepsilon g), m_{\varepsilon}(g,h) = rac{1}{\varepsilon}m(\varepsilon g, \varepsilon h).$$

Useful: put all of these into a (proper!) Lie groupoid over  $M \times \mathbb{R}$ :  $\tilde{\mathcal{G}} = \{(g, \varepsilon) \in E \times \mathbb{R} : \varepsilon g \in \mathcal{G}\},\$ 

with source, target, multiplication and inversion maps

 $\sigma(\boldsymbol{g},\varepsilon) = (\boldsymbol{s}_{\varepsilon}(\boldsymbol{g}),\varepsilon), \ \ \mu((\boldsymbol{g},\varepsilon),(\boldsymbol{h},\varepsilon)) = (\boldsymbol{m}_{\varepsilon}(\boldsymbol{g},\boldsymbol{h}),\varepsilon), \ \ \text{etc.}$ 

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Reduction to the fixed point case The fixed point case

The deformation cocycle

Roughly speaking, it is  $\frac{d}{d\epsilon}m_{\epsilon}$ .

Small problem: the domain of  $m_{\epsilon}$  varries with respect to  $\epsilon$ . Small sollution: for a groupoid  $\mathcal{H}$  over N, instead of using the multiplication map

m(g,h) = gh defined on  $\mathcal{H}^{(2)} = \{(g,h) \in \mathcal{H} \times \mathcal{H} : s(g) = t(h)\},$ use

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Note: the associativity for *m* translates into:

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Consider  $ar{m}_\epsilon$  instead of  $m_\epsilon$ ; "the deformation cocycle"  $\xi_\lambda$  (at  $\lambda$ ):

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i ({\pmb p},{\pmb q}) \mapsto \xi_\lambda({\pmb p},{\pmb q}) := rac{\mathrm{d}}{\mathrm{d}arepsilon}|_{arepsilon=\lambda} ar{m}_arepsilon({\pmb p},{\pmb q}) \in T_{ar{m}_\lambda({\pmb p},{\pmb q})} {\mathcal G}_\lambda.$$

The cocycle equation:  $\frac{d}{d\epsilon}$  of the associativity equation for  $\bar{m}_{\epsilon}$ :

#### Lemma

For any  $u, v, k \in \mathcal{G}_{\lambda}$  such that  $(u, k), (v, k) \in \mathcal{G}_{\lambda}^{[2]}$ ,

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Reduction to the fixed point case The fixed point case

Using multiplicative vector field

Look for multiplicative vector fields  $\tilde{X}$  on  $\tilde{\mathcal{G}}$  with second component  $\partial_{\varepsilon}$ 

 $\tilde{X}_{p,\lambda} = X_p^{\lambda} + \partial_{\lambda}$ 

(each  $X^{\lambda}$  is a vector field on  $\mathcal{G}_{\lambda}$ !).

## Lemma

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#### Conclusion

Hence we have  $\mathcal{G}_{\lambda}^{[2]} \ni (p,q) \mapsto \xi_{\lambda}(p,q) \in T_{\overline{m}_{\lambda}(p,q)}\mathcal{G}_{\lambda}$  satisfying the cocycle condition

 $(d\bar{m}_{\lambda})(\xi_{\lambda}(u,k),\xi_{\lambda}(v,k)) = \xi_{\lambda}(u,v) - \xi_{\lambda}(\bar{m}_{\lambda}(u,k),\bar{m}_{\lambda}(v,k))$ 

and we are looking for  $\mathcal{G}_{\lambda} \ni p \mapsto X^{\lambda}(p) \in T_{p}\mathcal{G}_{\lambda}$  satisfying  $(d\bar{m}_{\lambda})_{p,q}(X_{p}^{\lambda}, X_{q}^{\lambda}) = X_{\bar{m}_{\lambda}(p,q)}^{\lambda} - \xi_{\lambda}(p,q).$ 

... an this always has solution: use a Haar system and set:

$$X_{p}^{\lambda} = \int_{s(p)}^{\lambda} \xi_{\lambda}(m_{\lambda}(p,q),q) dq \in T_{p}\mathcal{G}_{\lambda}.$$

Reduction to the fixed point case The fixed point case

Conclusion

Hence we have  $\mathcal{G}_{\lambda}^{[2]} \ni (p,q) \mapsto \xi_{\lambda}(p,q) \in T_{\bar{m}_{\lambda}(p,q)}\mathcal{G}_{\lambda}$  satisfying the cocycle condition

 $(d\bar{m}_{\lambda})(\xi_{\lambda}(u,k),\xi_{\lambda}(v,k)) = \xi_{\lambda}(u,v) - \xi_{\lambda}(\bar{m}_{\lambda}(u,k),\bar{m}_{\lambda}(v,k))$ 

and we are looking for  $\mathcal{G}_{\lambda} \ni p \mapsto X^{\lambda}(p) \in T_{p}\mathcal{G}_{\lambda}$  satisfying  $(d\bar{m}_{\lambda})_{p,q}(X_{p}^{\lambda}, X_{q}^{\lambda}) = X_{\bar{m}_{\lambda}(p,q)}^{\lambda} - \xi_{\lambda}(p,q).$ 

... an this always has solution: use a Haar system and set:

$$X_{p}^{\lambda} = \int_{s(p)}^{\lambda} \xi_{\lambda}(m_{\lambda}(p,q),q) dq \in T_{p}\mathcal{G}_{\lambda}.$$

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Marius Crainic On the linearization of proper Lie groupoids

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