On the Nature of Transverse Poisson Structures to a Coadjoint Orbit

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Talk Structure

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   - Linear Poisson structures
   - Linearization theorems
   - Polynomial Poisson structures
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   - Polynomialization of TPS in $\mathfrak{e}(3)^*$

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Matrix of P in coordinates

$$\mathcal{P} = \left(\quad \{x_i, x_j\} \quad \right)$$

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- all these notions can be taken locally.

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All linear Poisson structures are of the form $(\mathfrak{g}^*, L)$.

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The dual space T_x^*M is therefore a Lie algebra, the Lie algebra associated to (M, P) at x.

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- the category (smooth or analytic) of the equivalence;
- on (M, P) itself.

Theorem (Conn, 1984, 1985)

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Definition

A Poisson structure (M,P) is polynomializable at a zero-rank point x, if it is (locally) equivalent to a polynomial Poisson structure.

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- are, typically, rational functions of linear coordinates on the affine subspace.

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Given $x \in (M, P)$ *with* $rank(P)_x = 2r$*, there exist:*

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  - $\omega_y(u, v) = \langle (P^\sharp)_y^{-1}(v), u \rangle$

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A2: T is built according to the following steps (Weinstein's construction, 1983):

Step 1 due to transversality of N, the decomposition[3]

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$$\begin{tikzcd}
T_y^*N \arrow[d, "\pi_y^*"'] & T_yN \\
T_y^*M \arrow[r, "P_y^\sharp"'] & T_yM \arrow[u, "\pi_y"']
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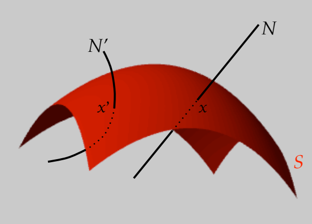

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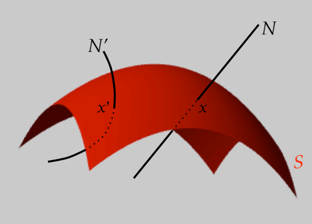

Any (N, T) as in the splitting theorem is known as a transverse Poisson structure to S.

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Following Weinstein's construction of T, and using the indicated identifications one arrives at:

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$$\begin{array}{rcl} T^\sharp_{\mu+\nu} : \mathfrak{g}_\mu & \longrightarrow & \mathfrak{g}^*_\mu \\ X & \longmapsto & \pi_{\mu+\nu}(ad^*_X \nu) \end{array}$$

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$$\mathcal{L} = \begin{pmatrix} . & -x_4 & -x_5 & x_2 & x_3 & . \\ x_4 & . & -x_6 & -x_1 & . & x_3 \\ x_5 & x_6 & . & . & -x_1 & -x_2 \\ -x_2 & x_1 & . & . & -x_6 & x_5 \\ -x_3 & . & x_1 & x_6 & . & -x_4 \\ . & -x_3 & x_2 & -x_5 & x_4 & . \end{pmatrix}$$

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$$N = \{(a, b, c + y_1, -c + y_2, b + y_3, -a + y_4) : y_i \in \mathbb{R}\}$$

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If $\mathfrak{h}$ is a complementary subspace of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ such that

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Consequently $T^\sharp_{\mu+\nu}(X) = \pi_{\mu+\nu}(ad_X^*\nu) = ad_X^*\nu$, and the result is linear[4]. □

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- $\mathfrak{g}_\mu^\perp$ stands for the orthogonal of $\mathfrak{g}_\mu$ with respect to B.

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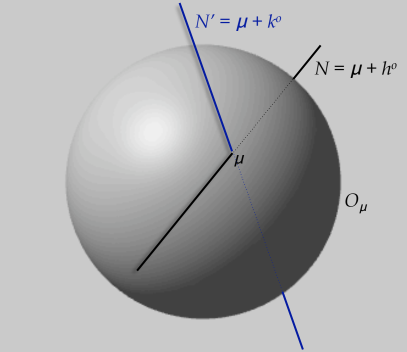
The right choice for $\mathfrak{h}$ (last example) would have been:

$$\mathfrak{h} = \langle c(X_1 - X_6) + a(X_4 - X_3), c(X_2 + X_5) + b(X_4 - X_3) \rangle$$

This shows that, at the same μ, changing $\mathfrak{h}$ can change the nature of the TPS.



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If μ and μ' belong to the same coadjoint orbit $\mathcal{O}$ of $(\mathfrak{g}^*, L)$ and if there is a linear TPS to $\mathcal{O}$ at μ, then there is also a linear TPS to $\mathcal{O}$ at μ'.

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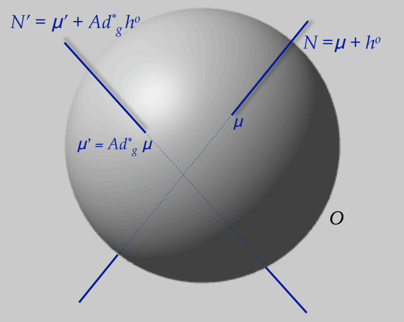

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Theorem (Cushman - Roberts, 2002)

If $\mathfrak{g}$ is semisimple then there is a polynomial TPS to any $\mathcal{O}_\mu$.

Example ($\mathfrak{e}(3)^*$)

Take $\mathfrak{g} = \mathfrak{e}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$.

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$$\mathcal{L} = \begin{pmatrix} . & x_3 & -x_2 & . & x_6 & -x_5 \\ -x_3 & . & x_1 & -x_6 & . & x_4 \\ x_2 & -x_1 & . & x_5 & -x_4 & . \\ . & x_6 & -x_5 & . & . & . \\ -x_6 & . & x_4 & . & . & . \\ x_5 & -x_4 & . & . & . & . \end{pmatrix}$$

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Again $\det(\mathcal{L}) = 0$, *and points of rank 4 (which exist) are regular.*

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$$N = \{(a, b, c + y_1, y_2, y_3, y_4) : y_i \in \mathbb{R}\}$$

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$$\mathcal{T} = \begin{pmatrix} 0 & \frac{c(cy_3 - by_4 + y_1 y_3)}{c+y_1} & \frac{c(ay_4 - cy_2 - y_1 y_2)}{c+y_1} & \frac{c(by_2 - ay_3)}{c+y_1} \\ * & 0 & -\frac{y_4^2}{c+y_1} & \frac{y_3 y_4}{c+y_1} \\ * & * & 0 & -\frac{y_2 y_4}{c+y_1} \\ * & * & * & 0 \end{pmatrix}$$

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The linear approximation at μ is:

$$\mathcal{T}^{(1)} = \begin{pmatrix} 0 & cy_3 - by_4 & ay_4 - cy_2 & by_2 - ay_3 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

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- regarding polynomiality, the results of Y. Oh and Cushman & Roberts do not apply;
- there are additional problems as $T^{(2)}, T^{(3)}, T^{(4)}, \ldots$ are not Poisson unless $a = b = 0$.

Example

In view of the last item we will consider $\mu = (0,0,1,0,0,0)$, which gives (on same N)

$$\mathcal{T} = \begin{pmatrix} 0 & y_3 & -y_2 & 0 \\ * & 0 & -\frac{y_4^2}{1+y_1} & \frac{y_3 y_4}{1+y_1} \\ * & * & 0 & -\frac{y_2 y_4}{1+y_1} \\ * & * & * & 0 \end{pmatrix}$$

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By parametrizing all possible complements $\mathfrak{h}$ *(9 parameters required) we proved that any polynomial TPS would have to be linear and hence does not exist.*

Example

Although there is no polynomial TPS to $\mathcal{O}_\mu$ in $\mathfrak{e}(3)^*$, the diffeomorphism

$$\varphi(y_1, y_2, y_3, y_4) = \left(1 - \frac{1}{1+y_1}, y_2(1+y_1), y_3(1+y_1), y_4(1+y_1)\right)$$

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$$\mathcal{P} = \begin{pmatrix} 0 & z_3(1-z_1)^2 & -z_2(1-z_1)^2 & 0 \\ * & 0 & -(1-z_1)\left(z_2^2+z_3^2+z_4^2\right) & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

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showing that the TPS to $\mathcal{O}_\mu$ is polynomializable (to degree 3).

References I

- A. Weinstein. The local structure of Poisson manifolds. *J. Diff. Geom.*, **18** (1983), 523–557.
- J. F. Conn. Normal forms for analytic Poisson structures. *Ann. of Maths.*, **119** (1984), 577–601.
- J. F. Conn. Normal forms for smooth Poisson structures. *Ann. of Maths.*, **121** (1985), 565–593.
- A. Weinstein. Errata and addenda. *J. Diff. Geom.*, **22** (1985), 255.
- P. Molino. Structure transverse aux orbites de la représentation coadjointe: le cas des orbites réductives. *Sémin. Géom. Différ., Univ. Sci. Tech. Languedoc, 1983-1984* (1984), 55–62.

References II

- Y.-G. Oh. Some remarks on the transverse Poisson structures of coadjoint orbits, *Lett. Math. Phys.*, **12** (1986), 87-91.
- J. P. Dufour. Linéarisation de certaines structures de Poisson. *J. Diff. Geom.*, **32** (1990), 415–428.
- Damianou, P.A. Transverse Poisson structures of coadjoint orbits, *Bull. Sci. Math.*, **120** (1996), 525–534.
- R. Cushman & M. Roberts. Poisson structures transverse to coadjoint orbits. *Bull. Sci. Math.*, **126** (7) (2002), 525–534.
- J. P. Dufour & N. T. Zung. Nondegeneraxy of the Lie algebra $\mathfrak{aff}(n)$. *C. R. Math. Acad. Sci.Paris*, **335** (12) (2002), 1043–1046.

References III

- I. Cruz & T. Fardilha. Linearity of the transverse Poisson structure to a coadjoint orbit, *Lett. Math. Phys.*, **65** (2003), 213-227.
- I. Cruz & T. Fardilha. On sufficient and necessary conditions for linearity of the transverse Poisson structure, *J. Geom. Phys.*, **60** (2010), 543-551.