On logarithmic Poisson cohomology and applications.

Dongho Joseph
University of Maroua, Cameroon

Abstract

We introduce the logarithmic Poisson structures and background, on a commutative ring $\mathcal{A}$ with singularities along an ideal $\mathcal{I}$ of $\mathcal{A}$, and we prove that such structure generalized Poisson structure induced by logsymplectic. We also prove that each logarithmic principal Poisson structure along $\mathcal{I}$ induce Lie-Rinehart structure on $\Omega_{\mathcal{A}}(\log \mathcal{I})$ with image in the module of logarithmic principal derivations. We define the notion of logarithmic Poisson cohomology and used it to prove Vaisman condition of prequantization of such Poisson structures.

1. Basic Definitions

In this section we will recall known structures such as Lie-Rinehart algebra structures on the module $\Omega_{\mathcal{A}}$ of Kälher differentials of a commutative Poisson Poisson algebra $\mathcal{A}$. Follows E. Okassa [1] and J. Huebshmann [2], we define Poisson cohomology of a Poisson algebra $(\mathcal{A}, \{-,-\})$. We also recall the notion of Kälher logarithmic differentials and logarithmic differential.

Let $(\mathcal{A}, \{-,-\})$ be a Poisson algebra and $\mathcal{I}$ an ideal of $\mathcal{A}$ generated by $S = \{u_1, ..., u_p\} \subset \mathcal{A}$ and L an $\mathcal{A}$-module which is also a Lie algebra with Lie bracket $[-,-]$. A structure of Lie-Rinehart algebra on L is an Lie algebra homomorphism $\rho : L \to Der_{\mathcal{A}}$ satisfying compatibly condition.

$$[l_1, al_2] = \rho(l_1)(a)l_2 + a[l_1, l_2] \tag{1}$$

More generally, let P be an $\mathcal{A}$-algebra and L a P-module of Lie. A structure of P-Lie-Rinehart algebra on L is a Lie algebra homomorphism $\rho : L \to \mathrm{Diff}_1(P, P)$ satisfying condition (1); where $\mathrm{Diff}_1(P, P)$ denoted the module of first order differential operators on P.

Let $H : \Omega_{\mathcal{A}} \to Der_{\mathcal{A}}$ be the Hamiltonian map associated to $\{-,-\}$ and ω the Poisson 2-form$\{-,-\}$. For all $a, b \in \mathcal{A}$, we define

$$[da, db] = d\{a, b\} \tag{2}$$

The following is well known.

THEOREM 1.1. *If $(\mathcal{A}, \{-,-\})$ is commutative Poisson algebra, then:*

a) $(\Omega_{\mathcal{A}}, [-,-])$ is a Lie Algebra

b) $H : \Omega_{\mathcal{A}} \to Der_{\mathcal{A}}$ is a structure of Lie-Rinehart algebra on $\Omega_{\mathcal{A}}$.

From this result, we deduce the following.

DEFINITION 1.2. *A Poisson cohomology of $(\mathcal{A}, \{-,-\})$ is that associated to the representation H.*

Observation. Since $\{-,-\}$ satisfy Jacobi identity, $[\omega, \omega]_{SN} = 0$ and then $\triangle^2 = 0$ where $[-,-]_{SN}$ is the Schouten-Nijenhuis bracket, $\triangle = i_\omega d - di_\omega$. Therefore, for all $\alpha \in \Omega^p_{\dashv}, \beta \in \bigoplus_{i \geqq 1} \Omega^i$, we have

$$[\alpha, \beta]_\triangle = [\alpha, \beta]_\omega = (-1)^p(\triangle(\alpha\beta) - \triangle(\alpha)\beta - (-1)^p\alpha \triangle (\beta))$$

(the bracket induced by the 2-order differential operator $\triangle$) satisfy the Jacobian identity (see [3]). The bracket defined by (2) is then $[\alpha, \beta]_\triangle$.
For all $adu, bdv \in \Omega_{\mathcal{A}}$, we have:

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd(\{u, v\}) \tag{3}$$

2. Logarithmic Poisson structures and first properties

DEFINITION 2.1. *A derivation D of $\mathcal{A}$ is saying logarithmic along $\mathcal{I}$ if $D(\mathcal{I}) \subset \mathcal{I}$. We denoted $Der_{\mathcal{A}}(\log \mathcal{I})$*

By definition, $Der_{\mathcal{A}}(\log \mathcal{I})$ is a Lie sub-algebra of $Der_{\mathcal{A}}$ and for all $D \in Der_{\mathcal{A}}, u \in \mathcal{S}$ we have; $uD(u) \in \mathcal{A}$. It result that the set $\widehat{Der_{\mathcal{A}}(\log \mathcal{I})}$ of logarithmic derivations δ such that for all $u \in \mathcal{S}, \delta(u) \in u\mathcal{A}$ is not trivial. In this note, we will called logarithmic principal derivation along $\mathcal{I}$ each element of $\widehat{Der_{\mathcal{A}}(\log \mathcal{I})}$. It is well known that

$$Der_{\mathcal{A}} \xrightarrow[\cong]{\sigma} \mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A}) \tag{4}$$

We denote $\Omega_{\mathcal{A}}(\log \mathcal{I})$ the $\mathcal{A}$-module generated by $\{\frac{du_i}{u_1}, 1 \leqq i \leqq p\} \cup \Omega_{\mathcal{A}}$. By definition

$$\Omega_{\mathcal{A}}(\log \mathcal{I}) \cong (\mathcal{A} - \mathcal{A}[\mathcal{S}])\{\frac{du}{u}, u \in \mathcal{S}\} \oplus \Omega_{\mathcal{A}} \tag{5}$$

$\Omega_{\mathcal{A}}(\log \mathcal{I})$ is called the $\mathcal{A}$-module of Kälher logaririthmic differential along $\mathcal{I}$. By construction, it is submodule of

$$(\mathcal{I}^* \cup 1_{\mathcal{A}})^{-1}\Omega_{\mathcal{A}} \cong (\mathcal{I}^* \cup 1_{\mathcal{A}})^{-1}\mathcal{A} \otimes \Omega_{\mathcal{A}}. \tag{6}$$

which is the module of rational kälher differential with poles along $\mathcal{S}$.

Observation. Let $a_0 \in \mathcal{A} - (O_{\mathcal{A}} \cup \mathcal{S})$ and $u \in \mathcal{S}$. Since $\frac{1}{a_0} \notin \mathcal{S}^{-1}\mathcal{A}$, then: $\frac{da_0}{a_0} + \frac{du}{du} \in (\mathcal{I}^* \cup 1_{\mathcal{A}})^{-1}\mathcal{A} - \Omega_{\mathcal{A}}(\log \mathcal{I})$. It follow from definition of logarithmic forms giving in [4] which clarify and comp let the one giving in [5] that the submodule of elements α of $(\mathcal{I}^* \cup 1_{\mathcal{A}})^{-1}\Omega_{\mathcal{A}}$ such that there is $u \in \mathcal{S}, u\alpha \in [\mathcal{A} - \mathcal{I}]^{-1}\mathcal{A} \otimes \Omega_{\mathcal{A}}$ is the suitable module of Kälher logarithmic differentials along $\mathcal{I}$. In other hand, it follow from definition of $\widehat{Der_{\mathcal{A}}(\log \mathcal{I})}$ that for all $\delta \in \widehat{Der_{\mathcal{A}}(\log \mathcal{I})}, \frac{1}{u}\delta(u) \in \mathcal{A}$. Therefore, the following map

$$\hat{\sigma} : \widehat{Der_{\mathcal{A}}(\log \mathcal{I})} \to \mathcal{H}_{\mathcal{A}}om(\Omega_{\mathcal{A}}, \mathcal{A}), a\frac{du}{u} \mapsto \frac{a}{u}\sigma(\delta)(du) \tag{7}$$

is an $\mathcal{A}$-modules homomorphism.

PROPOSITION 2.2. *Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ as above. The map and $\hat{\sigma}$ is an isomorphism of $\mathcal{A}$-modules.*

Proof. In is easy calculation. See [4] for more explanation. □

Let us introduce the definition of the main structure of this section.

DEFINITION 2.3. *A Poisson structure $\{-,-\}$ on $\mathcal{A}$ is logarithmic along $\mathcal{A}$ if for all $a \in \mathcal{A}$, the map*

$$\delta_a : x \mapsto \{a, x\}$$

is element of $Der_{\mathcal{A}}(\log \mathcal{I})$.
It is saying logarithmic principal if $\delta_a \in \widehat{Der_{\mathcal{A}}(\log \mathcal{I})}$.

THEOREM 2.4. *if $\{-,-\}$ is logarithmic principal Poisson structure along $\mathcal{I}$ on an integral algebra $\mathcal{A}$, then for all $u, v \in \mathcal{S}, \frac{1}{uv}\{u, v\} \in \mathcal{A}$.*

Proof. According to above definition of logarithmic principal derivation, for all $a \in \mathcal{A}$ and $u \in \mathcal{S}$, there is $\varphi_1(a) \in \mathcal{A}$ such that $\{a, u\} = u\varphi_1(a)$. Therefore, $u\varphi_1(v) = \{u, v\} = v\varphi_2(u)$. Then there is $a_2 \in \mathcal{A}$ such that $\varphi_2(u) = va_2$. Therefore $\frac{1}{uv}\{u, v\} = a_2$. □

When $\{-,-\}$ is a logarithmic principal Poisson structure along $\mathcal{I}$, then $(\mathcal{A}, \mathcal{I}, \{-,-\})$ is called logarithmic principal

COROLLARY 2.5. *Let $\{-,-\}$ be a logarithmic principal Poisson structure on $\mathcal{A}$ and H the associated Hamiltonian map. $H(\Omega_{\mathcal{A}}) \subset \widehat{Der_{\mathcal{A}}(\log \mathcal{I})}$ and for and for all $u \in \mathcal{S}, \frac{1}{u}H(du) = \frac{1}{u}\{u, -\} \subset \widehat{Der_{\mathcal{A}}(\log \mathcal{I})})$.*

We define $\tilde{H} : \Omega_{\mathcal{A}}(\log \mathcal{I}) \to \widehat{Der_{\mathcal{A}}(\log \mathcal{I})})$ by

$$\tilde{H}(a\frac{du}{u} + bdv) = \frac{a}{u}\{u, -\} + b\{v, -\}$$

DEFINITION 2.6. *$\tilde{H}$ is called logarithmic Hamiltonian map of logarithmic principal Poisson structure $\{-,-\}$.*

Observation. Let $(\mathcal{A}, \mathcal{I}, \{-,-\})$ be a logarithmic Poisson algebra; where $S = \{u^i; i \geqq 0.\}$. Denote by $M_S := S^{-1}\mathcal{A}$. It is well known that $\{a, \frac{b}{u}\}_s = \frac{1}{u}\{a, b\} - \frac{b}{u^2}\{a, u\}$ is the unique prolongation of $\{-,-\}$ on the fraction field of $\mathcal{A}$. By definition, elements of M_S are in the form $m = \frac{a}{u^n}; n \in \mathbb{Z}$. Let $m_p = \frac{a_p}{u_p^\lambda}$ and $m_q = \frac{a_q}{u_q^\lambda}$ two elements of M_S. We have: $\frac{dm_p}{m_p} = -\lambda_p\frac{du}{u} + \frac{da_p}{a_p} \in \Omega_{\mathcal{A}}(\log \mathcal{I})$. Since $\tilde{H}(\frac{dm_p}{m_p}) \in \widehat{Der_{\mathcal{A}}(\log \mathcal{I})})$, then we can computed the its image by $\hat{\sigma}$; which is element of the dual of $\Omega_{\mathcal{A}}(\log \mathcal{I})$. We defined

$$\{m_p, m_q\}_S = \begin{cases} \hat{\sigma}(\tilde{H}\frac{dm_p}{m_p})(\frac{dm_q}{m_q}) & \textbf{if} \quad m_i \in M_S - \mathcal{A} \\ \hat{\sigma}(Hdm_p)(\frac{dm_q}{m_q}) & \textbf{if} \quad m_q \in M_S - \mathcal{A} \quad \textbf{and} \quad m_p \in \mathcal{A} \\ \hat{\sigma}(Hdm_p)(dm_q) & \textbf{if} \quad m_i \in \mathcal{A} \end{cases} \tag{8}$$

We have:

PROPOSITION 2.7. *The bracket $\{-,-\}_S$ satisfy the following*

1) $\{-,-\}_D$ is R-bilinear skew-symmetric.

2)

$$\{m_p, m_q\}_S = \begin{cases} \frac{1}{m_p m_q}\{m_p, m_q\}_s & \textit{if} \quad m_i \in M_S - \mathcal{A} \\ \frac{1}{m_q}\{m_p, m_q\}_s & \textit{if} \quad m_q \in M_S - \mathcal{A} \quad \textit{and} \quad m_p \in \mathcal{A} \\ \{m_p, m_q\} & \textit{if} \quad m_i \in \mathcal{A} \end{cases} \tag{9}$$

3) $\{-,-\}_S$ is a logarithmic derivation of $M_S - \mathcal{A}$ in each components

4) For all $m_p, m_q \in M_S - \mathcal{A}, \frac{1}{m_p m_q}\{m_p, m_q\}_s \in \mathcal{A}$

COROLLARY 2.8. *$\{-,-\}_S$ is a Lie bracket on M_S.*

Proof. In the case where **if**$m_i \in M_S - \mathcal{A}$, we have:

$$\begin{aligned} \{u, \{v, a\}_S\}_S &= \{u, \frac{1}{v}\{v, a\}_s\}_D \\ &= \frac{1}{uv}\{u, \{v, a\}_s\}_s - \frac{1}{uv^2}\{u, v\}_s\{v, a\}_s \end{aligned}$$

Therefore

$$\{u, \{v, a\}_S\}_S + \circlearrowleft = \frac{1}{uv}\{u, \{v, a\}_s\}_s - \frac{1}{uv^2}\{u, v\}_s\{v, a\}_s + \frac{1}{uv}\{v, \{a, u\}_s\}_s$$
$$- \frac{1}{u^2v}\{a, u\}_s\{v, u\}_s + \frac{1}{uv}\{a, \{u, v\}_s\}_s - \frac{1}{uv^2}\{u, v\}_s\{a, v\}_s - \frac{1}{u^2v}\{u, v\}_s\{a, u\}_s$$

With the same methods, we prove other cases. □

We suppose that $S = \{u_i, 1 \leqq i \leqq p\}$ and that $\mathcal{I}$ is generated by S. Consider the bracket $[-,-]_S$ defined by:
$[\frac{du}{u}, \frac{dv}{v}]_S = d(\frac{1}{uv}\{u, v\})$; $[da, \frac{du}{u}]_S = d(\frac{1}{u}\{a, u\})$ and $[da, bb]_S = d(\{a, b\})$ For all $u, v \in S$ and $a, b \in \mathcal{A} - S$

PROPOSITION 2.9. *For all $u, v \in S$ and $a, b \in \mathcal{A} - S$, the bracket $[-,-]_S$ is a Lie bracket on $\Omega_{\mathcal{A}}(\log \mathcal{I})$.*

3. Logarithmic Poisson cohomology Example of computation

In this section we will prove that logarithmic Hamiltonian map is a Lie-Rinehart structure on $\Omega_{\mathcal{A}}(\log \mathcal{I})$ and we compute some example of associate cohomology which is called logarithmic cohomology.
We begin the section with the following proposition.

PROPOSITION 3.1. *Let $\{-,-\}_S$ be a logarithmic principal Poisson bracket along $\mathcal{I}$ and $\tilde{H}$ the associated logarithmic Hamiltonian map. $\tilde{H}$ satisfy the following properties:*

a) It is a Lie algebra homomorphism

b) It satisfy equation (1) for all $l_1, l_2 \in \Omega_{\mathcal{A}}(\log \mathcal{I})$.

DEFINITION 3.2. *Logarithmic Poisson cohomology of the logarithmic principal Poisson structure $\{-,-\}$ is the cohomology associated to the representation $\tilde{H}$ of $\Omega_{\mathcal{A}}(\log \mathcal{I})$.*

We will denoted cohomology groups by H^*_{SP}.

PROPOSITION 3.3. *Let $\mathcal{A}$ be the algebra of polynomials in two variables $\mathbb{C}[X, Y]$ and $\mathcal{S} = \{X\}$. The bracket $\{X, Y\} = X$ is logarithmic principal Poisson bracket along the ideal generated by $\mathcal{S}$. Its logarithmic Poisson cohomology is $H^0_{SP} \cong \mathbb{C}$, $H^1_{SP} \cong \mathbb{C}$, $H^2_{SP} \cong 0_{\mathcal{A}}$.*

PROPOSITION 3.4. *Let $\mathcal{A}$ be the algebra of polynomials in three variables $\mathcal{A} = \mathbb{C}[x, y, z]$ The bracket $\{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz$ is logarithmic principal Poisson bracket along the ideal generated by $\mathcal{S} = \{xyz\}$. We have:*

$$H^3_{PS} \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x]$$

And the associated Poisson cohomology is:

$$\begin{gathered} H^3_P \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus \\ xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z] \end{gathered}$$

4. Application of logarithmic Poisson cohomology

In this section, we give a little application of logarithmic Poisson cohomology to prequantization. We suppose that X is finale complex dimensional manifold and D is a reduced divisor of X. As in [5], we denoted $\Omega_{X,p}(\log D)$ the sheaf of germs of logarithmic form with poles along D. We also suppose that D is defined bay holomorphic map h. By definition, the usual De Rham complex of X is sub complex of the logarithmic De Rham complex.

$$0 \xrightarrow{d} \mathcal{O}_X \xrightarrow{d} \bigwedge^1 \Omega_X(\log D) \xrightarrow{d} \cdots \quad \xrightarrow{d} \bigwedge^i \Omega_X(\log D) \xrightarrow{d} \bigwedge^i \Omega_X(\log D) \xrightarrow{d} \cdots \tag{10}$$

A Poisson structure on X is saying logarithmic along D if it is logarithmic along ideal of definition $\mathcal{I}$ of D. Therefore the associated Poisson tensor π is a section of $\bigwedge^2 Der_X(\log D)$ where $Der_{X,p}(\log D)$ is the sheaf of germs at $p \in D$ of logarithmic vector fields. If there is a logarithmic Poisson structure on X, then, the logarithmic Hamiltonian map induce a morphism in cohomology

$$H^*_{Dr-S} \to H^*_{PS}$$

Where H^*_{Dr-S} is the cohomology of the complex 10. Therefore, we have the following sequence of morphism

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X) \longrightarrow H^*_{Dr-S} \longrightarrow H^2_{PS} \tag{11}$$

DEFINITION 4.1. *A closed section of $\bigwedge^2 \Omega_X(\log D)$ is saying integral if its cohomology class is in image of the composite map*

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X) \longrightarrow H^2_{Dr-S}$$

A logarithmic Poisson cocycle is saying integral if its cohomology class is in the image of

THEOREM 4.2. *Suppose that the divisor D satisfy hypotheses of Theorem 2.9 in [5] and that Definition 1.2 of [5] is modify as in [4]. A closed element $\omega = \frac{dh}{h} \wedge res(\omega) + \eta$ of $\bigwedge^2 \Omega_X(\log D)$ is integral iff $res(\omega)$ is exact and there is an integral element $[\omega_0]$ of $H^2(X)$ such that $[\eta] = [\omega_0]$.*

We can state now the main result of this section.

THEOREM 4.3. *Let D be as in Theorem 2.9 of [5]. A logarithmic principal Poisson structure on X; defined by π is prequantizable iff there exist an integral logarithmic 2-form ω_0 and a logarithmic vector field δ such that*

$$\pi + \partial(\delta) = \tilde{H}(\omega_0).$$

This theorem is the logarithmic version of Vaisman integral Theorem see [6].

Aknowledgements. I would like to thank Vladimir Rubtsov for suggesting these topics and for always useful discussions about this work.

References

[1] E.Okassa *algèbres de Poisson*, acte de la troisième rencontre de Géométrie et Topologie d'Afrique Centrale.

[2] J. Huebschmann. *Poisson cohomology and quantization*, J. Reine. Angew. Math., vol. 408, pages 57-113, 1990

[3] J. Koszul, *Crochet de Schouten-Nijenhuis*, Société Mathématique de France, Astérisque, hors série, 1985, p.257-271.

[4] J. Dongho, *Logarithmic Poisson cohomology: example of calculation and application to prequantization*, arXiv:1012.4683v2[math.DG]

[5] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, Sec. IA, J.Fac.Sci. Univ. Tokyo (1980) 265–291.

[6] I. Vaisman. *On the geometric quantization of Poisson manifolds*. J. Math. Phys 32(1991), 3339-3345.