Poisson Reduction

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We want to show that, given a Poisson Lie group acting on a Poisson manifold in such a way that action preserves the Poisson structure, we can reduce this manifold to another Poisson manifold. It is well known that, given a Lie group G acting on a symplectic manifold M, the orbit space M/G has a reduced Poisson structure. The Marsden-Weinstein reduction [1] gives a description of the symplectic leaves on M/G in case of an action induced by an equivariant momentum map. By applying such a prescription, it is possible to reduce the phase space and obtain another symplectic manifold in which the symmetries are divided out. We want to generalize this procedure to the case of a Poisson Lie group G acting on a Poisson manifold M. Our main result is the definition of a foliation of the reduced space M/G that inherits a Poisson structure from M. Given a Poisson action $\Phi : G \times M \to M$ with a momentum map $\boldsymbol{\mu} : M \to G^*$, we define a G-invariant foliation $\mathcal{F}$ of M. The leaves are not Poisson manifolds, but considering the action of G on the space of the leaves, we can prove that the Poisson structure on M induces a Poisson structure on the orbit space $\mathcal{L}/G_{\mathcal{L}}$. This shows that we can reduce M to another Poisson manifold $\mathcal{L}/G_{\mathcal{L}}$ that we define as the **Poisson reduced space**.

Poisson manifold and Splitting Theorem

Recall some basic notions of Poisson manifolds which will be used in the following.
A **Poisson manifold** is a pair $(M, \{\cdot,\cdot\})$, where M is a manifold and $\{\cdot,\cdot\}$ is a bilinear operation on $C^\infty(M)$ such that $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra and $\{\cdot,\cdot\}$ is a derivation in each argument. The pair $(C^\infty(M), \{\cdot,\cdot\})$ is called **Poisson algebra** [2].
In terms of local coordinates (x_i) on M

$$\{f, g\}(m) = \pi^{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

where the tensor π^{ij} is called the **Poisson tensor** of M. The vector bundle map $\pi^\sharp : T^*M \to TM$ naturally associated to π is defined by

$$\pi(m)(\alpha_m, \beta_m) = \langle \alpha_m, \pi^\sharp(\beta_m)\rangle.$$

Intuitively, any Poisson manifold is obtained by gluing together symplectic manifolds. More precisely, the local structure of a Poisson manifold at $O \in M$ is described by the **Splitting Theorem** [3], [4]: on a Poisson manifold (M, π), any point $O \in M$ has a neighborhood with coordinates (q, p, y) centered at O, such that

$$\pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{i,j}\phi_{ij}(y)\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \quad \phi_{ij}(0) = 0.$$

For any point O of the Poisson manifold, if (q, p, y) are the normal coordinates, then the symplectic leaf through O is given locally by the equation $y = 0$. Hence, for any point $m \in M$, we have a symplectic leaf through it. Locally, this leaf has canonical coordinates (q, p), where the bracket is given by canonical symplectic relations. The term

$$\frac{1}{2}\sum_{i,j}\phi_{ij}(y)\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

is called the **transverse Poisson structure** of dimension l. Locally, the transverse structure is determined by the structure functions $\pi_{ij}(y) = \{y_i, y_j\}$ which vanishes at $y = 0$.

Poisson action and Momentum map

A **Poisson Lie group** (G, π) is a Lie group equipped with a Poisson structure π such that the multiplication $G \times G \to G$ is a Poisson map, where $G \times G$ is given the product Poisson structure. In the following we will consider G compact,connected and simply connected.

The action of (G, π_G) on (M, π) is called **Poisson action** if the map $\Phi : G \times M \to M$ is Poisson, i.e. preserves the Poisson structure, where $G \times M$ is given the product Poisson structure $\pi_G \oplus \pi$.
Let $\Phi : G \times M \to M$ be a Poisson action of a Poisson Lie group (G, π_G) on the Poisson manifold (M, π). Let G^* be the dual Poisson Lie group of G. For each $\xi \in \mathfrak{g}$, let θ_ξ be the left invariant 1-form on G^* with value ξ at e and $\xi_M \in \Lambda^1(M)$ the infinitesimal generator of the action.
Given the map $\alpha : \mathfrak{g} \to \Omega^1(M)$, the dual map of α defines a $\mathfrak{g}^*$-valued one-form on M by $\alpha_\xi = \alpha(\xi)$ that satisfies the Maurer-Cartan equation.

A **momentum map** [5] is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\xi_M = i_{\alpha_\xi}\pi^\sharp$$

where $\alpha_\xi = \boldsymbol{\mu}^*(\theta_\xi)$ is a $\mathfrak{g}^*$-valued one-form on M and $\boldsymbol{\mu}^*$ is the cotangent lift of $\boldsymbol{\mu}$.

The dressing action of G on G^* is a notable example of Poisson action, where the momentum mapping is given by the identity map. It's important to recall that the symplectic leaves of G (resp., G^*) are the connected components of the orbits of the dressing action of G^* (resp., G).
Recall that a momentum map is G-equivariant if and only if it is a Poisson map, i.e. $\boldsymbol{\mu}_*\pi = \pi_{G^*}$.

Poisson structure on orbit space

Given a Poisson Lie group G acting on a Poisson manifold M, if the orbit space is a smooth manifold, it still carries a Poisson structure such that the natural projection $\mathrm{pr} : M \to M/G$ is a Poisson mapping [6].
Consider a Poisson Lie group (G, π_G) acting freely on a Poisson manifold (M, π). Assume that this is a Poisson action induced by an equivariant momentum map $\boldsymbol{\mu} : M \to G^*$. From the Splitting Theorem we know that, on the symplectic leaves of the dual Poisson Lie group (G^*, π^*), the transverse Poisson structure $\pi^*_{ij}(y) = \{y_i, y_j\}$ vanishes. Since the orbits of the dressing action of G on G^* are the same as symplectic leaves of π^*, then the generic orbit $G \cdot x$ is a closed submanifold of G^* with local transversal coordinates y_i and the Poisson structure π^* has zero transversal structure on it. Moreover, the momentum map is a submersion, then the image of M is an open neighborhood of $G \cdot x$.
Let us define the set of functions $H_i \in C^\infty(M)$ as a pullback by $\boldsymbol{\mu}$ of the transversal coordinates y_i to the orbit on G^*

$$H_i := y_i \circ \boldsymbol{\mu}.$$

Since $\{H_i, H_j\}$ vanish on the preimage $\boldsymbol{\mu}^{-1}(G \cdot x)$ and using the fact that we can rewrite the one-form $\alpha_\xi = \boldsymbol{\mu}^*(\theta_\xi)$ in terms of dH_i, we get an expression of the infinitesimal generator of the action as a linear combination of Hamiltonian vector fields:

$$\xi_M = \pi^\sharp(\boldsymbol{\mu}^*(\alpha_\xi)) = \sum_i c_i(\xi)\{H_j, \cdot\}.$$

Using this observation, we can easily review the procedure of Poisson reduction [6]. In fact, let $f, g \in C^\infty(M)^G$, then $\xi_M[f] = \xi_M[g] = 0$ for any $\xi \in \mathfrak{g}$. Using the relation above, we get

$$\xi_M\left[\{f, g\}\right] = 0.$$

Hence $\{f, g\}$ is G-invariant and we can conclude that $C^\infty(M)^G$ is a Lie subalgebra in $C^\infty(M)$.

Poisson Reduction for Poisson Lie group actions

Consider the $\mathfrak{g}^*$-valued one-form α_ξ defined by $\boldsymbol{\mu}$ such that $\xi_M = i_{\alpha_\xi}\pi^\sharp$ for $\xi \in \mathfrak{g}$. The distribution $\{\alpha_\xi | \xi \in \mathfrak{g}\}$ defines a G-invariant foliation $\mathcal{F}$ on M. In fact, considering the distribution $\mathcal{D}_\alpha$ spanned by the vector fields in the kernel of the one-forms α_ξ, the foliation $\mathcal{F}$ associated to the distribution $\mathcal{D}_\alpha$ has the property that for each $m \in M$, with $\mathcal{L}$ passing through m,

$$T_m\mathcal{L} = \mathcal{D}_\alpha(m).$$

Since α_ξ is a linear combination of dH_i, it is obvious that $\mathcal{F}$ is given by the kernel of dH_i. Let $\mathcal{L} = \boldsymbol{\mu}^{-1}(x)$. The leaf $\mathcal{L}$ is not a Poisson submanifold but we can prove that, considering the action of G on the space of leaves, the quotient $\mathcal{L}/G_{\mathcal{L}}$ inherits a Poisson structure by M, where

$$G_{\mathcal{L}} = \{g \in G | g \cdot \mathcal{L} = \mathcal{L}\}$$

is the stabilizer of the action of G on $\mathcal{L}$.
Consider $x \in G^*$ a regular value of $\boldsymbol{\mu}$; the preimage $N = \boldsymbol{\mu}^{-1}(G \cdot x)$ of the symplectic orbit of G in G^* is a closed G-invariant submanifold of M. By definition, H_i are defined locally in a G-invariant open neighborhood U of $N = \boldsymbol{\mu}^{-1}(G \cdot x)$.
Let $\mathcal{I}$ be the ideal generated by H_i. We can show that $\mathcal{I}$ is well defined and is closed under Poisson bracket. Finally, notice the following identifications

$$C^\infty(\mathcal{L}/G_{\mathcal{L}}) = C^\infty(N/G) = (C^\infty(U)/\mathcal{I})^G.$$

Under the assumption of compactness of G we can also identify $(C^\infty(U)/\mathcal{I})^G$ with $(C^\infty(U)^G + \mathcal{I})/\mathcal{I}$ and show that the Poisson bracket of M induces a well defined Poisson bracket on $(C^\infty(U)^G + \mathcal{I})/\mathcal{I}$.
We can conclude that, given a free Poisson action of (G, π_G) on a Poisson manifold (M, π) with equivariant momentum map $\boldsymbol{\mu} : M \to G^*$, the orbit space $\mathcal{L}/G_{\mathcal{L}}$ has a Poisson structure induced by π.

References

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