Global action-angle variables for non-commutative integrable systems

Rui Loja Fernandes

Departamento de Matemática
Instituto Superior Técnico

Poisson Geometry and Applications - Figueira da Foz
Goals

- Understand non-commutative integrable systems on Poisson manifolds.
- Describe obstructions to the existence of global action-angle variables.
Goals

- Understand non-commutative integrable systems on Poisson manifolds.

- Describe obstructions to the existence of global action-angle variables.
Goals

- Understand non-commutative integrable systems on Poisson manifolds.
- Describe obstructions to the existence of global action-angle variables.
Goals

- Understand non-commutative integrable systems on Poisson manifolds.

- Describe obstructions to the existence of global action-angle variables.

Ongoing joint work with:

- Raquel Caseiro
- Rui Loja Fernandes
- Camille Laurent-Gengoux
- Pol Vanhaecke.
### Integrable systems: definition

**Definition**

An **integrable system** on a symplectic manifold \((M^{2n}, \omega)\) is a hamiltonian system \(X_h\) admitting a family of first integrals \(\{f_1, \ldots, f_n\}\) satisfying:

1. **involution**: \(\{f_i, f_j\} = 0\) for all \(i, j\);
2. **independence**: \(df_1 \wedge \cdots \wedge df_n \neq 0\).

This definition involves naturally the Poisson bracket, not the symplectic form: \(\Rightarrow\) Poisson manifolds.

Such a system can be integrated by quadratures. There are other examples of systems integrated by quadratures where the Poisson brackets do not commute: \(\Rightarrow\) non-commutative integrable systems.
An integrable system on a symplectic manifold \( (\mathcal{M}^{2n}, \omega) \) is a hamiltonian system \( X_h \) admitting a family of first integrals \( \{f_1, \ldots, f_n\} \) satisfying:

1. involution: \( \{f_i, f_j\} = 0 \) for all \( i, j \);

2. independence: \( df_1 \wedge \cdots \wedge df_n \neq 0 \).

This definition involves naturally the Poisson bracket, not the symplectic form: \( \Rightarrow \) Poisson manifolds.

Such a system can be integrated by quadratures. There are other examples of systems integrated by quadratures where the Poisson brackets do not commute: \( \Rightarrow \) non-commutative integrable systems.
An **integrable system** on a symplectic manifold \((M^{2n}, \omega)\) is a hamiltonian system \(X_h\) admitting a family of first integrals \(\{f_1, \ldots, f_n\}\) satisfying:

1. **involution:** \(\{f_i, f_j\} = 0\) for all \(i, j\);
2. **independence:** \(df_1 \wedge \cdots \wedge df_n \neq 0\).

- This definition involves naturally the Poisson bracket, not the symplectic form: \(\Rightarrow\) Poisson manifolds.
- Such a system can be integrated by quadratures. There are other examples of systems integrated by quadratures where the Poisson brackets do not commute: \(\Rightarrow\) non-commutative integrable systems.
Let $M = T^* T^n$ with canonical symplectic form:

$$\omega = \sum_{i=1}^{n} ds^i \wedge d\theta^i$$

- Any $h = h(s^1, \ldots, s^n)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^n)$.
- The angle variables $(\theta^1, \ldots, \theta^n)$ evolve linearly in time.

How far is an integrable system from this example?
Integrable systems: example (“normal form”)

Let $M = T^* \mathbb{T}^n$ with canonical symplectic form:

$$\omega = \sum_{i=1}^{n} ds^i \wedge d\theta^i$$

- Any $h = h(s^1, \ldots, s^n)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^n)$.
- The angle variables $(\theta^1, \ldots, \theta^n)$ evolve linearly in time.

How far is an integrable system from this example?
Integrable systems: example ("normal form")

Let $M = T^* \mathbb{T}^n$ with canonical symplectic form:

$$\omega = \sum_{i=1}^{n} ds^i \wedge d\theta^i$$

- Any $h = h(s^1, \ldots, s^n)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^n)$.
- The angle variables $(\theta^1, \ldots, \theta^n)$ evolve linearly in time.

How far is an integrable system from this example?
Integrable systems: existence of normal form

- **Arnold-Liouville Theorem**: under connecteness and compactness assumptions every integrable systems is locally of this form.

- **Duistermaat**: Obstructions to the existence of global action-angle variables can be described.
Integrable systems: existence of normal form

- **Arnold-Liouville Theorem**: under connecteness and compactness assumptions every integrable system is locally of this form.

- **Duistermaat**: Obstructions to the existence of global action-angle variables can be described.
Integrable systems: existence of normal form

- **Arnold-Liouville Theorem**: under connecteness and compactness assumptions every integrable systems is locally of this form.

- **Duistermaat**: Obstructions to the existence of global action-angle variables can be described.
Integrable systems: existence of normal form

- **Arnold-Liouville Theorem**: under connecteness and compactness assumptions every integrable systems is locally of this form.

- **Duistermaat**: Obstructions to the existence of global action-angle variables can be described.

The local problem, the global problem and their solutions are best described using groupoid language.
Integrable systems & fibrations

To every integrable system \((f_1, \ldots, f_n)\) on a symplectic manifold \((M^{2n}, \omega)\) there is associated a Lagrangian fibration:

\[
\phi : M^{2n} \to \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))
\]

**Proposition**

Conversely, every Lagrangian fibration

\[
\phi : (M^{2n}, \omega) \to B^n
\]

is locally of this form.

Notice that these are Poisson fibrations if we equip the base with the trivial bracket:

\[
\phi : (M^{2n}, \omega^{-1}) \to (B^n, \pi \equiv 0)
\]
To every integrable system \((f_1, \ldots, f_n)\) on a symplectic manifold \((M^{2n}, \omega)\) there is associated a Lagrangian fibration:

\[
\phi : M^{2n} \rightarrow \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))
\]

**Proposition**

Conversely, every Lagrangian fibration

\[
\phi : (M^{2n}, \omega) \rightarrow B^n
\]

is locally of this form.

Notice that these are Poisson fibrations if we equip the base with the trivial bracket:

\[
\phi : (M^{2n}, \omega^{-1}) \rightarrow (B^n, \pi \equiv 0)
\]
To every integrable system \((f_1, \ldots, f_n)\) on a symplectic manifold \((M^{2n}, \omega)\) there is associated a **Lagrangian fibration**:

\[
\phi : M^{2n} \to \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))
\]

**Proposition**

Conversely, every Lagrangian fibration

\[
\phi : (M^{2n}, \omega) \to B^n
\]

*is locally of this form.*

Notice that these are **Poisson fibrations** if we equip the base with the trivial bracket:

\[
\phi : (M^{2n}, \omega^{-1}) \to (B^n, \pi \equiv 0)
\]
Integrable systems & fibrations

To every integrable system \((f_1, \ldots, f_n)\) on a symplectic manifold \((M^{2n}, \omega)\) there is associated a Lagrangian fibration:

\[
\phi : M^{2n} \rightarrow \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))
\]

**Proposition**

Conversely, every Lagrangian fibration

\[
\phi : (M^{2n}, \omega) \rightarrow B^n
\]

is locally of this form.

Notice that these are Poisson fibrations if we equip the base with the trivial bracket:

\[
\phi : (M^{2n}, \omega^{-1}) \rightarrow (B^n, \pi \equiv 0)
\]
Every complete Poisson fibration $\phi : (M, \pi_M) \to (B, \pi_B)$ gives rise to:

- A Lie algebroid action of $T^*B$ on $\phi : M \to B$:
  $\alpha \mapsto \pi_M^\#(\phi^*\alpha)$.

- A symplectic groupoid action of $\Sigma(B)$ on $\phi : M \to B$. 
Every complete Poisson fibration $\phi : (M, \pi_M) \to (B, \pi_B)$ gives rise to:

- A Lie algebroid action of $T^*B$ on $\phi : M \to B$:
  \[
  \alpha \mapsto \pi_M^\#(\phi^*\alpha).
  \]

- A symplectic groupoid action of $\Sigma(B)$ on $\phi : M \to B$. 
Every \textbf{complete} Poisson fibration $\phi : (M, \pi_M) \to (B, \pi_B)$ gives rise to:

- A Lie algebroid action of $T^*B$ on $\phi : M \to B$:
  \[ \alpha \mapsto \pi^\#_M(\phi^* \alpha). \]

- A symplectic groupoid action of $\Sigma(B)$ on $\phi : M \to B$. 
Every **complete** Poisson fibration $\phi : (M, \pi_M) \to (B, \pi_B)$ gives rise to:

- A Lie algebroid action of $T^*B$ on $\phi : M \to B$:
  \[ \alpha \mapsto \pi_M^\sharp(\phi^*\alpha). \]

- A symplectic groupoid action of $\Sigma(B)$ on $\phi : M \to B$.

**Proposition**

For a Lagrangian fibration with compact connected fibers, the kernel of the symplectic action $T^*B \rightrightarrows B$ on the fibration $\phi : M \to B$ is a Lagrangian, full rank, lattice $\Lambda \subset T^*B$. 
Conclusion

For any Lagrangian fibration \( \phi : (M, \omega) \rightarrow B \) with compact, connected fibers:

(i) There exists a full rank, Lagrangian, lattice \( \Lambda \subset T^*B \);

(ii) \( T^*B/\Lambda \rightarrow B \) is a symplectic groupoid integrating \( (B, \pi_B = 0) \) which acts freely and properly in a symplectic manner in the fibration \( \phi : (M, \omega) \rightarrow B \).

These facts form the basis to understand the existence of both local and global normal forms of Lagrangian fibrations/integrable systems.
Conclusion

For any Lagrangian fibration \( \phi : (M, \omega) \rightarrow B \) with compact, connected fibers:

(i) There exists a full rank, Lagrangian, lattice \( \Lambda \subset T^*B \);

(ii) \( T^*B/\Lambda \Rightarrow B \) is a symplectic groupoid integrating \( (B, \pi_B = 0) \) which acts freely and properly in a symplectic manner in the fibration \( \phi : (M, \omega) \rightarrow B \).

These facts form the basis to understand the existence of both local and global normal forms of Lagrangian fibrations/integrable systems.
Conclusion

For any Lagrangian fibration $\phi : (M, \omega) \to B$ with compact, connected fibers:

(i) There exists a full rank, Lagrangian, lattice $\Lambda \subset T^* B$;

(ii) $T^* B / \Lambda \cong B$ is a symplectic groupoid integrating $(B, \pi_B = 0)$ which acts freely and properly in a symplectic manner in the fibration $\phi : (M, \omega) \to B$.

These facts form the basis to understand the existence of both local and global normal forms of Lagrangian fibrations/integrable systems.
For any Lagrangian fibration $\phi : (M, \omega) \to B$ with compact, connected fibers:

(i) There exists a full rank, Lagrangian, lattice $\Lambda \subset T^*B$;

(ii) $T^*B/\Lambda \to B$ is a symplectic groupoid integrating $(B, \pi_B = 0)$ which acts freely and properly in a symplectic manner in the fibration $\phi : (M, \omega) \to B$.

These facts form the basis to understand the existence of both local and global normal forms of Lagrangian fibrations/integrable systems.
Proposition

Given a Poisson action of a Poisson groupoid $\mathcal{G} \rightrightarrows B$ on a Poisson fibration $\phi : M \to B$, every (local) coisotropic section $\sigma : B \to M$ determines a (local) Poisson map:

$$\mathcal{G} \to M, \quad g \mapsto g \cdot \sigma(s(g)).$$

Proof. An exercise in coisotropic calculus!

Corollary (Arnol’d-Liouville Theorem)

Every Lagrangian fibration is locally isomorphic to $(T^*\mathbb{T}^n, \omega_{\text{can}}) \to \mathbb{R}^n$. 
**Proposition**

Given a Poisson action of a Poisson groupoid $\mathcal{G} \rightrightarrows B$ on a Poisson fibration $\phi : M \to B$, every (local) coisotropic section $\sigma : B \to M$ determines a (local) Poisson map:

$$\mathcal{G} \to M, \quad g \mapsto g \cdot \sigma(s(g)).$$

**Proof.** An exercise in coisotropic calculus!

**Corollary (Arnol’d-Liouville Theorem)**

Every Lagrangian fibration is locally isomorphic to $(\mathbb{T}^*\mathbb{T}^n, \omega_{can}) \to \mathbb{R}^n$. 

Rui Loja Fernandes

Global action-angle variables for non-commutative integrable systems
Proposition

Given a Poisson action of a Poisson groupoid $\mathcal{G} \rightrightarrows B$ on a Poisson fibration $\phi : M \to B$, every (local) coisotropic section $\sigma : B \to M$ determines a (local) Poisson map:

$$
\mathcal{G} \to M, \quad g \mapsto g \cdot \sigma(s(g)).
$$

Proof. An exercise in coisotropic calculus!

Corollary (Arnol’d-Liouville Theorem)

Every Lagrangian fibration is locally isomorphic to $(T^*\mathbb{T}^n, \omega_{can}) \to \mathbb{R}^n$. 
Obstructions to triviality of a Lagrangian fibration $\phi : M \rightarrow B$:

1. **Vanishing of Hamiltonian monodromy**: The holonomy of the cover $\Lambda \rightarrow B$ must be trivial (obstruction for the $\mathbb{T}^n$-fibration to be a principal $\mathbb{T}^n$-bundle).

2. **Vanishing of the Lagrangian Chern class**: the class $c(\phi) \in \check{H}(B; \Gamma_{\text{Lagr}}(T^*B/\Lambda))$ must be trivial (obstruction to existence of a global Lagrangian section).

$\Rightarrow$ (H. Duistermaat, 1983) obstructions to existence of global action-angle variables
Groupoid actions and global canonical forms

Obstructions to triviality of a Lagrangian fibration \( \phi : M \to B \):

1. **Vanishing of Hamiltonian monodromy**: The holonomy of the cover \( \Lambda \to B \) must be trivial (obstruction for the \( \mathbb{T}^n \)-fibration to be a principal \( \mathbb{T}^n \)-bundle).

2. **Vanishing of the Lagrangian Chern class**: the class \( c(\phi) \in \tilde{H}(B; \Gamma_{\text{Lagr}}(T^*B/\Lambda)) \) must be trivial (obstruction to existence of a global Lagrangian section).

\( \Rightarrow \) (H. Duistermaat, 1983) obstructions to existence of global action-angle variables
Obstructions to triviality of a Lagrangian fibration $\phi : M \to B$:

1. **Vanishing of Hamiltonian monodromy**: The holonomy of the cover $\Lambda \to B$ must be trivial (obstruction for the $\mathbb{T}^n$-fibration to be a principal $\mathbb{T}^n$-bundle).

2. **Vanishing of the Lagrangian Chern class**: the class $c(\phi) \in \tilde{H}(B; \Gamma_{\text{Lagr}}(T^*B/\Lambda))$ must be trivial (obstruction to existence of a global Lagrangian section).

$\Rightarrow$ (H. Duistermaat, 1983) obstructions to existence of global action-angle variables
Groupoid actions and global canonical forms

Obstructions to triviality of a Lagrangian fibration $\phi : M \to B$:

1. **Vanishing of Hamiltonian monodromy**: The holonomy of the cover $\Lambda \to B$ must be trivial (obstruction for the $\mathbb{T}^n$-fibration to be a principal $\mathbb{T}^n$-bundle).

2. **Vanishing of the Lagrangian Chern class**: the class $c(\phi) \in \check{H}(B; \Gamma_{\text{Lagr}}(T^* B/\Lambda))$ must be trivial (obstruction to existence of a global Lagrangian section).

$\Rightarrow$ (H. Duistermaat, 1983) obstructions to existence of global action-angle variables
We would like to implement this program for non-commutative integrable systems (relax the commutativity condition on the first integrals);

We would like to study singularities of integrable systems (relax the independence condition on the first integrals).
We would like to implement this program for non-commutative integrable systems (relax the commutativity condition on the first integrals);

We would like to study singularities of integrable systems (relax the independence condition on the first integrals).
Non-commutative integrable systems: definition

Definition

A non-commutative integrable system of rank $r$ on a Poisson manifold $(M^m, \pi)$ is a Hamiltonian system with Hamiltonian $X_h$ admitting a family of first integrals $\{f_1, \ldots, f_s\}$, $r + s = m$, satisfying:

1. involution: $\{f_i, f_j\} = 0$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$;
2. independence: $df_1 \wedge \cdots \wedge df_s \neq 0$.

We shall also assume the non-degeneracy condition:

- the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ are independent.

Note: When $r = s$ we obtain a classical integrable system on a symplectic manifold.
Definition

A non-commutative integrable system of rank \( r \) on a Poisson manifold \((M^m, \pi)\) is a hamiltonian system \(X_h\) admitting a family of first integrals \(\{f_1, \ldots, f_s\}\), \(r + s = m\), satisfying:

1. involution: \(\{f_i, f_j\} = 0\) for all \(1 \leq i \leq r\) and \(1 \leq j \leq s\);
2. independence: \(df_1 \wedge \cdots \wedge df_s \neq 0\).

We shall also assume the non-degeneracy condition:

- the hamiltonian vector fields \(X_{f_1}, \ldots, X_{f_r}\) are independent.

Note: When \(r = s\) we obtain a classical integrable system on a symplectic manifold.
Non-commutative integrable systems: definition

Definition

A non-commutative integrable system of rank \( r \) on a Poisson manifold \((M^m, \pi)\) is a Hamiltonian system \(X_h\) admitting a family of first integrals \( \{f_1, \ldots, f_s\} \), \( r + s = m \), satisfying:

1. involution: \( \{f_i, f_j\} = 0 \) for all \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \);
2. independence: \( df_1 \wedge \cdots \wedge df_s \neq 0 \).

We shall also assume the non-degeneracy condition:

- the Hamiltonian vector fields \( X_{f_1}, \ldots, X_{f_r} \) are independent.

Note: When \( r = s \) we obtain a classical integrable system on a symplectic manifold.
Non-commutative integrable systems: examples

- Non-commutative integrable systems are integrable by quadratures.

- Examples of non-commutative integrable systems include:
  - Natural mechanical systems such as the Kepler system and the Euler-Poinsot rigid body.
  - Classes of systems invariant under a hamiltonian group action (collective motions)

- Non-commutative integrable systems are examples of superintegrable systems (motion occurs in lower dimension tori).
Non-commutative integrable systems: examples

- Non-commutative integrable systems are integrable by quadratures.

- Examples of non-commutative integrable systems include:
  
  - Natural mechanical systems such as the Kepler system and the Euler-Poinsot rigid body.
  
  - Classes of systems invariant under a hamiltonian group action (collective motions)

- Non-commutative integrable systems are examples of superintegrable systems (motion occurs in lower dimension tori).
Non-commutative integrable systems: examples

- Non-commutative integrable systems are integrable by quadratures.

- Examples of non-commutative integrable systems include:
  - Natural mechanical systems such as the Kepler system and the Euler-Poinsot rigid body.
  - Classes of systems invariant under a Hamiltonian group action (collective motions)

- Non-commutative integrable systems are examples of superintegrable systems (motion occurs in lower dimension tori).
Let $M = T^*T^r \times \mathbb{R}^{s-r}$ with Poisson structure:

$$\omega = \sum_{i=1}^{r} \frac{\partial}{\partial s^i} \wedge \frac{\partial}{\partial \theta^i} + \sum_{j,k=1}^{s-r} \varphi_{j,k}^{(z)} \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k}$$

- Any $h = h(s^1, \ldots, s^r)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^r)$.
- The angle variables $(\theta^1, \ldots, \theta^r)$ evolve linearly in time.
Non-commutative integrable systems: example ("normal form")

Let $M = T^* \mathbb{T}^r \times \mathbb{R}^{s-r}$ with Poisson structure:

$$\omega = \sum_{i=1}^{r} \frac{\partial}{\partial s^i} \wedge \frac{\partial}{\partial \theta^i} + \sum_{j,k=1}^{s-r} \varphi^{jk}(z) \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k}$$

- Any $h = h(s^1, \ldots, s^r)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^r)$.
- The angle variables $(\theta^1, \ldots, \theta^r)$ evolve linearly in time.
Non-commutative integrable systems: example ("normal form")

Let $M = T^* T^r \times \mathbb{R}^{s-r}$ with Poisson structure:

$$
\omega = \sum_{i=1}^{r} \frac{\partial}{\partial s^i} \wedge \frac{\partial}{\partial \theta^i} + \sum_{j,k=1}^{s-r} \varphi^{jk}(z) \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k}
$$

- Any $h = h(s^1, \ldots, s'^r)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s'^r)$.
- The angle variables $(\theta^1, \ldots, \theta'^r)$ evolve linearly in time.
Non-commutative integrable systems: existence of normal form

- Local normal form: by the work of Laurent-Gengoux, Miranda and Vanhaecke, under connecteness and compactness assumptions, every non-commutative integrable system is locally of this form (Arnold-Liouville Theorem).

- What are the obstructions to the existence of global action-angle variables?
Local normal form: by the work of Laurent-Gengoux, Miranda and Vanhaecke, under connecteness and compactness assumptions, every non-commutative integrable system is locally of this form (Arnold-Liouville Theorem).

What are the obstructions to the existence of global action-angle variables?
Non-commutative integrable systems: existence of normal form

- Local normal form: by the work of Laurent-Gengoux, Miranda and Vanhaecke, under connecteness and compactness assumptions, every non-commutative integrable system is locally of this form (Arnold-Liouville Theorem).

- What are the obstructions to the existence of global action-angle variables?
Non-commutative integrable systems: existence of normal form

- Local normal form: by the work of Laurent-Gengoux, Miranda and Vanhaecke, under connecteness and compactness assumptions, every non-commutative integrable system is locally of this form (Arnold-Liouville Theorem).

- What are the obstructions to the existence of global action-angle variables?

**Note:** In a fundamental paper, Dazord and Delzant have study in detail the case where $M$ is symplectic.
To every non-commutative integrable system there is associated two Poisson fibrations:

\[(M, \pi_M) \xrightarrow{\phi} (\mathbb{R}^s, \pi_B)\]
\[(\mathbb{R}^r, 0) \xrightarrow{\psi} (f_1(x), \ldots, f_s(x))\]

\[x \xrightarrow{\phi} (f_1(x), \ldots, f_s(x))\]

The fibers of these fibrations are isotropic/coisotropic since they satisfy:

\[\ker d\phi = \pi_M^\sharp(\ker d\psi)^0.\]

**Note:** The choice of commuting functions \(f_1, \ldots, f_r\) may vary, so the large fibration is not fixed.
To every non-commutative integrable system there is associated two Poisson fibrations:

\[(\mathcal{M}, \pi_M) \xrightarrow{\phi} (\mathbb{R}^S, \pi_B) \xleftarrow{\psi} (\mathbb{R}^r, 0)\]

\[x \mapsto (f_1(x), \ldots, f_s(x)) \xleftarrow{\psi} (f_1(x), \ldots, f_r(x))\]

The fibers of these fibrations are isotropic/coisotropic since they satisfy:

\[\ker d\phi = \pi_M^\#(\ker d\psi)^0.\]

**Note:** The choice of commuting functions \(f_1, \ldots, f_r\) may vary, so the large fibration is not fixed.
To every non-commutative integrable system there is associated two Poisson fibrations:

\[ (\mathcal{M}, \pi_M) \xrightarrow{\phi} (\mathbb{R}^s, \pi_B) \]

\[ x \xrightarrow{\phi} (f_1(x), \ldots, f_s(x)) \]

\[ \psi \xrightarrow{\phi} (\mathbb{R}^r, 0) \]

\[ (f_1(x), \ldots, f_r(x)) \]

The fibers of these fibrations are isotropic/coisotropic since they satisfy:

\[ \text{Ker } d\phi = \pi_M^\#(\text{Ker } d\psi)^0. \]

**Note:** The choice of commuting functions \( f_1, \ldots, f_r \) may vary, so the large fibration is not fixed.
To every non-commutative integrable system there is associated two Poisson fibrations:

\[
(M, \pi_M) \xrightarrow{\phi} (\mathbb{R}^s, \pi_B) \quad \text{and} \quad x \xrightarrow{\phi} (f_1(x), \ldots, f_s(x))
\]

\[
\psi \quad \text{and} \quad \psi \quad \downarrow \quad \downarrow
\]

\[
(\mathbb{R}^r, 0) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (f_1(x), \ldots, f_r(x))
\]

The fibers of these fibrations are isotropic/coisotropic since they satisfy:

\[
\text{Ker } d\phi = \pi_M^\#(\text{Ker } d\psi)^0.
\]

**Note:** The choice of commuting functions \( f_1, \ldots, f_r \) may vary, so the large fibration is not fixed.
Definition of abstract non-commutative integrable systems

Definition

A fibration \( \phi : (M^m, \pi_M) \to (B^s, \pi_B) \) is called a non-degenerate isotropic fibration or an abstract non-commutative integrable system of rank \( r := m - s \) if there is a \( r \)-distribution \( D \subset TB \) such that:

\[
\pi^\# (\phi^*(D^0)) = \text{Ker} \, d\phi.
\]

Notes:

- When \((M, \pi_M)\) is symplectic, the distribution \(D\) is uniquely defined and the definition corresponds to Delzant and Dazord notion of symplectically complete isotropic fibrations.
- In the Poisson case, there can be several choices of \(D\). Notice that we always have \( \text{Im} \, \pi_B^\# \subset D \) (equivalently, \( D^0 \subset \text{Ker} \, \pi_B^\# \)).
Definition of abstract non-commutative integrable systems

**Definition**

A fibration \( \phi : (M^m, \pi_M) \to (B^s, \pi_B) \) is called a non-degenerate isotropic fibration or an abstract non-commutative integrable system of rank \( r := m - s \) if there is a \( r \)-distribution \( D \subset TB \) such that:

\[
\pi^\#(\phi^*(D^0)) = \text{Ker} \, d\phi.
\]

**Notes:**

- When \((M, \pi_M)\) is symplectic, the distribution \(D\) is uniquely defined and the definition corresponds to Delzant and Dazord notion of symplectically complete isotropic fibrations.
- In the Poisson case, there can be several choices of \(D\). Notice that we always have \( \text{Im} \, \pi^\#_B \subset D \) (equivalently, \( D^0 \subset \text{Ker} \, \pi^\#_B \)).
Definition of abstract non-commutative integrable systems

**Definition**

A fibration \( \phi : (M^m, \pi_M) \to (B^s, \pi_B) \) is called a non-degenerate isotropic fibration or an abstract non-commutative integrable system of rank \( r := m - s \) if there is a \( r \)-distribution \( D \subset TB \) such that:

\[
\pi^\#(\phi^*(D^0)) = \text{Ker } d\phi.
\]

**Notes:**

- When \((M, \pi_M)\) is symplectic, the distribution \(D\) is uniquely defined and the definition corresponds to Delzant and Dazord notion of symplectically complete isotropic fibrations.

- In the Poisson case, there can be several choices of \(D\). Notice that we always have \(\text{Im } \pi_B^\# \subset D\) (equivalently, \(D^0 \subset \text{Ker } \pi_B^\#\)).
Given an abstract non-commutative integrable system $\phi : M \to B$ we obtain a

- A Lie algebroid action of $D^0 \subset T^*B$ on $\phi : M \to B$:
  $$\alpha \mapsto \pi^h_M(\phi^* \alpha).$$

- A groupoid action of $\mathcal{G}(D^0) = (D^0, +)$ on $\phi : M \to B$. 
Given an abstract non-commutative integrable system $\phi : M \to B$ we obtain a

- A Lie algebroid action of $D^0 \subset T^*B$ on $\phi : M \to B$:
  $$\alpha \mapsto \pi^B_M(\phi^*\alpha).$$

- A groupoid action of $\mathcal{G}(D^0) = (D^0, +)$ on $\phi : M \to B$. 
Given an abstract non-commutative integrable system \( \phi : M \to B \) we obtain a

- A Lie algebroid action of \( D^0 \subset T^* B \) on \( \phi : M \to B \):
  \[
  \alpha \mapsto \pi^\sharp_M (\phi^* \alpha).
  \]

- A groupoid action of \( \mathcal{G}(D^0) = (D^0, +) \) on \( \phi : M \to B \).
Given an abstract non-commutative integrable system \( \phi : M \rightarrow B \) we obtain a

- A Lie algebroid action of \( D^0 \subset T^*B \) on \( \phi : M \rightarrow B \):

\[
\alpha \mapsto \pi^M_M(\phi^*\alpha).
\]

- A groupoid action of \( G(D^0) = (D^0, +) \) on \( \phi : M \rightarrow B \).

**Proposition**

*For an abstract non-commutative integrable system \( \phi : M \rightarrow B \) with compact connected fibers, the kernel of the action \( D^0 \Rightarrow B \) on the fibration \( \phi : M \rightarrow B \) is a full rank, lattice \( \Lambda \subset D^0 \).*
We conclude that an abstract non-commutative integrable system \( \phi : M \to B \) with compact connected fibers

- is locally isomorphic to \( D^0/\Lambda \to B \) (hence it is a \( \mathbb{T}^r \)-fibration);
- is globally isomorphic to \( D^0/\Lambda \to B \), provided the Chern class vanishes (i.e., if it has a global section).
- is a principal \( \mathbb{T}^r \)-bundle if the monodromy of \( \Lambda \) is trivial.
We conclude that an abstract non-commutative integrable system \( \phi : M \rightarrow B \) with compact connected fibers

- is locally isomorphic to \( D^0/\Lambda \rightarrow B \) (hence it is a \( T^r \)-fibration);
- is globally isomorphic to \( D^0/\Lambda \rightarrow B \), provided the Chern class vanishes (i.e., if it has a global section).
- is a principal \( T^r \)-bundle if the monodromy of \( \Lambda \) is trivial.
We conclude that an abstract non-commutative integrable system \( \phi : M \to B \) with compact connected fibers

- is locally isomorphic to \( D^0/\Lambda \to B \) (hence it is a \( \mathbb{T}^r \)-fibration);
- is globally isomorphic to \( D^0/\Lambda \to B \), provided the Chern class vanishes (i.e., if it has a global section).

- is a principal \( \mathbb{T}^r \)-bundle if the monodromy of \( \Lambda \) is trivial.
We conclude that an abstract non-commutative integrable system $\phi : M \to B$ with compact connected fibers

- is locally isomorphic to $D^0/\Lambda \to B$ (hence it is a $\mathbb{T}^r$-fibration);
- is globally isomorphic to $D^0/\Lambda \to B$, provided the Chern class vanishes (i.e., if it has a global section).
- is a principal $\mathbb{T}^r$-bundle if the monodromy of $\Lambda$ is trivial.
We conclude that an abstract non-commutative integrable system $\phi : M \to B$ with compact connected fibers

- is locally isomorphic to $D^0/\Lambda \to B$ (hence it is a $\mathbb{T}^r$-fibration);
- is globally isomorphic to $D^0/\Lambda \to B$, provided the Chern class vanishes (i.e., if it has a global section).
- is a principal $\mathbb{T}^r$-bundle if the monodromy of $\Lambda$ is trivial.

... but the Poisson geometry is more complicated because $D^0 \Rightarrow M$ is not a Poisson groupoid.
Proposition

Fix a (local) coisotropic section $\sigma : B \to M$ of the abstract non-commutative integrable system $\phi : M \to B$. Then:

1. The Dirac structure $L := \sigma^* L_{\text{graph}(\pi_M)}$ takes the form:

$$L = \text{graph}(\pi_B) \oplus \text{hor},$$

where hor is an integrable distribution such that $TB = D \oplus \text{hor}$.

2. The map $D^0 \ni \alpha \mapsto \alpha \cdot \sigma(p(\alpha)) \in M$ gives a local Poisson diffeomorphism if we equip $D^0$ with the Poisson structure whose graph is $e^\omega p^* L$. Moreover, $\Lambda$ becomes coisotropic.
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol’d-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:

1. trivial monodromy of \( \Lambda \);
2. trivial (ordinary) Chern class \( c \in H^2(B, \Lambda) \);
3. existence of a global coisotropic section \( \sigma : B \to M \);

(e.g., the Poisson structure \( \pi_B \) must admit an extension to a special Dirac structure as before).
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol'd-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:

1. trivial monodromy of $\Lambda$;
2. trivial (ordinary) Chern class $c \in H^2(B, \Lambda)$;
3. existence of a global coisotropic section $\sigma : B \to M$;

(e.g., the Poisson structure $\pi_B$ must admit an extension to a special Dirac structure as before).
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol’d-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:

1. trivial monodromy of $\Lambda$;
2. trivial (ordinary) Chern class $c \in H^2(B, \Lambda)$;
3. existence of a global coisotropic section $\sigma : B \to M$;

(e.g., the Poisson structure $\pi_B$ must admit an extension to a special Dirac structure as before).
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol’d-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:

1. trivial monodromy of $\Lambda$;
2. trivial (ordinary) Chern class $c \in H^2(B, \Lambda)$;
3. existence of a global coisotropic section $\sigma : B \to M$;

(e.g., the Poisson strucutre $\pi_B$ must admit an extension to a special Dirac structure as before).
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol’d-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:

1. trivial monodromy of \( \Lambda \);
2. trivial (ordinary) Chern class \( c \in H^2(B, \Lambda) \);
3. existence of a global coisotropic section \( \sigma : B \rightarrow M \);

(e.g., the Poisson structure \( \pi_B \) must admit an extension to a special Dirac structure as before).
Canonical form for the Poisson structure: global obstructions

⇒ Local normal form (Arnol’d-Liouville theorem for non-commutative integrable systems).

⇒ Obstructions to existence of global action-angle variables:
   1. trivial monodromy of $\Lambda$;
   2. trivial (ordinary) Chern class $c \in H^2(B, \Lambda)$;
   3. existence of a global coisotropic section $\sigma : B \to M$;

(e.g., the Poisson structure $\pi_B$ must admit an extension to a special Dirac structure as before).
Work in progress

- give examples where all possible combinations of the obstructions above exist;
- determine if existence of global isotropic section can be expressed in cohomological terms;
- understand if existence of certain type of singularities imply vanishing of (some of) the obstructions;
- ...
Work in progress

- give examples where all possible combinations of the obstructions above exist;
- determine if existence of global isotropic section can be expressed in cohomological terms;
- understand if existence of certain type of singularities imply vanishing of (some of) the obstructions
- ...
Work in progress

- give examples where all possible combinations of the obstructions above exist;
- determine if existence of global isotropic section can be expressed in cohomological terms;
- understand if existence of certain type of singularities imply vanishing of (some of) the obstructions

...
Work in progress

- give examples where all possible combinations of the obstructions above exist;
- determine if existence of global isotropic section can be expressed in cohomological terms;
- understand if existence of certain type of singularities imply vanishing of (some of) the obstructions
- ...