Global action-angle variables for non-commutative integrable systems

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Poisson Geometry and Applications - Figueira da Foz

Goals

- Understand non-commutative integrable systems on Poisson manifolds.
- Describe obstructions to the existence of global action-angle variables.

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Ongoing joint work with:

- Raquel Caseiro
- Rui Loja Fernandes
- Camille Laurent-Gengoux
- Pol Vanhaecke.

Integrable systems: definition

Definition

An integrable system on a symplectic manifold (M^{2n}, ω) is a hamiltonian system X_h admitting a family of first integrals $\{f_1, \ldots, f_n\}$ satisfying:

1. involution: $\{f_i, f_j\} = 0$ for all i, j;
2. independence: $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$.

- This definition involves naturally the Poisson bracket, not the symplectic form: $\Rightarrow$ Poisson manifolds.
- Such a system can be integrated by quadratures. There are other examples of systems integrated by quadratures where the Poisson brackets do not commute: $\Rightarrow$ non-commutative integrable systems.

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Integrable systems: example (“normal form”)

Let $M = T^*\mathbb{T}^n$ with canonical symplectic form:

$$\omega = \sum_{i=1}^{n} \mathrm{d}s^i \wedge \mathrm{d}\theta^i$$

- Any $h = h(s^1, \ldots, s^n)$ defines an integrable system with first integrals the action variables $(s^1, \ldots, s^n)$.
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Integrable systems: existence of normal form

- Arnold-Liouville Theorem: under connecteness and compactness assumptions every integrable systems is locally of this form.
- Duistermaat: Obstructions to the existence of global action-angle variables can be described.

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The local problem, the global problem and their solutions are best described using groupoid language.

Integrable systems & fibrations

- To every integrable system $(f_1, \ldots, f_n)$ on a symplectic manifold (M^{2n}, ω) there is associated a Lagrangian fibration:

$$\phi : M^{2n} \to \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))$$

Proposition

Conversely, every Lagrangian fibration

$$\phi : (M^{2n}, \omega) \to B^n$$

is locally of this form.

Notice that these are Poisson fibrations if we equip the base with the trivial bracket:

$$\phi : (M^{2n}, \omega^{-1}) \to (B^n, \pi \equiv 0)$$

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Poisson fibrations & Groupoids

Every complete Poisson fibration $\phi : (M, \pi_M) \to (B, \pi_B)$ gives rise to:

- A Lie algebroid action of T^*B on $\phi : M \to B$:

$$\alpha \mapsto \pi_M^\sharp(\phi^*\alpha).$$

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Proposition

For a Lagragian fibration with compact connected fibers, the kernel of the symplectic action $T^*B \rightrightarrows B$ on the fibration $\phi : M \to B$ is a Lagrangian, full rank, lattice $\Lambda \subset T^*B$.

Poisson fibrations & Groupoids

Conclusion

For any Lagrangian fibration $\phi : (M, \omega) \to B$ with compact, connected fibers:

(i) *There exists a full rank, Lagrangian, lattice $\Lambda \subset T^*B$;*

(ii) *$T^*B/\Lambda \rightrightarrows B$ is a symplectic groupoid integrating $(B, \pi_B = 0)$ which acts freely and properly in a symplectic manner in the fibration $\phi : (M, \omega) \to B$.*

- These facts form the basis to understand the existence of both local and global normal forms of Lagrangian fibrations/integrable systems.

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Groupoid actions and canonical forms

Proposition

Given a Poisson action of a Poisson groupoid $\mathcal{G} \rightrightarrows B$ on a Poisson fibration $\phi : M \to B$, every (local) coisotropic section $\sigma : B \to M$ determines a (local) Poisson map:

$$\mathcal{G} \to M, \qquad g \mapsto g \cdot \sigma(\mathbf{s}(g)).$$

Proof. An exercise in coisotropic calculus!

Corollary (Arnol'd-Liouville Theorem)

Every Lagrangian fibration is locally isomorphic to $(T^*\mathbb{T}^n, \omega_{can}) \to \mathbb{R}^n$.

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Groupoid actions and global canonical forms

Obstructions to triviality of a Lagrangian fibration $\phi : M \to B$:

1. **Vanishing of Hamiltonian monodromy**: The holonomy of the cover $\Lambda \to B$ must be trivial (obstruction for the $\mathbb{T}^n$-fibration to be a principal $\mathbb{T}^n$-bundle).
2. **Vanishing of the Lagrangian Chern class**: the class $c(\phi) \in \check{H}(B; \Gamma_{\mathrm{Lagr}}(T^*B/\Lambda))$ must be trivial (obstruction to existence of a global Lagrangian section).

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Non-commutative integrable systems: definition

Definition

A non-commutative integrable system of rank r on a Poisson manifold (M^m, π) is a hamiltonian system X_h admitting a family of first integrals $\{f_1, \dots, f_s\}$, $r + s = m$, satisfying:

1. involution: $\{f_i, f_j\} = 0$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$;
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We shall also assume the non-degeneracy condition:

- the hamiltonian vector fields $X_{f_1}, \dots, X_{f_r}$ are independent.

Note: When $r = s$ we obtain a classical integrable system on a symplectic manifold.

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Non-commutative integrable systems: examples

- Non-commutative integrable systems are integrable by quadratures.

- Examples of non-commutative integrable systems include:

- Natural mechanical systems such as the Kepler system and the Euler-Poinsot rigid body.
- Classes of systems invariant under a hamiltonian group action (collective motions)

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Non-commutative integrable systems: example ("normal form")

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$$\omega = \sum_{i=1}^{r} \frac{\partial}{\partial s^i} \wedge \frac{\partial}{\partial \theta^i} + \sum_{j,k=1}^{s-r} \varphi^{jk}(z) \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k}$$

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- Local normal form: by the work of Laurent-Gengoux, Miranda and Vanhaecke, under connecteness and compactness assumptions, every non-commutative integrable system is locally of this form (Arnold-Liouville Theorem).

- What are the obstructions to the existence of global action-angle variables?

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Note: In a fundamental paper, Dazord and Delzant have study in detail the case where M is symplectic.

Non-commutative integrable systems & fibrations

- To every non-commutative integrable system there is associated two Poisson fibrations:

$$\begin{tikzcd} (M,\pi_M) \arrow[r, "\phi"] \arrow[dr, "\psi"'] & (\mathbb{R}^s,\pi_B) \arrow[d, "q"] \\ & (\mathbb{R}^r,0) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "\phi"] \arrow[dr, mapsto, "\psi"'] & (f_1(x),\dots,f_s(x)) \\ & (f_1(x),\dots,f_r(x)) \end{tikzcd}$$

- The fibers of these fibrations are isotropic/coisotropic since they satisfy:

$$\operatorname{Ker} \mathrm{d}\phi = \pi_M^\sharp(\operatorname{Ker} \mathrm{d}\psi)^0.$$

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Definition of abstract non-commutative integrable systems

Definition

A fibration $\phi : (M^m, \pi_M) \to (B^s, \pi_B)$ is called a non-degenerate isotropic fibration or an abstract non-commutative integrable system of rank $r := m - s$ if there is a r-distribuition $D \subset TB$ such that:

$$\pi^\sharp(\phi^*(D^0)) = \mathrm{Ker}\, d\phi.$$

Notes:

- When (M, π_M) is symplectic, the distribuition D is uniquely defined and the definition corresponds to Delzant and Dazord notion of symplectically complete isotropic fibrations.
- In the Poisson case, there can be several choices of D. Notice that we always have $\mathrm{Im}\, \pi_B^\sharp \subset D$ (equivalently, $D^0 \subset \mathrm{Ker}\, \pi_B^\sharp$).

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Abstract non-commutative integrable systems & groupoids

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- A groupoid action of $\mathcal{G}(D^0) = (D^0, +)$ on $\phi : M \to B$.

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- A Lie algebroid action of $D^0 \subset T^*B$ on $\phi : M \to B$:

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Proposition

For an abstract non-commutative integrable system $\phi : M \to B$ with compact connected fibers, the kernel of the action $D^0 \rightrightarrows B$ on the fibration $\phi : M \to B$ is a full rank, lattice $\Lambda \subset D^0$.

Canonical forms of the fibration

We conclude that an abstract non-commutative integrable system $\phi : M \to B$ with compact connected fibers

- is locally isomorphic to $D^0/\Lambda \to B$ (hence it is a $\mathbb{T}^r$-fibration);
- is globally isomorphic to $D^0/\Lambda \to B$, provided the Chern class vanishes (i.e., if it has a global section).
- is a principal $\mathbb{T}^r$-bundle if the monodromy of Λ is trivial.

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... but the Poisson geometry is more complicated because $D^0 \rightrightarrows M$ is not a Poisson groupoid.

Canonical form for the Poisson structure

Proposition

Fix a (local) coisotropic section $\sigma : B \to M$ of the abstract non-commutative integrable system $\phi : M \to B$. Then:

1. *The Dirac structure $L := \sigma^* L_{graph(\pi_M)}$ takes the form:*

$$L = graph(\pi_B) \oplus hor,$$

where hor is an integrable distribuition such that $TB = D \oplus hor$.

2. *The map $D^0 \ni \alpha \mapsto \alpha \cdot \sigma(p(\alpha)) \in M$ gives a local Poisson diffeomorphism if we equip D^0 with the Poisson structure whose graph is $e^\omega p^* L$. Moreover, Λ becomes coisotropic.*

Canonical form for the Poisson structure: global obstructions

$\Rightarrow$ Local normal form (Arnol'd-Liouville theorem for non-commutative integrable systems).

$\Rightarrow$ Obstructions to existence of global action-angle variables:

1. trivial monodromy of Λ;
2. trivial (ordinary) Chern class $c \in H^2(B, \Lambda)$;
3. existence of a global coisotropic section $\sigma : B \to M$;

(e.g., the Poisson stucture π_B must admit an extension to a special Dirac structure as before).

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Work in progress

- give examples where all possible combinations of the obstructions above exist;
- determine if existence of global isotropic section can be expressed in cohomological terms;
- understand if existence of certain type of singularities imply vanishing of (some of) the obstructions
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