Implicit difference equations

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Outline

- Implicit differential equations. Integrability
- Lie groupoids
- Implicit difference equations. Extracting the integrable part
- Application: Discrete Lagrangian mechanics

Implicit differential equations

Definition

- An *implicit first-order differential equation* on a manifold Q is a submanifold $D \subset TQ$.
- A curve $\gamma : I \to Q$ is called a *solution* of an implicit differential equation $D \subset TQ$ if $\dot{\gamma}(s) \in D$ for any $s \in I$.
- D is said to be *explicit* if it is the image of a differentiable vector field $X : U \to TQ$ ($U \subset Q$ open submanifold).

• An implicit differential equation is said to be *integrable at* $v \in D$ if there is a solution $\gamma : I \to Q$ such that $\dot{\gamma}(0) = v$.
• It is *integrable in a subset* $S \subset D$ if it is integrable at each $v \in S$.
• It is *integrable* if it is integrable at each $v \in D$.

$Q = \mathbb{R}^2$

$$D = \{(x, y, \dot{x}, \dot{y}) \in TQ | y = 0,\ \dot{x} = 0,\ \dot{y} > 0\}$$

has no solutions

$$D = \{(x, y, \dot{x}, \dot{y}) \in TQ | x^2 + y^2 + (\dot{x} - 1)^2 + \dot{y}^2 = 1\}$$

is not integrable in the set

$$\{(x, y, \dot{x}, \dot{y}) \in TQ | x^2 + y^2 = 1, x \neq 0, \dot{x} = 1, \dot{y} = 0\}$$

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Integrability algorithm for implicit differential equations

$D \subset TQ$ implicit differential equation

$$\begin{aligned} D^0 = D, &\quad C^0 = \tau_Q(D), \\ D^k = D^{k-1} \cap TC^{k-1}, &\quad C^k = \tau_Q(D^k), \end{aligned}$$

$\tau_Q : TQ \to Q$ is the canonical projection.

$$D^k \subseteq D^{k-1}, \qquad C^{k-1} \subseteq C^k, \text{ for any } k$$

Assume that there exists k_f such that $D^{k_f} = D^{k_f - 1}$.

Theorem

$D^{k_f} \subseteq D$ is the (possibly empty) integrable part of D

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Relation with Lagrangian mechanics

$L\colon TP \to \mathbb{R}$ Lagrangian function

$S_L = A_P(dL(TP)) \subseteq T(T^*P)$ implicit differential equation

First step of integrability algorithm will give us the solution of the Euler-Lagrange equations.

$$S_L = \left\{(q^i, \frac{\partial L}{\partial v^i}; v^i, \frac{\partial L}{\partial q^i})\right\},$$

$$(S_L)^1 = \left\{\left(q^i, \frac{\partial L}{\partial v^i}; v^i, \frac{\partial L}{\partial q^i}\right) \middle| \ \dot{q}^i = v^i, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i}\right\}.$$

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Extension to Lie algebroids

$\tau : E \to Q$ Lie algebroid with anchor map $\rho : E \to TQ$ and $S \subset E$ be a submanifold of E (not necessarily a vector subbundle)

$$
\begin{aligned}
S_0 &= S \\
S_1 &= S_0 \cap \rho^{-1}\big(T\tau(S_0)\big) \\
&\vdots \\
S_{k+1} &= S_k \cap \rho^{-1}\big(T\tau(S_k)\big) \\
&\vdots
\end{aligned}
$$

In our case, $S_k = \rho^{-1}(TQ_k)$ (equivalently, $Q_k = \tau(S_k)$).

Lie groupoids

$\mathcal{G} \rightrightarrows M$

- $\mathbf{s} : \mathcal{G} \to M$ and $\mathbf{t} : \mathcal{G} \to M$ source and target map

$$\mathcal{G}_2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} \,|\, \mathbf{s}(g) = \mathbf{t}(h)\}$$

- $m : \mathcal{G}_2 \to \mathcal{G}$ multiplication
  - $\mathbf{s}(gh) = \mathbf{s}(h)$ and $\mathbf{t}(gh) = \mathbf{t}(g)$, for any $(g, h) \in \mathcal{G}_2$,
  - $g(hk) = (gh)k$, for any $g, h, k \in \mathcal{G}$
- $\epsilon : M \to \mathcal{G}$ identity section
  - $g\epsilon(\mathbf{s}(g)) = g$ and $\epsilon(\mathbf{t}(g))g = g$, for any $g \in \mathcal{G}$.
- $\iota : \mathcal{G} \to \mathcal{G}$ inversion
  - $gg^{-1} = \epsilon(\mathbf{t}(g))$ and $g^{-1}g = \epsilon(\mathbf{s}(g))$, for any $g \in \mathcal{G}$.

Examples of Lie groupoids

1.- *Lie groups* (Lie groupoid over a single point)

2.- *Banal groupoid*

M smooth manifold $\Rightarrow M \times M \rightrightarrows M$ Lie groupoid

$$\begin{array}{lll} \mathbf{s} : M \times M \to M & ; & (x,y) \mapsto y \\ \mathbf{t} : M \times M \to M & ; & (x,y) \mapsto x \\ m : (M \times M)_2 \to M \times M & ; & ((x,y),(y,z)) \mapsto (x,z) \end{array}$$

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Lie algebroid associated with Lie groupoid $\mathcal{G}$

$\mathcal{G} \rightrightarrows M$ Lie groupoid $\Rightarrow \tau : A\mathcal{G} \to M$ Lie algebroid

$$(A\mathcal{G})_x := \operatorname{Ker} \mathrm{d}_{\epsilon(x)}\mathbf{t}$$

$$\operatorname{Sec}(\tau) \cong \mathfrak{X}^L(\mathcal{G})$$

$$X \in \operatorname{Sec}(\tau) \Rightarrow \overleftarrow{X}(g) = (\mathrm{d}_{\epsilon(\mathbf{s}(g))} l_g)(X(\mathbf{s}(g))),$$

$(\llbracket\cdot,\cdot\rrbracket, \rho)$ Lie algebroid on $A\mathcal{G} \to M$

$$\begin{cases} \overleftarrow{\llbracket X_1, X_2 \rrbracket} = [\overleftarrow{X}_1, \overleftarrow{X}_2] \\ \rho(X)(x) = \mathrm{d}_{\epsilon(x)}\mathbf{s}(X_x) \end{cases}$$

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Cotangent groupoid
$\mathcal{G} \rightrightarrows M$ Lie groupoid $\Rightarrow T^*\mathcal{G} \rightrightarrows A^*\mathcal{G}$ Lie groupoid

$$\tilde{\mathbf{s}}(\mu_g)(X) = \mu_g((\mathrm{d}_{\epsilon(\mathbf{s}(g))} l_g)(X)), \text{ for } \mu_g \in T^*_g\mathcal{G} \text{ and } X \in A_{\mathbf{s}(g)}\mathcal{G},$$
$$\tilde{\mathbf{t}}(\nu_h)(Y) = \nu_h((\mathrm{d}_{\epsilon(\mathbf{t}(h))} r_h)(Y - (\mathrm{d}_{\epsilon(\mathbf{t}(h))}(\epsilon \circ \mathbf{s}))(Y))),$$
$$\text{for } \nu_h \in T^*_h\mathcal{G} \text{ and } Y \in A_{\mathbf{t}(h)}\mathcal{G},$$
$$(\mu_g \oplus_{T^*\mathcal{G}} \nu_h)(\mathrm{d}_{(g,h)} m(X_g, Y_h)) = \mu_g(X_g) + \nu_h(Y_h),$$
$$\text{for } (X_g, Y_h) \in T_{(g,h)}\mathcal{G}_2,$$

Remark
$\tilde{\mathbf{t}} : T^*\mathcal{G} \to A^*\mathcal{G}$ Poisson map and $\tilde{\mathbf{s}} : T^*\mathcal{G} \to A^*\mathcal{G}$ anti-Poisson

Implicit difference equations

Definition

- An *implicit difference equation* is a submanifold E of $\mathcal{G}$.
- An *admissible sequence* on $\mathcal{G}$ is a mapping $\gamma : I \cap \mathbb{Z} \longrightarrow \mathcal{G}$ such that $\mathbf{s}(\gamma(i)) = \mathbf{t}(\gamma(i+1))$ for all $i, i+1 \in I \cap \mathbb{Z}$. Here I is an interval on $\mathbb{R}$.
- A *solution* of an implicit difference equation $E \subset \mathcal{G}$ is an admissible sequence : $I \cap \mathbb{Z} \longrightarrow \mathcal{G}$ such that $\gamma(i) \in E$, for all $i \in I \cap \mathbb{Z}$.

$E \subset \mathcal{G}$ is said to be

1) *forward integrable at* $g \in E$ if there is a solution $\gamma : \mathbb{Z}^+ \longrightarrow E \subseteq \mathcal{G}$ with $\gamma(0) = g$.
2) *backward integrable at* $g \in E$ if there is a solution $\gamma : \mathbb{Z}^- \longrightarrow E \subseteq \mathcal{G}$ with $\gamma(0) = g$.
3) *integrable at* $g \in E$ if if there is a solution $\gamma : \mathbb{Z} \longrightarrow E \subseteq \mathcal{G}$ with $\gamma(0) = g$.

If these conditions hold for all g, we say that E is forward integrable, backward integrable or integrable, respectively.

Definition

Let E be a submanifold of $\mathcal{G}$. Then,

1) E is forward integrable if and only if for each $g \in E$ exists at least an $h \in E$ such that $\mathbf{s}(g) = \mathbf{t}(h)$.
2) E backward integrable if and only if for each $g \in E$ exists at least an $h' \in E$ such that $\mathbf{s}(h') = \mathbf{t}(g)$.

The sets

$$\begin{aligned} \tilde{E}_f &= \{g \in E \mid E \text{ is } \textit{forward integrable at } g\}, \\ \tilde{E}_b &= \{g \in E \mid E \text{ is } \textit{backward integrable at } g\}, \\ \tilde{E}_{fb} &= \{g \in E \mid E \text{ is } \textit{integrable at } g\}, \end{aligned}$$

are called the *forward integrable*, *backward integrable* and *integrable parts* of E, respectively.

Example: Bisections on groupoids

$\mathcal{G} \rightrightarrows M$ Lie groupoid
$\sigma\colon M \to \mathcal{G}$ bisection

$$\mathbf{t} \circ \sigma = \mathrm{Id}, \qquad \mathbf{s} \circ \sigma \text{ diffeomorphism}$$

$E_\sigma = \sigma(M) \subset \mathcal{G}$ implicit difference equation

$g = \sigma(x) \in E_\sigma$

$$\gamma(i) = \sigma((\mathbf{s} \circ \sigma)^i(x)), \qquad \text{for } i \in \mathbb{Z},$$

is a solution with $\gamma(0) = g$. Thus, E_σ is integrable.

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Algorithm for extracting the forward integrable part

$E \subset \mathcal{G}$ implicit difference equation

$$C_f^0 = \mathbf{t}(E), \quad C_f^1 = \mathbf{t}(E \cap \mathbf{s}^{-1}(C_f^0)), \dots, C_f^k = \mathbf{t}(E \cap \mathbf{s}^{-1}(C_f^{k-1})), \dots$$

$$E_f^0 = E, \quad E_f^1 = E \cap \mathbf{t}^{-1}(C_f^0), \dots, E_f^k = E \cap \mathbf{t}^{-1}(C_f^{k-1}), \dots$$

$$\mathbf{s}(E_f^k) = C_f^k, \qquad C_f^k \subseteq C_f^{k-1}, \qquad E_f^k \subseteq E_f^{k-1}$$

Assume that there exists k_f such that

$$C_f^{k_f} = C_f^{k_f+1} = \bar{C}_f$$

Then

$$E_f^{k_f+1} = E_f^{k_f+2} = \bar{E}_f$$

Theorem

The forward integrable part of E is $\bar{E}_f$.

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Application: Implicit systems defined by Discrete Lagrangian functions

Discrete Lagrangian $L\colon \mathcal{G} \to \mathbb{R}$.

Discrete action sum:

$$\mathcal{S}L(\gamma) = \sum_{i=1}^{N} L(\gamma(i)), \qquad \gamma \text{ admissible sequence}$$

Discrete Euler-Lagrange equations

$$\mathrm{d}\left[L \circ l_{g_k} + L \circ r_{g_{k+1}} \circ i\right](\epsilon(x_k))_{\big|A_{x_k}\mathcal{G}} = 0, \quad g_k = \gamma(k), \; \mathbf{s}(g_k) = x_k$$

Example: $Q \times Q$

$$D_2L(q_0, q_1) + D_1L(q_1, q_2) = 0$$

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Discrete Lagrangian evolution operator for L (*discrete flow*):
$\Upsilon\colon \mathcal{G} \to \mathcal{G}$ smooth map such that:

- $\mathbf{s}(g) = \mathbf{t}(\Upsilon(g))$, for all $g \in \mathcal{G}$, (Υ is a *second order operator*).
- $(g, \Upsilon(g))$ is a solution of the discrete Euler-Lagrange equations, for all $g \in \mathcal{G}$.

Discrete Legendre transformations:
$\mathbb{F}^{-}L\colon \mathcal{G} \to A^*\mathcal{G}$ and $\mathbb{F}^{+}L\colon \mathcal{G} \to A^*\mathcal{G}$

$$
\begin{aligned}
(\mathbb{F}^{-}L)(g) &= \tilde{\mathbf{t}}(\mathrm{d}L(g)),\\
(\mathbb{F}^{+}L)(g) &= \tilde{\mathbf{s}}(\mathrm{d}L(g))
\end{aligned}
$$

A discrete Lagrangian $L\colon \mathcal{G} \to \mathbb{R}$ is said to be *regular* if and only if the Legendre transformation $\mathbb{F}^{-}L$ is a local diffeomorphism.

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If $\mathbb{F}^+L$ and $\mathbb{F}^-L$ are global diffeomorphisms (that is, L is *hyperregular*) then $\Upsilon_L = (\mathbb{F}^-L)^{-1} \circ \mathbb{F}^+L$.

If $L\colon \mathcal{G} \to \mathbb{R}$ is a hyperregular Lagrangian function, the *discrete Hamiltonian evolution operator*, $\tilde{\Upsilon}_L : A^*\mathcal{G} \to A^*\mathcal{G}$ is given by

$$\tilde{\Upsilon}_L = \mathbb{F}^{\pm}L \circ \Upsilon_L \circ (\mathbb{F}^{\pm}L)^{-1} \;= \mathbb{F}^+L \circ (\mathbb{F}^-L)^{-1}.$$

Proposition

Let $L\colon \mathcal{G} \to \mathbb{R}$ be a discrete Lagrangian. The first step, $(S_L)^1_f$, to obtain the forward integrable part of $S_L = \mathrm{d}L(\mathcal{G}) \subset T^*G$ are the points $\mathrm{d}L(g) \in S_L$ such that there exists an element h satisfying that (g,h) is a solution of the Discrete Euler-Lagrange equations

$$(S_L)^1_f \quad = \quad \{(g,h) \in \mathcal{G}_2 \,|\, (g,h) \text{ is a solution of the DEL eqns.}\}$$

As a consequence, if $\tilde{\mathbf{s}}$ (resp. $\tilde{\mathbf{t}}$) is the target (resp. source) map of the groupoid $T^*\mathcal{G}$

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$L\colon \mathcal{G} \to \mathbb{R}$ hyperregular discrete Lagrangian function.
$g \in \mathcal{G}$

$\Upsilon_{T^*(\mathcal{G})}\colon \mathbb{Z} \to S_L = \mathrm{d}L(\mathcal{G}) \subseteq T^*(\mathcal{G})$ defined by

$$\Upsilon_{T^*(\mathcal{G})}(i) = \mathrm{d}L(\Upsilon_L^i(g)), \quad \text{for } i \in \mathbb{Z},$$

is a solution of S_L and $\Upsilon_{T^*(\mathcal{G})}(0) = \mathrm{d}L(g)$.

Corollary

Let $L\colon \mathcal{G} \to \mathbb{R}$ be a discrete hyperregular Lagrangian function.Then, the implicit difference equation $S_L = \mathrm{d}L(\mathcal{G})$ is integrable.

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$$\underbrace{\begin{array}{c} S_L \text{ Lagrangian submanifold of } T^*(\mathcal{G}) \\ (\tilde{\mathbf{t}}, \tilde{\mathbf{s}}) : T^*(\mathcal{G}) \to A^*\mathcal{G} \times \overline{A^*\mathcal{G}} \text{ is a Poisson map} \end{array}}$$

$$\Downarrow$$

$$(\tilde{\mathbf{t}}, \tilde{\mathbf{s}})(S_L) \text{ is a coisotropic submanifold of } A^*\mathcal{G} \times \overline{A^*\mathcal{G}}.$$

Proposition

Let $L : \mathcal{G} \to \mathbb{R}$ be a discrete hyperregular Lagrangian.Then, the discrete Hamiltonian evolution operation $\tilde{\Upsilon}_L$ preserves the Poisson structure on $A^*\mathcal{G}$.

Example: a discrete Singular Lagrangian

$$L_d^h : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad L_d^h(x_1, y_1, x_2, y_2) = \frac{1}{2}\left(\frac{x_2 - x_1}{h}\right)^2 + \frac{1}{2}x_1^2 y_1$$

Then $S_{L_d^h} \subset T^*(\mathbb{R}^2 \times \mathbb{R}^2)$ defined by

$$S_{L_d^h} = \left\{(x_1, y_1, x_2, y_2; -\frac{x_2 - x_1}{h^2} + x_1 y_1, \frac{1}{2}x_1^2, \frac{x_2 - x_1}{h^2}, 0)\right\}$$

The algorithm produces the integrable part

$$\begin{aligned} \left(S_{L_d^h}\right)_f^2 &= \left\{(0, y_1, 0, 0; 0, y_2, 0, 0) \mid (y_1, y_2) \in \mathbb{R}^2\right\} \\ C_f^3 &= \tilde{\mathfrak{t}}(\left(S_{L_d^h}\right)_f^2) = \{(0, y; 0, 0) \mid y \in \mathbb{R})\} \end{aligned}$$

Implicit systems defined by discrete nonholonomic Lagrangian systems

$(L, \mathcal{M}_c, \mathcal{D}_c)$ discrete nonholonomic Lagrangian system

$$\begin{cases} L : \mathcal{G} \to \mathbb{R} \\ i_{\mathcal{M}_c} : \mathcal{M}_c \to \mathcal{G} \\ i_{\mathcal{D}_c} : \mathcal{D}_c \to A\mathcal{G} \end{cases}$$

$S_{(L,\mathcal{M}_c)} = \mathrm{d}L(\mathcal{M}_c) \subseteq T^*\mathcal{G}$, that is,

$$S_{(L,\mathcal{M}_c)} = \{\mathrm{d}L(i_{\mathcal{M}_c}(g)) \,|\, g \in \mathcal{M}_c\}.$$

Apply the forward integrability algorithm to $S_{(L,\mathcal{M}_c)}$ in $T^*\mathcal{G}$, but changing the maps $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{t}}$ for the new maps $\tilde{\mathbf{s}}_{\mathcal{D}_c}$ and $\tilde{\mathbf{t}}_{\mathcal{D}_c}$,

$$\tilde{\mathbf{s}}_{\mathcal{D}_c} = i^*_{\mathcal{D}_c} \circ \tilde{\mathbf{s}} : T^*\mathcal{G} \to \mathcal{D}^*_c$$
$$\tilde{\mathbf{t}}_{\mathcal{D}_c} = i^*_{\mathcal{D}_c} \circ \tilde{\mathbf{t}} : T^*\mathcal{G} \to \mathcal{D}^*_c.$$

Proposition

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $(L, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system on $\mathcal{G}$. The first step, $(S_{(L,\mathcal{M}_c)})^1_f$, to obtain the forward integrable part of $S_{(L,\mathcal{M}_c)} = \mathrm{d}L(\mathcal{M}_c)$ for the maps $\tilde{\mathbf{s}}_{\mathcal{D}_c}, \tilde{\mathbf{t}}_{\mathcal{D}_c} : T^*\mathcal{G} \to \mathcal{D}_c^*$ are the points $\mathrm{d}L(g) \in S_{(L,\mathcal{M}_c)}$ such that there exists an element $h \in \mathcal{M}_c$, satisfying $(g,h) \in \mathcal{G}_2$ and, in addition, (g,h) is a solution of the discrete non-holonomic Euler-Lagrange equations, that is,

$$\mathrm{d}\left[L \circ l_g + L \circ r_h \circ i\right](\epsilon(x))_{|(\mathcal{D}_c)_x} = 0,$$

where $x = \mathbf{s}(g) = \mathbf{t}(h)$.
In other words,

$$\begin{aligned}
(S_L)^1_f &= \{(g,h) \in \mathcal{G}_2 \cap (\mathcal{M}_c \times \mathcal{M}_c) \,| \\
&\quad (g,h) \text{ is a solution of the D. NH E-L. eqns.}\} \\
&= \{(g,h) \in \mathcal{G}_2 \cap (\mathcal{M}_c \times \mathcal{M}_c) \,|\, \tilde{\mathbf{s}}_{\mathcal{D}_c}(\mathrm{d}L(g)) = \tilde{\mathbf{t}}_{\mathcal{D}_c}(\mathrm{d}L(h))\}
\end{aligned}$$