On Dirac structures and Dirac pairs

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A story of pairs

This is a story about pairs.

• In the first part we shall show how, on some Courant algebroids, we can characterize

the characteristic pair that defines a Dirac structure

in terms of Terashima's "Poisson functions" which generalize the "Hamiltonian operators" of Liu–Weinstein–Xu which themselves generalize the "Poisson bivectors".

• In the second part, we shall introduce Dirac pairs, defined in terms of Nijenhuis relations, and we shall show that the notion of Dirac pairs unifies

- Hamiltonian pairs (bi-Hamiltonian structures),
- $P\Omega$-structures, and
- a restricted class of ΩN-structures.

We shall give explicit examples on 4-dimensional flat manifolds.

• The third part, if I had time..., would deal with Nijenhuis structures in Courant algebroids.

Part I. Characteristic pairs of Dirac structures

– Briefly recall:

Courant algebroids.

Dirac sub-bundles.

Double of a Lie algebroid,

a Lie bialgebroid,

a proto-bialgebroid.

– Introduce Poisson functions and presymplectic functions.

– Theorem (generalizes LWX [1997])

The graph of a bivector is Dirac iff the bivector is a Poisson function,

The graph of a 2-form is Dirac iff the 2-form is a presymplectic function.

– Introduce characteristic pairs of Dirac structures in the double of a proto-bialgebroid.

– Theorem (Yin and He [2006], yks [2011], generalizes Liu [2000]).

Courant algebroids

Let $(E, \langle \ , \ \rangle)$ be a vector bundle, over a manifold M, equiped with a symmetric fiberwise bilinear form.

Let $\{ \ , \ \}$ be the even graded Poisson bracket on the algebra $\mathcal{F}$ of functions on the minimal symplectic realization of E.

A Courant algebroid structure on $(E, \langle \ , \ \rangle)$ is defined by an element Θ of degree 3 in $\mathcal{F}$ such that $\boxed{\{\Theta, \Theta\} = 0.}$

Consider the following derived brackets:

$\{\{X, \Theta\}, Y\} = [X, Y]$ defines the Dorfman bracket on ΓE, a Leibniz (Loday) bracket on ΓE.

Recall that in a Leibniz (Loday) algebra the bracket is not in general skew-symmetric, it satisfies the Jacobi identity,

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

$\{\{X, \Theta\}, f\} = \rho(X)f$ defines the anchor $\rho : E \to TM$ (a vector bundle morphism that induces a Lie algebra morphism from ΓE to $\Gamma(TM)$ equipped with the Lie bracket of vector fields.).

Dirac sub-bundles of Courant algebroids

Definition

A Dirac sub-bundle of $(E, \langle\ ,\ \rangle, \Theta)$ is a Lagrangian (*i.e.*, maximally isotropic) sub-bundle of E whose space of sections is closed under the Dorfman bracket.

The double of a Lie algebroid

Assume that $E = A \oplus A^*$ where A is a vector bundle and $\langle\ ,\ \rangle$ is the canonical symmetric fiberwise bilinear form.

Then $\mathcal{F}$ is the algebra of functions on the supermanifold $T^*[2]A[1]$ and the even Poisson bracket on $\mathcal{F}$ is the big bracket.

A Lie algebroid structure on A is defined by an element $\mu \in \mathcal{F}$, of bidegree $(1, 2)$, such that $\boxed{\{\mu, \mu\} = 0.}$

Then $(E = A \oplus A^*, \langle\ ,\ \rangle)$ with $\Theta = \mu$ is a Courant algebroid, called the double of A.

A Dirac sub-bundle of $E = A \oplus A^*$ is called a Dirac structure in A.

If, in particular, $A = TM$, $E = TM \oplus T^*M$ is called the generalized tangent bundle of M, and a Dirac sub-bundle of the Courant algebroid $TM \oplus T^*M$ is called a Dirac structure on M.

Graphs of Poisson bivectors and of presymplectic forms

We consider a bivector π as the map, denoted by the same letter, from A^* to A defined by $\pi\xi = i_\xi \pi$ for all $\xi \in A^*$.

• The graph of a bivector $\pi \in \Gamma(\wedge^2 A)$ is a Dirac sub-bundle of $E = A \oplus A^*$ if and only if π is a Poisson bivector ($[\pi, \pi] = 0$, where $[\ ,\ ]$ is the Schouten–Nijenhuis bracket of multivectors extending the Lie bracket of sections of A as a graded biderivation).

We consider a 2-form $\omega \in \Gamma(\wedge^2 A^*)$ as the map, denoted by the same letter, $\omega : A \to A^*$ defined by $\omega x = -i_x \omega$ for all $x \in A$.

• The graph of a 2-form $\omega \in \Gamma(\wedge^2 A^*)$ is a Dirac sub-bundle of $E = A \oplus A^*$ if and only if ω is a presymplectic form (ω is d-closed, where d is the differential on the sections of $\wedge^\bullet(A^*)$ defined by μ).

Proto-bialgebroids

Let A be a vector bundle. A proto-bialgebroid is defined by elements of $\mathcal{F}$, μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, ϕ of bidegree $(3,0)$, ψ of bidegree $(0,3)$, such that

$$\boxed{\{\Theta,\Theta\} = 0,}$$

where $\boxed{\Theta = \psi + \mu + \gamma + \phi.}$

The case of Lie-quasi bialgebroids is that of $\psi = 0$.
The case of quasi-Lie bialgebroids is that of $\phi = 0$.
The case of Lie bialgebroids is that of $\phi = \psi = 0$.
The case of Lie algebroids is that of $\gamma = \phi = \psi = 0$.

Then $E = A \oplus A^*$, with the canonical symmetric fiberwise bilinear form and $\Theta = \psi + \mu + \gamma + \phi$, is a Courant algebroid, called the double of of the proto-bialgebroid.

Graphs in the double of a Lie bialgebroid

In the case of the double of a Lie bialgebroid,
Liu, Weinstein and Xu [1997] defined "Hamiltonian operators" as the solutions of the "Maurer–Cartan type equation",

$$\mathrm{d}_\gamma \pi + \frac{1}{2}[\pi, \pi] = 0.$$

Then they proved:

Theorem

The graph of a bivector is a Dirac sub-bundle if and only if it is a Hamiltonian operator.

More generally, on a proto-bialgebroid, in order to characterize graphs of bivectors (resp., 2-forms) as Dirac structures we need to define Poisson functions (resp., presymplectic functions). They are defined as the solutions of a "generalized Maurer–Cartan equation".

Twisting

- A function of bidegree $(2,0)$ is a bivector π on V.
- A function of bidegree $(0,2)$ is a 2-form ω on V.

Twisted structures

$\Theta_\pi = \psi_\pi + \mu_\pi + \gamma_\pi + \phi_\pi$ and $\Theta_\omega = \psi_\omega + \mu_\omega + \gamma_\omega + \phi_\omega$ with

$$\begin{cases} \phi_\pi = \phi - \{\gamma, \pi\} + \frac{1}{2}\{\{\mu, \pi\}, \pi\} - \frac{1}{6}\{\{\{\psi, \pi\}, \pi\}, \pi\} \ , \\ \gamma_\pi = \gamma - \{\mu, \pi\} + \frac{1}{2}\{\{\psi, \pi\}, \pi\} \\ \mu_\pi = \mu - \{\psi, \pi\} \ , \\ \psi_\pi = \psi \ . \end{cases}$$

$$\begin{cases} \phi_\omega = \phi \ , \\ \gamma_\omega = \gamma - \{\phi, \omega\} \ , \\ \mu_\omega = \mu - \{\gamma, \omega\}, + \frac{1}{2}\{\{\phi, \omega\}, \omega\} \ , \\ \psi_\omega = \psi - \{\mu, \omega\} + \frac{1}{2}\{\{\gamma, \omega\}, \omega\} - \frac{1}{6}\{\{\{\phi, \omega\}, \omega\}, \omega\} \ . \end{cases}$$

Definition of Poisson functions and of presymplectic functions

[Roytenberg 2002] [Terashima 2008]
[yks 2007 (Prog. Math. 287, 2011)]
Consider a proto-bialgebroid defined by $(\psi, \mu, \gamma, \phi)$.
(i) A *Poisson function* is a bivector π such that $\phi_\pi = 0$, *i.e.*,.

$$\phi - \{\gamma, \pi\} + \frac{1}{2}\{\{\mu, \pi\}, \pi\} - \frac{1}{6}\{\{\{\psi, \pi\}, \pi\}, \pi\} = 0.$$

(ii) A *presymplectic function* is a 2-form ω such that $\psi_\omega = 0$, *i.e.*,

$$\psi - \{\mu, \omega\} + \frac{1}{2}\{\{\gamma, \omega\}, \omega\} - \frac{1}{6}\{\{\{\phi, \omega\}, \omega\}, \omega\} = 0.$$

• Fact In a Lie bialgbroid, a bivector is a Poisson function if and only if it is a Hamiltonian operator.
Proof $\mathrm{d}_\gamma \pi = \{\gamma, \pi\}$ and $[\pi, \pi] = \{\{\pi, \mu\}, \pi\} = -\{\{\mu, \pi\}, \pi\}$. □

Graphs in the double of a proto-bialgebroid

Let A be a proto-bialgebroid defined by $(\psi, \mu, \gamma, \phi)$.
The following theorem [Roytenberg] [Terashima] [yks] extends the theorem of Liu–Weinstein–Xu.

Theorem
(i) The graph of a bivector π is a Dirac structure if and only if π is a Poisson function.
(ii) The graph of a 2-form ω is a Dirac structure if and only if ω is a presymplectic function.

Summary When $\pi : A^* \to A$, and A is a
Lie algebroid, $\boxed{\text{graph}\,\pi \text{ is Dirac iff } \pi \text{ is a Poisson bivector,}}$
Lie bialgebroid, $\boxed{\text{graph}\,\pi \text{ is Dirac iff } \pi \text{ is a Hamiltonian operator,}}$
proto-bialgebroid, $\boxed{\text{graph}\,\pi \text{ is Dirac iff } \pi \text{ is a Poisson function.}}$

Characteristic pairs

• Ted Courant, case of $TM \oplus T^*M$ [1990],
• Diatta and Medina, case of $\mathfrak{g} \oplus \mathfrak{g}^*$ [CRAS 1999],
• Zhang-Ju Liu, case of Lie bialgebroids [Poisson Geometry, Banach Center Publ., 2000].

For $D \subset A$ a sub-bundle and π a bivector on A, consider

$$L = \{(X + \pi\xi, \xi) \,|\, X \in D, \xi \in D^\perp\}.$$

Then L is a Lagrangian sub-bundle of $A \oplus A^*$. The pair (D, π) is called a characteristic pair of L.
Problem Characterize Dirac structures in terms of characteristic pairs, *i.e.*, find conditions on D and π for L to be a Dirac sub-bundle.
Answer Liu [2000] for the double of a Lie bialgebroid, Yanbin Yin and Long-Guang He 2006] for the double of a proto-bialgebroid, yks [2011] in terms of Poisson functions.

Poisson functions $\mathrm{mod}D$

Let $D \subset A$ be a sub-bundle of a proto-bialgebroid, $(A, \mu, \gamma, \phi, \psi)$, let $D^\perp$ be its orthogonal in A^* (conormal bundle), and let π be a bivector on A.

Definition

A bivector π is a *Poisson function* $\mathrm{mod}D$ if $\phi_\pi \in \Gamma(\wedge^3 D)$ and $\psi_\pi \in \Gamma(\wedge^3(D^\perp))$, where $\psi_\pi = \psi$ and

$$\phi_\pi = \phi - \{\gamma, \pi\} + \frac{1}{2}\{\{\mu, \pi\}, \pi\} - \frac{1}{6}\{\{\{\psi, \pi\}, \pi\}, \pi\}.$$

Dirac structures defined by a characteristic pair.

Theorem

Let $D \subset A$ be a sub-bundle and let π be a bivector on A. Let L be the Lagrangian sub-bundle of $A \oplus A^*$,

$$L = \{(X + \pi\xi, \xi) \,|\, X \in D, \xi \in D^\perp\}.$$

L is a Dirac structure if and only if
(i) ΓD is closed under μ_π,
(ii) $\Gamma(D^\perp)$ is closed under γ_π, and
(iii) π is a Poisson function $\mathrm{mod} D$.

Proof Find necessary and sufficient conditions for

$$\{\{X + \pi\xi + \xi, \phi + \gamma + \mu + \psi\}, Y + \pi\eta + \eta\},$$

for $X, Y \in \Gamma D$ and $\xi, \eta \in \Gamma(D^\perp)$ to be equal to $Z + \pi\zeta + \zeta$, with $\zeta \in \Gamma(D^\perp)$ and $Z \in \Gamma D$.
Case $X = \eta = 0$ yields (i). Case $\xi = Y = 0$ yields (ii).
Case $\xi = \eta = 0$ (condition on ϕ_π) and case $X = Y = 0$ (condition on ψ_π) yield the two conditions for (iii). $\square$

Dual characteristic pairs

There is a dual result for Dirac structures defined by a sub-bundle and a 2-form Let F be a sub-bundle of A^*, and let $F^\perp$ be its orthogonal in A^*.

Definition

A 2-form ω is a *presymplectic function* $\mathrm{mod} F$ if $\phi_\omega \in \Gamma(\wedge^3(F^\perp))$ and $\psi_\omega \in \Gamma(\wedge^3 F)$, where $\phi_\omega = \phi$, and
$\psi_\omega = \psi - \{\mu, \omega\} + \frac{1}{2}\{\{\gamma, \omega\}, \omega\} - \frac{1}{6}\{\{\{\phi, \omega\}, \omega\}, \omega\}$.

Theorem

Let ω be a bivector on A. Let L be the Lagrangian sub-bundle of $A \oplus A^*$,

$$L = \{(X, \xi + \omega X) \,|\, X \in F^\perp, \xi \in F\}.$$

L is a Dirac structure if and only if
ΓF is closed under γ_ω, $\Gamma(F^\perp)$ is closed under μ_ω, and ω is a presymplectic function $\mathrm{mod} F$.

Part II. Dirac pairs in the double of a Lie algebroid

Dirac pairs generalize the familiar compatible structures, such as bi-Hamiltonian structures, etc. defined by Magri in the early 80's. They were defined by Dorfman in 1987, following her work with Gelfand [1979][1980].

- Definitions concerning relations in sets and in vector bundles.
- Torsion of a relation, Nijenhuis relations.
- Dirac pairs defined in terms of Nijenhuis relations.
- The aim is to prove that the notion of Dirac pairs unifies
  Hamiltonian pairs (bi-Hamiltonian structures),
  PΩ-structures,
  a restricted class of ΩN-structures.
- Examples.

For Dirac structures L and L' in the double of a Lie algebroid $A \oplus A^*$ to form a Dirac pair, they must satisfy a compatibility condition, which is a condition on a relation in A.

Relations

When U, V and W are sets, the *composition*, $\mathbf{R}' * \mathbf{R}$, of relations $\mathbf{R} \subset U \times V$ and $\mathbf{R}' \subset V \times W$ is

$$\mathbf{R}' * \mathbf{R} = \{(u, w) \in U \times W \,|\, \exists v \in V, (u, v) \in \mathbf{R} \,\mathrm{and}\, (v, w) \in \mathbf{R}'\}.$$

The *transpose* of a relation $\mathbf{R} \subset U \times V$ is the relation

$$\overline{\mathbf{R}} = \{(v, u) \in V \times U \,|\, (u, v) \in \mathbf{R}\}.$$

If $\phi : U \to V$ and $\phi' : V \to W$ are maps, and if $\mathbf{R} = \mathrm{graph}\, \phi$ and $\mathbf{R}' = \mathrm{graph}\, \phi'$, then

$$\mathbf{R}' * \mathbf{R} = \mathrm{graph}(\phi' \circ \phi).$$

If $\phi : U \to V$ is invertible,

$$\overline{\mathrm{graph}\, \phi} = \mathrm{graph}(\phi^{-1}).$$

Relations in vector spaces and vector bundles

Let U and V be vector spaces. The *dual* of a relation $\mathbf{R} \subset U \times V$ is the relation $\mathbf{R}^* \subset V^* \times U^*$ defined by

$$\mathbf{R}^* = \{(\beta, \alpha) \in V^* \times U^* \mid \langle \alpha, u \rangle = \langle \beta, v \rangle, \forall (u, v) \in \mathbf{R}\}.$$

If $\mathbf{R} = \operatorname{graph} \phi$, where ϕ is a linear map from U to V, then $\mathbf{R}^*$ is the graph of the dual map, ϕ^*.

Convention When U and V are vector bundles over a manifold M, and $\mathbf{R} \subset U \times V$ is a relation, we denote by the same letter the relation on sections induced by $\mathbf{R}$.

Nijenhuis relations in Leibniz algebras

Let $\mathbf{N}$ be a relation in a Leibniz algebra $(E, [\ ,\ ])$.
Consider the real-valued function defined on a subset of $E \times E \times E \times E \times E^* \times E^* \times E^*$ by

$$\mathbf{T}(\mathbf{N})(u_1, v_1, u_2, v_2, \alpha, \alpha', \alpha'')$$
$$= \langle \alpha, [v_1, v_2] \rangle - \langle \alpha', [v_1, u_2] + [u_1, v_2] \rangle + \langle \alpha'', [u_1, u_2] \rangle,$$

for all $u_1, v_1, u_2, v_2 \in E, \alpha, \alpha', \alpha'' \in E^*$ such that $(u_1, v_1) \in \mathbf{N}, (u_2, v_2) \in \mathbf{N}$, $(\alpha, \alpha') \in \mathbf{N}^*, (\alpha', \alpha'') \in \mathbf{N}^*$.

The function $\mathbf{T}(\mathbf{N})$ is called the torsion of the relation $\mathbf{N}$.

Definition

A Nijenhuis relation in E is a subset $\mathbf{N}$ of $E \times E$ such that its torsion, $\mathbf{T}(\mathbf{N})$, vanishes.

Nijenhuis relations generalize Nijenhuis tensors

Proposition

Let $(E, [\ ,\ ])$ *be a Leibniz algebra. A linear map,* $N : E \to E$*, is a Nijenhuis tensor if and only if* $\mathrm{graph}\, N$ *is a Nijenhuis relation in* E*.*

Proof The graph of N is the relation,

$$\mathrm{graph}\, N = \{(u, Nu) \in E \times E \mid u \in E\},$$

and its dual is the graph of the dual N^* of N,

$$\mathrm{graph}(N^*) = \{(\alpha, N^*\alpha) \in E^* \times E^* \mid \alpha \in E^*\}.$$

Therefore, $\mathrm{graph}\, N$ is a Nijenhuis relation if and only if, for all $u_1, u_2 \in E$, $\alpha \in E^*$,

$$\langle \alpha, [Nu_1, Nu_2] \rangle - \langle N^*\alpha, [Nu_1, u_2] + [u_1, Nu_2] \rangle + \langle (N^*)^2 \alpha, [u_1, u_2] \rangle = 0,$$

which is equivalent to

$$\langle \alpha, [Nu_1, Nu_2] - N([Nu_1, u_2] + [u_1, Nu_2]) + N^2[u_1, u_2] \rangle = 0,$$

i.e., $\langle \alpha, TN(u_1, u_2) \rangle = 0$, where TN is the Nijenhuis torsion of the linear map N. □

More generally,

Proposition

If E is a Leibniz algebroid, a vector bundle morphism, $N : E \to E$, is a Nijenhuis tensor if and only if $\mathrm{graph}\, N$ *defines a Nijenhuis relation in ΓE.*

Remark. The torsion as a relation

For a relation $\mathbf{R} \subset U \times U$, set

$$\mathbf{R}^{(2)} = \{(u, u', u'') \in U \times U \times U \,|\, (u, u') \in \mathbf{R} \text{ and } (u', u'') \in \mathbf{R}\}.$$

With this notation, the vanishing of $\mathbf{T}(\mathbf{N})$ defines a relation,

$$\widehat{\mathbf{T}(\mathbf{N})} \subset (\mathbf{N} \times \mathbf{N}) \times (\mathbf{N}^*)^{(2)}.$$

Hamiltonian pairs

Let (A, μ) be a Lie algebroid. Recall that a bivector π is a Poisson structure on A if and only if, for all $\xi_1, \xi_2 \in \Gamma(A^*)$,

$$\boxed{[\pi\xi_1, \pi\xi_2] = \pi[\xi_1, \xi_2]_\pi,}$$

where $[\ ,\ ]_\pi$ is the bracket of sections of A^* defined by μ and π,

$$[\xi_1, \xi_2]_\pi = L_{\pi\xi_1}\xi_2 - L_{\pi\xi_2}\xi_1 + d(\pi(\xi_1, \xi_2)).$$

Definition

Poisson structures π and π' on A are said to be *compatible* if $\pi + \pi'$ is a Poisson structure. When Poisson structures π and π' are compatible, (π, π') is said to be a *bi-Hamiltonian structure* or a *Hamiltonian pair*.

Fact Poisson structures π and π' constitute a Hamiltonian pair if and only if $[\pi, \pi'] = 0$. where $[\ ,\ ]$ is the Schouten–Nijenhuis bracket.

The relation defined by a Hamiltonian pair

For bivectors π and π', set

$$\boxed{\mathbf{N}(\pi, \pi') = \operatorname{graph}\pi * \overline{\operatorname{graph}\pi'}.}$$

Theorem

Let π and π' be bivectors. The torsion of the relation $\mathbf{N}(\pi, \pi')$ satisfies the equation

$$2\mathbf{T}(\mathbf{N}(\pi, \pi'))(\xi_1, \xi_2, \xi, \xi', \xi'')$$

$$\langle \xi, [\pi, \pi](\xi_1, \xi_2)\rangle + \langle \xi'', [\pi', \pi'](\xi_1, \xi_2)\rangle - 2\langle \xi', [\pi, \pi'](\xi_1, \xi_2)\rangle.$$

for all $\xi_1, \xi_2, \xi, \xi', \xi'' \in \Gamma(A^*)$ such that $\pi\xi = \pi'\xi'$ and $\pi\xi' = \pi'\xi''$.

Proof Use $[\pi\xi_1, \pi\xi_2] = \pi[\xi_1, \xi_2]_\pi$ and the skew-symmetry of π and π'. $\square$

Hamiltonian pairs and Poisson pairs

Corollary

If (π, π') is a Hamiltonian pair, then $\mathbf{N}(\pi, \pi')$ is a Nijenhuis relation.

Let us call Poisson bivectors π and π' on A such that $\mathbf{N}(\pi, \pi')$ is a Nijenhuis relation a *Poisson pair*. Then we can state:

$$\boxed{\text{Any Hamiltonian pair is a Poisson pair.}}$$

In order to state a converse, let us set

$$K = \pi^{-1}(\mathrm{Im}\,\pi') \cap \pi'^{-1}(\mathrm{Im}\,\pi) \subset A^*.$$

Corollary

(i) If (π, π') is a Poisson pair, then $i_\xi[\pi, \pi'] = 0$ for all $\xi \in K$.
(ii) If, in addition, $K = A^*$, then (π, π') is a Hamiltonian pair.

In particular,

$$\boxed{\text{Any non-degenerate Poisson pair is a Hamiltonian pair.}}$$

(Non-degenerate means that both bivectors are non-degenerate.)

Hierarchies of Poisson structures

The preceding results imply the well known proposition [Fuchssteiner–Fokas, Dorfman, yks–Magri, etc.],

Proposition

(i) Assume that (π, π') *is a Hamiltonian pair, where* π *is non-degenerate. Then* $N = \pi' \pi^{-1}$ *is a Nijenhuis tensor.*
(ii) Assume that π *and* π' *are non-degenerate Poisson structures and that* $N = \pi' \pi^{-1}$ *is a Nijenhuis tensor. Then* (π, π') *is a Hamiltonian pair. More generally, all* $(N^k \pi, N^\ell \pi)$ $(k, \ell \in \mathbb{N})$ *are Hamiltonian pairs.*

Dirac pairs

Let A be a vector bundle, and let A^* be the dual vector bundle. For relations $L \subset A \times A^*$ and $L' \subset A \times A^*$, we consider the relation in A,

$$\boxed{\mathbf{N}_{L,L'} = \overline{L} * L'.}$$

Assume that (A, μ) is a Lie algebroid, and that $E = A \oplus A^*$ is equipped with the Dorfman bracket.

Definition

Dirac structures L and L' in A are said to be a *Dirac pair* if $\mathbf{N}_{L,L'}$ is a Nijenhuis relation in A.

Poisson pairs and Dirac pairs

If $L = \overline{\mathrm{graph}\,\pi}$ and $L' = \overline{\mathrm{graph}\,\pi'}$, then

$$\mathbf{N}_{L,L'} = \mathrm{graph}\,\pi * \overline{\mathrm{graph}\,\pi'} = \mathbf{N}(\pi, \pi').$$

Theorem

(i) Bivectors π and π' constitute a Poisson pair if and only if their graphs constitute a Dirac pair.
(ii) If (π, π') is a Hamiltonian pair, then $(\overline{\mathrm{graph}\,\pi}, \overline{\mathrm{graph}\,\pi'})$ is a Dirac pair.
(iii) Conversely, if $(\overline{\mathrm{graph}\,\pi}, \overline{\mathrm{graph}\,\pi'})$ is a Dirac pair and if π and π' are non-degenerate bivectors, then (π, π') is a Hamiltonian pair.

Presymplectic pairs

Definition

If ω and ω' are presymplectic structures whose graphs constitute a Dirac pair, (ω, ω') is called a *presymplectic pair*. If, in addition, ω and ω' are non-degenerate, (ω, ω') is called a *symplectic pair*.

For $L = \mathrm{graph}\,\omega$, $L' = \mathrm{graph}\,\omega'$,

$$\mathbf{N}_{L,L'} = \overline{\mathrm{graph}\,\omega} * \mathrm{graph}\,\omega'.$$

Theorem

Symplectic pairs are in one-to-one correspondence with non-degenerate Poisson pairs.

Examples from the theory of Monge-Ampère operators

See Kushner–Lychagin–Rubtsov [2007] and Lychagin–Rubtsov–Chekalov [1993]. See yks–Roubtsov [2010].

Let $M = T^*\mathbb{R}^2$ and let Ω be the canonical symplectic form on M. Here $A = TM$. In canonical coordinates (q^1, q^2, p_1, p_2) on M, $\Omega = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_2$.
Examples of presymplectic pairs (Ω, ω) are defined by

$$\omega = \omega_H = \mathrm{d}q^1 \wedge \mathrm{d}p_1 - \mathrm{d}q^2 \wedge \mathrm{d}p_2,$$

$$\omega = \omega_E = \mathrm{d}q^1 \wedge \mathrm{d}p_2 - \mathrm{d}q^2 \wedge \mathrm{d}p_1,$$

$$\omega = \omega_P = \mathrm{d}q^1 \wedge \mathrm{d}p_2.$$

The pair (Ω, ω_E) is a 'conformal symplectic couple' as defined by Geiges (Duke [1996], 4-manifolds), *i.e.*, it is a closed, effective 2-form ($\Omega \wedge \omega = 0$), with Pfaffian equal to 1 ($\omega \wedge \omega = \Omega \wedge \Omega$).

PΩ-structures

Definition

A bivector π and a 2-form ω define a *PΩ-structure* on a Lie algebroid (A, μ) if π is a Poisson bivector, and both ω and ω_N are closed, where $N = \pi \circ \omega$ and $\omega_N = \omega \circ N$.

Proposition

Let π be a Poisson bivector and let ω be a presymplectic form. Then $(\overline{\mathrm{graph}\,\pi}, \mathrm{graph}\,\omega)$ is a Dirac pair if and only if $\pi \circ \omega$ is a Nijenhuis tensor.

Proof If $L = \overline{\mathrm{graph}\,\pi}$ and $L' = \mathrm{graph}\,\omega$, then

$$\mathbf{N}_{L,L'} = \mathrm{graph}\,(\pi \circ \omega).$$

□

Dirac pairs and PΩ-structures

Theorem

(i) If a Poisson structure π and a presymplectic structure ω constitute a PΩ-structure, their graphs constitute a Dirac pair.
(ii) Conversely, if the graphs of a Poisson structure π and a presymplectic structure ω constitute a Dirac pair, and if π is non-degenerate, then π and ω constitute a PΩ-structure.

The proof of (ii) uses the fact that if π is a non-degenerte Poisson bivector, ω is closed and $N = \pi \circ \omega$ is a Nijenhuis tensor, then $\{\pi, \mathrm{d}(\omega_N)\} = 0$.

ΩN-structures

Let N be a $(1,1)$-tensor and ω a 2-form on (A,μ) such that $\omega \circ N = N^* \circ \omega$. Then ω_N defined by $\omega_N = \omega \circ N$ is a 2-form.

Definition

A 2-form ω and a $(1,1)$-tensor N define an *ΩN-structure* on a Lie algebroid (A,μ) if $\omega \circ N = N^* \circ \omega$, N is a Nijenhuis tensor, and both ω and ω_N are closed, where $\omega_N = \omega \circ N$.

Examples

In the notation of the previous example, in coordinates on $T^*\mathbb{R}^2$, (q^1, q^2, p_1, p_2), let $N_H = \Omega^{-1} \circ \omega_H$ and $N_E = \Omega^{-1} \circ \omega_E$, so that

$$N_H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad N_E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then (Ω, N_H) and (Ω, N_E) are ΩN-structures on $T^*\mathbb{R}^2$, with $N_H^2 = \mathrm{Id}$ and $N_E^2 = -\mathrm{Id}$. Thus N_E is a complex structure, and N_H is a product structure on $T^*(\mathbb{R}^2)$.

Let $N_P = \Omega^{-1} \circ \omega_P$, so that $N_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then (Ω, N_P) is an ΩN-structure with $N_P^2 = 0$, so that N_P is a tangent structure.

The non-degenerate case

Proposition

Let ω be a non-degenerate 2-form and N a $(1,1)$-tensor such that $\omega_N = \omega \circ N$ is skew-symmetric. Then (ω, N) is an ΩN-structure if and only if $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ is a Dirac pair.

Proof When $L = \mathrm{graph}\,\omega$ and $L' = \mathrm{graph}\,\omega_N$,

$$\mathbf{N}_{LL'} = \{(x,y) \in A \times A \,|\, \omega_N x = \omega y\}.$$

Therefore, when ω is invertible, $\mathbf{N}_{LL'} = \mathrm{graph}\,N$. $\square$

Example The pairs $(\mathrm{graph}\,\Omega\,,\ \mathrm{graph}\,\omega_H)$, $(\mathrm{graph}\,\Omega\,,\ \mathrm{graph}\,\omega_E)$ and $(\mathrm{graph}\,\Omega, \mathrm{graph}\,\omega_P)$ are the Dirac pairs associated with the ΩN-structures described in the previous example.

Weak ΩN-structures

In the next theorem (yks [2011]), the 2-form ω is not assumed to be non-degenerate. Cf. also Dorfman [1993].
Let ω be a 2-form and N a $(1,1)$-tensor such that $\omega_N = \omega \circ N$ is skew-symmetric.
We shall call (ω, N) a *weak ΩN-structure* if ω and ω_N are closed 2-forms, and the torsion of N takes values in the kernel of ω.

We set $\mathbf{N} = \mathbf{N}_{LL'} = \{(x, y) \in A \times A \,|\, \omega_N x = \omega y\}$ and

$$\mathbf{N}^+ = \{(\omega x, \omega_N x) \in A^* \times A^* \,|\, x \in A\}.$$

The relation $\mathbf{N}^+$ is the restriction of the graph of N^* to the image of ω, and a subset of $\mathbf{N}^*$.

Dirac pairs and ΩN-structures

Theorem

(i) If (ω, N) is an ΩN-structure, and if $\mathbf{N}^+ = \mathbf{N}^*$, then $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ is a Dirac pair.
(ii) Conversely, if $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ is a Dirac pair, then (ω, N) is a weak ΩN-structure.

Proof Evaluate $\mathrm{d}\omega$, $\mathrm{d}\omega_N$ and $\mathrm{d}\omega_{N^2}$ on well chosen triples of vectors [...]. $\square$

More generally, all 2-forms $\omega \circ N^2$, $\omega \circ N^3$, $\ldots$, $\omega \circ N^p, \ldots$ are closed. Whence a hierarchy of Dirac pairs.

This property is the basis of the construction of a sequence of integrals in involution for bi-Hamiltonian systems, and for the extension of this property to systems associated to a Dirac pair.

Conclusion

• Generalized geometry appears more and more frequently in the physics literature. Last February, I heard a lecture at IHP (Institut Henri Poincaré) in Paris on .. supergravity in terms of "generalized connections" (by Daniel Waldram). (Cf. the earlier Gabella et al., on "type IIB supergravity and generalized complex geometry" [2010]).

• Dirac pairs are the basis of Dorfman's work on integrable systems. More recently, see Barakat–De Sole–Kac [2009].

• Search for new examples and applications.

• Relate Dirac pairs and the Dirac–Nijenhuis manifolds of Long-Guang He and Bao-Kang Liu [2006].

• Extend the theory of Dirac pairs to more general doubles (Lie bialgebroid, proto-bialgebroid).

• Define and study Dirac pairs on Courant algebroids in general.

• Relate Dirac pairs with Dirac-Nijenhuis structures (Cariñena–Grabowski–Marmo [2004], Clemente-Gallardo–Nunes da Costa [2004]) and "weak deforming tensors".

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